INTUITIONISTIC FUZZY $\theta$-CLOSURE AND $\theta$-INTERIOR

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ABSTRACT. The concept of intuitionistic fuzzy $\theta$-interior operator is introduced and discussed in intuitionistic fuzzy topological spaces. As applications of this concept, intuitionistic fuzzy strongly $\theta$-continuous, intuitionistic fuzzy $\theta$-continuous, and intuitionistic fuzzy weakly continuous functions are characterized in terms of intuitionistic fuzzy $\theta$-interior operator.

1. Introduction

As a generalization of fuzzy sets, the concept of intuitionistic fuzzy set was introduced by Atanassov [1]. Recently, Çoker and his colleagues [2, 3, 4] introduced intuitionistic fuzzy topological space using intuitionistic fuzzy sets. Mukherjee introduced the concepts of fuzzy $\theta$-closure operator in [9] and the notions of fuzzy $\theta$-continuous and fuzzy weakly continuous functions in [8]. Hanafy et al. introduced and investigated intuitionistic fuzzy $\theta$-closure operator, intuitionistic fuzzy strongly $\theta$-continuous, intuitionistic fuzzy $\theta$-continuous and intuitionistic fuzzy weakly continuous functions in [6]. In this paper, we define intuitionistic fuzzy $\theta$-interior operator and study the properties of intuitionistic fuzzy $\theta$-interior operator in intuitionistic fuzzy topological spaces. As applications of this concept, intuitionistic fuzzy strongly $\theta$-continuous, intuitionistic fuzzy $\theta$-continuous, and intuitionistic fuzzy weakly continuous functions are characterized in terms of intuitionistic fuzzy $\theta$-interior operator.

2. Preliminaries

Let $X$ be a nonempty set and $I$ the unit interval [0,1]. An intuitionistic fuzzy set (IFS for short) $A$ is an object having the form

$$A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\},$$

where the functions $\mu_A : X \to I$ and $\gamma_A : X \to I$ denote the degree of membership and the degree of nonmembership, respectively, and $\mu_A + \gamma_A \leq 1$. 

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Sometimes we denote $A = (\mu_A, \gamma_A)$ for simplicity. Let $I(X)$ denote the set of all intuitionistic fuzzy sets in $X$.

Obviously, every fuzzy set $\mu_A$ in $X$ is an intuitionistic fuzzy set of the form 
$\{ (x, \mu_A(x), 1 - \mu_A(x)) : x \in X \}$.

**Definition 2.1** ([1]). Let $X$ be a nonempty set and the IFSs $A$ and $B$ be of the form 
$A = \{ (x, \mu_A(x), \gamma_A(x)) : x \in X \}, \quad B = \{ (x, \mu_B(x), \gamma_B(x)) : x \in X \}$.

Then

1. $A \leq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$,
2. $A = B$ if and only if $A \leq B$ and $B \leq A$,
3. $A^c = \{ (x, \gamma_A(x), \mu_A(x)) : x \in X \}$,
4. $A \cap B = \{ (x, \mu_A \wedge \mu_B(x), \gamma_A \vee \gamma_B(x)) : x \in X \}$,
5. $A \cup B = \{ (x, \mu_A \vee \mu_B(x), \gamma_A \wedge \gamma_B(x)) : x \in X \}$,
6. $0_+ = \{ (x, 0, 1) : x \in X \}$ and $1_- = \{ (x, 1, 0) : x \in X \}$.

**Definition 2.2** ([2]). Let $X$ and $Y$ be two nonempty sets, and let $f : X \to Y$ be a function.

1. If $B = \{ (y, \mu_B(y), \gamma_B(y)) : y \in Y \}$ is an IFS in $Y$, then the preimage of $B$ under $f$, denoted by $f^{-1}(B)$, is the IFS in $X$ defined by 
$f^{-1}(B) = \{ (x, f^{-1}(\mu_B(x)), f^{-1}(\gamma_B(x))) : x \in X \}$.
2. If $A = \{ (x, \lambda_A(x), \delta_A(x)) : x \in X \}$ is an IFS in $X$, then the image of $A$ under $f$, denoted by $f(A)$, is the IFS in $Y$ defined by 
$f(A) = \{ (y, f(\lambda_A(y)), (1 - f(1 - \delta_A))(y)) : y \in Y \}$, where

$$f(\lambda_A)(y) = \begin{cases} 
\sup_{x \in f^{-1}(y)} \lambda_A(x) & \text{if } f^{-1}(y) \neq \emptyset \\
0 & \text{otherwise,}
\end{cases}$$

and

$$f(1 - \delta_A)(y) = \begin{cases} 
\inf_{x \in f^{-1}(y)} \lambda_A(x) & \text{if } f^{-1}(y) \neq \emptyset \\
1 & \text{otherwise.}
\end{cases}$$

**Theorem 2.3** ([2]). Let $A$ and $A_j (j \in J)$ be IFSs in $X$, $B$ and $B_j (j \in K)$ IFSs in $Y$. Let $f : X \to Y$ be a function. Then

1. $A_1 \leq A_2 \Rightarrow f(A_1) \leq f(A_2)$,
2. $B_1 \leq B_2 \Rightarrow f^{-1}(B_1) \leq f^{-1}(B_2)$,
3. $A \leq f^{-1}(f(A))$ (If $f$ is injective, then $A = f^{-1}(f(A))$),
4. $f(f^{-1}(B)) \leq B$ (If $f$ is surjective, then $B = f(f^{-1}(B))$),
5. $f^{-1}(\bigcup B_j) = \bigcup f^{-1}(B_j)$,
6. $f^{-1}(\bigcap B_j) = \bigcap f^{-1}(B_j)$,
7. $f(\bigcup A_j) = \bigcup f(A_j)$,
8. $f(\bigcap A_j) \leq \bigcap f(A_j)$, (If $f$ is injective, then $f(\bigcap A_j) = \bigcap f(A_j)$),
9. $f^{-1}(1) = 1$, if $f$ is surjective,
10. $f(0) = 0$,
11. $f(A)^c \leq f(A^c)$, if $f$ is surjective,
(12) \( f^{-1}(B^c) = f^{-1}(B)^c \).

**Definition 2.4** ([2]). An *intuitionistic fuzzy topology* (IFT for short) on a nonempty set \( X \) is a family \( \mathcal{T} \) of IFSs in \( X \) which satisfies the following axioms:

1. \( 0_x, 1_x \in \mathcal{T} \).
2. \( G_1 \cap G_2 \in \mathcal{T} \) for any \( G_1, G_2 \in \mathcal{T} \).
3. \( \bigcup G_i \in \mathcal{T} \) for any arbitrary \( \{G_i : i \in J\} \subseteq \mathcal{T} \).

In this case the pair \( (X, \mathcal{T}) \) is called an *intuitionistic fuzzy topological space* (IFTS for short) and any IFS in \( \mathcal{T} \) is known as an *intuitionistic fuzzy open set* (IFOS for short) in \( X \).

**Definition 2.5** ([2]). Let \( (X, \mathcal{T}) \) be an IFTS and \( A = \langle x, \mu_A, \lambda_A \rangle \) an IFS in \( X \). Then the *intuitionistic fuzzy interior of \( A \)* and the *intuitionistic fuzzy closure of \( A \)* are defined by

\[
\text{int}(A) = \bigcup \{G \mid G \subseteq A, G \in \mathcal{T}\}
\]

and

\[
\text{cl}(A) = \bigcap \{K \mid A \subseteq K, K^c \in \mathcal{T}\}.
\]

**Theorem 2.6** ([2]). For any IFS \( A \) in \( (X, \mathcal{T}) \), we have

\[
\text{cl}(A^c) = (\text{int}(A))^c \quad \text{and} \quad \text{int}(A^c) = (\text{cl}(A))^c.
\]

**Definition 2.7** ([3, 4]). Let \( \alpha, \beta \in [0, 1] \) and \( \alpha + \beta \leq 1 \). An *intuitionistic fuzzy point* (IFP for short) \( x_{(\alpha, \beta)} \) of \( X \) is an IFS in \( X \) defined by

\[
x_{(\alpha, \beta)}(y) = \begin{cases} \alpha, & y = x, \\ 0, & y \neq x. \end{cases}
\]

In this case, \( x \) is called the *support* of \( x_{(\alpha, \beta)} \), \( \alpha \) the value of \( x_{(\alpha, \beta)} \) and \( \beta \) the *nonvalue* of \( x_{(\alpha, \beta)} \). An IFP \( x_{(\alpha, \beta)} \) is said to belong to an IFS \( A = (\mu_A, \gamma_A) \) in \( X \), denoted by \( x_{(\alpha, \beta)} \in A \), if \( \alpha \leq \mu_A(x) \) and \( \beta \geq \gamma_A(x) \).

**Remark 2.8.** If we consider an IFP \( x_{(\alpha, \beta)} \) as an IFS, then we have the relation \( x_{(\alpha, \beta)} \in A \) if and only if \( x_{(\alpha, \beta)} \leq A \).

**Definition 2.9** ([3, 4]). Let \( x_{(\alpha, \beta)} \) be an IFP in \( X \) and \( U = (\mu_U, \gamma_U) \) an IFS in \( X \). Suppose further that \( \alpha \) and \( \beta \) are real numbers between 0 and 1. The IFP \( x_{(\alpha, \beta)} \) is said to be *properly contained* in \( U \) if and only if \( \alpha < \mu_U(x) \) and \( \beta > \gamma_U(x) \).

**Definition 2.10** ([4]). (1) An IFP \( x_{(\alpha, \beta)} \) is said to be *quasi-coincident* with the IFS \( U = (\mu_U, \gamma_U) \), denoted by \( x_{(\alpha, \beta)} \sim U \), if and only if \( \alpha > \gamma_U(x) \) or \( \beta < \mu_U(x) \).

(2) Let \( U = (\mu_U, \gamma_U) \) and \( V = (\mu_V, \gamma_V) \) be two IFSs in \( X \). Then \( U \) and \( V \) are said to be *quasi-coincident*, denoted by \( U \sim V \), if and only if there exists an element \( x \in X \) such that \( \mu_U(x) > \gamma_V(x) \) or \( \gamma_U(x) < \mu_V(x) \).

The word ‘not quasi-coincident’ will be abbreviated as \( \not\sim \).

**Proposition 2.11** ([4]). Let \( U, V \) be IFSs and \( x_{(\alpha, \beta)} \) an IFP in \( X \). Then
(1) \( U \tilde{q} V^c \iff U \leq V \),
(2) \( U q V \iff U \not\leq V^c \),
(3) \( x_{(\alpha, \beta)} \leq U \iff x_{(\alpha, \beta)} \tilde{q} U^c \),
(4) \( x_{(\alpha, \beta)} q U \iff x_{(\alpha, \beta)} \not\leq U^c \).

**Definition 2.12** ([4]). Let \((X, T)\) be an IFTS and \(x_{(\alpha, \beta)}\) an IFP in \(X\). An IFS \(A\) is called a neighborhood \((q\)-neighborhood, respectively\) of \(x_{(\alpha, \beta)}\), if there exists an IFOS \(U\) in \(X\) such that \(x_{(\alpha, \beta)} \in U \leq A\) \((x_{(\alpha, \beta)} q U \leq A\), respectively\). The family of all neighborhoods \((q\)-neighborhoods, respectively\) of \(x_{(\alpha, \beta)}\) will be denoted by \(N(x_{(\alpha, \beta)})(N^q(x_{(\alpha, \beta)}),\) respectively\).

3. **Intuitionistic fuzzy \(\theta\)-closure and \(\theta\)-interior**

In this section, we study some properties of intuitionistic fuzzy \(\theta\)-interior.

**Definition 3.1** ([6]). An IFP \(x_{(\alpha, \beta)}\) is said to be intuitionistic fuzzy \(\theta\)-cluster point of an IFS \(U\) if and only if \(\text{cl}(A) \tilde{q} U^c\) for each \(q\)-neighborhood \(A\) of \(x_{(\alpha, \beta)}\). The set of all intuitionistic fuzzy \(\theta\)-cluster points of \(U\) is called the intuitionistic fuzzy \(\theta\)-closure of \(U\) and denoted by \(\text{cl}_\theta(U)\). An IFS \(U\) will be called intuitionistic fuzzy \(\theta\)-closed (IF\(\theta\)CS for short) if and only if \(U = \text{cl}_\theta(U)\). The complement of an IF\(\theta\)CS is called an intuitionistic fuzzy \(\theta\)-open set (IF\(\theta\)OS for short).

**Remark 3.2.** Usually, the complement of a fuzzy set \(A\) is defined by \(1 - A\), but the complement of an intuitionistic fuzzy set \(A = \langle x, \mu_A, \gamma_A \rangle\) is defined by \(A^c = \langle x, \gamma_A, \mu_A \rangle\). So \(1 - A = \langle x, 1 - \mu_A, 1 - \gamma_A \rangle \neq \langle x, \gamma_A, \mu_A \rangle = A^c\).

Moreover, although \(A\) is an intuitionistic fuzzy set, the set \(1 - A\) is not necessarily an IFS. In [6], Hanafy defined the intuitionistic fuzzy \(\theta\)-interior of \(U\) by
\[
\text{int}_\theta(U) = 1 - \text{cl}_\theta(1 - U).
\]
This definition could be misunderstood because of the expression \(1 - U\). So we rephrase the definition of intuitionistic fuzzy \(\theta\)-interior as follows.

**Definition 3.3.** Let \((X, T)\) be an IFTS and \(U\) an IFS in \(X\). The intuitionistic fuzzy \(\theta\)-interior of \(U\) is denoted and defined by
\[
\text{int}_\theta(U) = (\text{cl}_\theta(U^c))^c.
\]

From the above definition, we have the following relations:
(1) \(\text{cl}_\theta(U^c) = (\text{int}_\theta(U))^c\),
(2) \((\text{cl}_\theta(U))^c = \text{int}_\theta(U^c)\).

**Lemma 3.4.** Let \(U, V\) and \(A\) be IFSs in an IFTS \((X, T)\). If \(A q(U \cup V)\), then \(A q U\) or \(A q V\).

**Proof.** Suppose that \(A q U\) and \(A q V\). Then \(A \leq U^c\) and \(A \leq V^c\). Thus \(A \leq U^c \cap V^c = (U \cup V)^c\). Hence \(A q(U \cup V)\). \(\Box\)
Theorem 3.5. Let $U$ and $V$ be two IFSs in an IFTS $(X, T)$. Then we have the following:

1. $\text{cl}_0(0_-) = 0_-$,
2. $U \leq \text{cl}_0(U)$,
3. $U \leq V \Rightarrow \text{cl}_0(U) \leq \text{cl}_0(V)$,
4. $\text{cl}_0(U) \cup \text{cl}_0(V) = \text{cl}_0(U \cup V)$,
5. $\text{cl}_0(U \cap V) \leq \text{cl}_0(U) \cap \text{cl}_0(V)$.

Proof. (1) Obvious.

(2) Suppose that there is an IFP $x_{(a, \beta)}$ in $X$ such that $x_{(a, \beta)} \notin \text{cl}_0(U)$ and $x_{(a, \beta)} \in U$. Then there is a $q$-neighborhood $A$ of $x_{(a, \beta)}$ such that $\text{cl}(A)qU$. Thus $A \leq U^c$. Since $A$ is a $q$-neighborhood of $x_{(a, \beta)}$, there is an IFOS $V$ such that $x_{(a, \beta)} \notin V$. Since $A \leq U^c$, we have $x_{(a, \beta)} \notin V$, and hence $x_{(a, \beta)} \in U^c$. On the other hand, we have $x_{(a, \beta)} \leq U$, because $x_{(a, \beta)} \in U$. It is a contradiction.

(3) Let $x_{(a, \beta)}$ be an IFP in $X$ such that $x_{(a, \beta)} \notin \text{cl}_0(V)$. Then there is a $q$-neighborhood $A$ of $x_{(a, \beta)}$ such that $\text{cl}(A)qV$. Since $U \leq V$, we have $\text{cl}(A)qU$. Therefore $x_{(a, \beta)} \notin \text{cl}_0(U)$.

(4) Since $U \leq U \cup V$, $\text{cl}_0(U) \leq \text{cl}_0(U \cup V)$. Similarly, $\text{cl}_0(V) \leq \text{cl}_0(U \cup V)$. Hence $\text{cl}_0(U) \cup \text{cl}_0(V) \leq \text{cl}_0(U \cup V)$. On the other hand, take any $x_{(a, \beta)} \in \text{cl}_0(U \cup V)$. Then for any $q$-neighborhood $A$ of $x_{(a, \beta)}$, $\text{cl}(A)q(U \cup V)$. By Lemma 3.4, $\text{cl}(A)qU$ or $\text{cl}(A)qV$. Therefore $x_{(a, \beta)} \in \text{cl}_0(U)$ or $x_{(a, \beta)} \in \text{cl}_0(V)$. Hence $\text{cl}_0(U \cup V) \leq \text{cl}_0(U) \cup \text{cl}_0(V)$.

(5) Since $U \cap V \leq U$, $\text{cl}_0(U \cap V) \leq \text{cl}_0(U)$. Similarly, $\text{cl}_0(U \cap V) \leq \text{cl}_0(V)$. Therefore $\text{cl}_0(U \cap V) \leq \text{cl}_0(U) \cap \text{cl}_0(V)$. \(\square\)

Remark 3.6. For an IFS $A$ in an IFTS $(X, T)$, intuitionistic fuzzy $\theta$-closure $\text{cl}_0(A)$ is not necessarily an IFCS, and hence $\text{cl}_0(\text{cl}_0(A)) \neq \text{cl}_0(A)$, which is shown in the following example. Thus $\text{cl}_0$ operator does not satisfies the Kuratowski closure axioms.

Example 3.7. Let $X = \{a, b, c\}$ and $U = \langle (0.5, 0.3, 0.2), (0.6, 0.7, 0.4) \rangle, V = \langle (0.4, 0.5, 0.1), (0.5, 0.7, 0.3) \rangle$. Then the family $T = \{U, V\}$ of IFSs of $X$ is an IFP on $X$. Let $A = \langle (0.3, 0.4, 0.5), (0.4, 0.7, 0.3) \rangle$ be an IFS in $X$. Then $a_{(0.8, 0.1)} \notin \text{cl}_0(A)$ and $a_{(0.6, 0.4)} \in \text{cl}_0(A)$. But $a_{(0.8, 0.1)} \in \text{cl}_0(a_{(0.6, 0.4)}) \leq \text{cl}_0(\text{cl}_0(A))$. Hence $\text{cl}_0(\text{cl}_0(A)) \neq \text{cl}_0(A)$.

Remark 3.8 (\cite{6}). For any IFS $U$ in IFTS $(X, T)$, $\text{cl}(U) \leq \text{cl}_0(U)$. Moreover $\text{cl}(U) = \text{cl}_0(U)$ for an IFOS. Thus for any IFS $U$ in IFTS $(X, T)$,

$$\text{cl}_0(U) = \bigcap \{\text{cl}(A) \mid A \in T, U \leq A\}$$

$$= \bigcap \{\text{cl}(A) \mid A \in T, U \leq A\}.$$

So, in an intuitionistic fuzzy regular space $(X, T)$, every IFCS is an IFCS and hence for any IFS $U$ in $X$, $\text{cl}_0(U)$ is an IFCS.

Clearly, $U$ is an IFCS if and only if $\text{int}_0(U) = U$. Also we have following properties for the interior operator.
Theorem 3.9. Let $U$ and $V$ be two IFSs in an IFTS $(X, T)$. Then we have the following:

1. $\text{int}_\theta(1_\sim) = 1_\sim$,
2. $\text{int}_\theta(U) \subseteq U$,
3. $U \subseteq V \Rightarrow \text{int}_\theta(U) \subseteq \text{int}_\theta(V)$,
4. $\text{int}_\theta(U \cap V) = \text{int}_\theta(U) \cap \text{int}_\theta(V)$,
5. $\text{int}_\theta(U) \cup \text{int}_\theta(V) \subseteq \text{int}_\theta(U \cup V)$.

Proof. (1) Obvious.

(2) Let $x_{(\alpha, \beta)} \in \text{int}_\theta(U)$. From the fact that $\text{int}_\theta(U) = (\text{cl}_\theta(U^c))^c = (x, \gamma_{\text{cl}_\theta(U^c)}, \mu_{\text{cl}_\theta(U^c)})$, we have $\alpha \leq \gamma_{\text{cl}_\theta(U^c)}(x)$ and $\beta \geq \mu_{\text{cl}_\theta(U^c)}(x)$. Since $U^c \subseteq \text{cl}_\theta(U^c)$, we have $\mu_{U^c} \leq \mu_{\text{cl}_\theta(U^c)}$ and $\gamma_{U^c} \geq \gamma_{\text{cl}_\theta(U^c)}$. Thus $\alpha \leq \gamma_{U^c}(x) = \mu_{U^c}(x)$ and $\beta \geq \mu_{U^c}(x) = \gamma_{U^c}(x)$. Hence $x_{(\alpha, \beta)} \in U$.

(3) Let $U \subseteq V$. Then $U^c \supseteq V^c$. By Theorem 3.5, $\text{cl}_\theta(U^c) \supseteq \text{cl}_\theta(V^c)$. Thus $(\text{cl}_\theta(U^c))^c \subseteq (\text{cl}_\theta(V^c))^c$. Hence $\text{int}_\theta(U) = (\text{cl}_\theta(U^c))^c \subseteq (\text{cl}_\theta(V^c))^c = \text{int}_\theta(V)$.

(4) $\text{int}_\theta(U \cap V) = (\text{cl}_\theta((U \cap V)^c))^c = (\text{cl}_\theta(U^c \cup V^c))^c = (\text{cl}_\theta(U^c) \cup \text{cl}_\theta(V^c))^c = (\text{cl}_\theta(U^c))^c \cap (\text{cl}_\theta(V^c))^c = \text{int}_\theta(U) \cap \text{int}_\theta(V)$.

(5) Since $U \subseteq U \cup V$, we have $\text{int}_\theta(U) \subseteq \text{int}_\theta(U \cup V)$. Since $V \subseteq U \cup V$, we have $\text{int}_\theta(V) \subseteq \text{int}_\theta(U \cup V)$. Therefore $\text{int}_\theta(U) \cup \text{int}_\theta(V) \subseteq \text{int}_\theta(U \cup V)$. □

Corollary 3.10. For an IFS $U$, $\text{int}_\theta(U) \subseteq \text{int}(U)$.

Proof. Let $U$ be an IFS. Then $U^c$ is an IFS. Thus $\text{cl}(U^c) \subseteq \text{cl}_\theta(U^c)$ by [6, Theorem 3.3 (iii)]. Hence $\text{int}_\theta(U) = (\text{cl}_\theta(U^c))^c \subseteq (\text{cl}(U^c))^c = \text{int}(U)$. □

Theorem 3.11. If $U$ is an IFCS in an IFTS $(X, T)$, then $\text{int}_\theta(U) = \text{int}(U)$.

Proof. Let $U$ be an IFCS. Then $U^c$ is an IFOS. Thus $\text{cl}(U^c) = \text{cl}_\theta(U^c)$ by [6, Theorem 3.6]. Hence $\text{int}_\theta(U) = (\text{cl}_\theta(U^c))^c = (\text{cl}(U^c))^c = \text{int}(U)$. □

Theorem 3.12. Let $U$ be an IFS in an IFTS $(X, T)$. Then

$$\text{int}_\theta(U) = \bigvee \{\text{int}_\theta(A) \mid A^c \in T, A \subseteq U\}$$

$$= \bigvee \{\text{int}(A) \mid A^c \in T, A \subseteq U\}.$$ 

Proof. Using [6, Theorem 3.15], we have

$$\text{int}_\theta(U) = (\text{cl}_\theta(U^c))^c = (\bigcap \{\text{cl}_\theta(B) \mid B \in T, U^c \subseteq B\})^c$$

$$= \bigvee \{\text{cl}_\theta(B)^c \mid B \in T, U^c \subseteq B\}$$

$$= \bigvee \{\text{int}_\theta(B^c) \mid B \in T, U^c \subseteq B\}.$$ 

Let $A = B^c$. Then

$$\text{int}_\theta(U) = \bigvee \{\text{int}_\theta(A) \mid A^c \in T, A \subseteq U\}.$$ 

The second equality holds from Theorem 3.11. □

Corollary 3.13. For an IFS $U$ in an IFTS $(X, T)$, $\text{int}_\theta(U)$ is an IFOS.
Remark 3.14. For an IFS \( U \) in an IFTS \((X,T)\), \( \text{int}_\theta(U) \) is not necessarily IF\(\theta\)OS.

4. Characterizations for some types of functions

Hanafy et al. already characterized some types of functions by intuitionistic fuzzy \( \theta \)-closure. Here, we will characterize an intuitionistic fuzzy strongly \( \theta \)-continuous, intuitionistic fuzzy \( \theta \)-continuous, and intuitionistic fuzzy weakly continuous functions in terms of intuitionistic fuzzy \( \theta \)-interior.

**Lemma 4.1.** Let \( f : (X, T) \to (Y, T') \) be a function and \( U, V \) be an IFSs. If \( UqV \), then \( f(U)qf(V) \).

**Proof.** Suppose that \( f(U)\tilde{q}f(V) \). Then \( f(U) \leq (f(V))^c \). Since \( U \leq f^{-1}(f(U)) \), we have \( U \leq f^{-1}(f(U)) \leq f^{-1}((f(V))^c) \). Thus we have \( U\tilde{q}f^{-1}((f(V))^c) = f^{-1}(((f(V))^c))^c = f^{-1}(f(V)). \) Since \( V \leq f^{-1}(f(V)) \) and \( U\tilde{q}f^{-1}(f(V)) \), we have \( UqV \). \( \square \)

Recall that a function \( f : (X, T) \to (Y, T') \) is said to be intuitionistic fuzzy strongly \( \theta \)-continuous if and only if for each IFP \( x_{(\alpha,\beta)} \) in \( X \) and \( V \in N^q(f(x_{(\alpha,\beta)})) \), there exists \( U \in N^q(x_{(\alpha,\beta)}) \) such that \( f(U) \leq V \) (See [6]).

**Theorem 4.2.** Let \( f : (X, T) \to (Y, T') \) be a function. Then the following statements are equivalent:

1. \( f \) is an intuitionistic fuzzy strongly \( \theta \)-continuous function.
2. \( f(\text{cl}_\theta(U)) \leq \text{cl}(f(U)) \) for each IFS \( U \) in \( X \).
3. \( \text{cl}_\theta(f^{-1}(V)) \leq f^{-1}(\text{cl}(V)) \) for each IFS \( V \) in \( Y \).
4. \( f^{-1}(V) \) is an IFCS in \( X \) for each IFCS \( V \) in \( Y \).
5. \( f^{-1}(V) \) is an IF\(\theta\)OS in \( X \) for each IFOS \( V \) in \( Y \).
6. \( f^{-1}(\text{int}(V)) \leq \text{int}_\theta(f^{-1}(V)) \) for each IFS \( V \) of \( Y \).

**Proof.** (1) \( \Leftrightarrow \) (2) \( \Leftrightarrow \) (3) \( \Leftrightarrow \) (4) \( \Leftrightarrow \) (5). See [6].

(3) \( \Rightarrow \) (6). Let \( V \) be an IFS in \( Y \). Then \( V^c \) is an IFS in \( Y \). Since \( f \) is an intuitionistic fuzzy strongly \( \theta \)-continuous function, by the hypothesis, \( \text{cl}_\theta(f^{-1}(V^c)) \leq f^{-1}(\text{cl}(V^c)) \). Thus

\[
\text{cl}_\theta(f^{-1}(V^c)) = (\text{cl}_\theta(f^{-1}(V)))^c = (\text{cl}(f^{-1}(V)))^c
\]

\[
\leq (\text{cl}_\theta(f^{-1}(V))) = (\text{cl}_\theta(f^{-1}(V)))^c
\]

\[
\leq f^{-1}(\text{int}(V)) \leq \text{int}_\theta(f^{-1}(V^c)).
\]

(6) \( \Rightarrow \) (3). Let \( V \) be an IFS in \( Y \). Then \( V^c \) is an IFS in \( Y \). By the hypothesis, \( f^{-1}(\text{int}(V)) \leq \text{int}_\theta(f^{-1}(V^c)) \). Thus

\[
\text{cl}_\theta(f^{-1}(V)) = (\text{cl}_\theta(f^{-1}(V)))^c = (\text{int}_\theta(f^{-1}(V)))^c
\]

\[
\leq f^{-1}(\text{int}(V^c)) = f^{-1}(\text{int}(V)) = f^{-1}(\text{cl}(V)). \quad \square
\]

**Theorem 4.3.** Let \( f : (X, T) \to (Y, T') \) be a bijection. Then the following statements are equivalent:

1. \( f \) is an intuitionistic fuzzy strongly \( \theta \)-continuous function.
(2) \( f^{-1}(\text{int}(V)) \leq \text{int}_\theta(f^{-1}(V)) \) for each IFS \( V \) of \( Y \).
(3) \( \text{int}(f(U)) \leq f(\text{int}_\theta(U)) \) for each IFS \( U \) in \( X \).

**Proof.** By Theorem 4.4, it suffices to show that (2) is equivalent to (3).

(2) \( \Rightarrow \) (3). Let \( U \) be an IFS in \( X \). Then \( f(U) \) is an IFS in \( Y \). By the hypothesis, \( f^{-1}(\text{int}(f(U))) \leq \text{int}_\theta(f^{-1}(f(U))) \). Since \( f \) is one-to-one,
\[
 f^{-1}(\text{int}(f(U))) \leq \text{int}_\theta(f^{-1}(f(U))) = \text{int}_\theta(U).
\]

Since \( f \) is onto,
\[
 \text{int}(f(U)) = f(f^{-1}(\text{int}(f(U)))) \leq f(\text{int}_\theta(U)).
\]

(3) \( \Rightarrow \) (2). Let \( V \) be an IFS in \( Y \). Then \( f^{-1}(V) \) is an IFS in \( Y \). By the hypothesis, \( \text{int}(f(f^{-1}(V))) \leq f(\text{int}_\theta(f^{-1}(V))) \). Since \( f \) is onto,
\[
 \text{int}(V) \leq f(\text{int}_\theta(f^{-1}(V))).
\]

Since \( f \) is one-to-one,
\[
 f^{-1}(\text{int}(V)) \leq f^{-1}(f(\text{int}_\theta(f^{-1}(V)))) = \text{int}_\theta(f^{-1}(V)). \quad \Box
\]

Recall that function \( f : (X, T) \to (Y, T') \) is said to be an intuitionistic fuzzy \( \theta \)-continuous if and only if for each IFPF \( x_{(\alpha, \beta)} \) in \( X \) and \( V \in N^\theta(x_{(\alpha, \beta)}) \), there exists \( U \in N^\theta(x_{(\alpha, \beta)}) \) such that \( f(\text{cl}(U)) \leq \text{cl}(V) \) (See [6]).

**Theorem 4.4 ([6]).** Let \( f : (X, T) \to (Y, T') \) be a function. Then the following statements are equivalent:

1. \( f \) is an intuitionistic fuzzy \( \theta \)-continuous function.
2. \( f(\text{cl}_\theta(U)) \leq \text{cl}_\theta(f(U)) \) for each IFS \( U \) in \( X \).
3. \( \text{cl}_\theta(f^{-1}(V)) \leq f^{-1}(\text{cl}_\theta(V)) \) for each IFS \( V \) in \( Y \).
4. \( \text{cl}_\theta(f^{-1}(V)) \leq f^{-1}(\text{cl}(V)) \) for each IFPS \( V \) in \( Y \).
5. \( f^{-1}(\text{int}_\theta(V)) \leq \text{int}_\theta(f^{-1}(V)) \) for each IFS \( V \) of \( Y \).

**Proof.** (1) \( \Leftrightarrow \) (2) \( \Leftrightarrow \) (3) \( \Leftrightarrow \) (4). See [6].

(3) \( \Rightarrow \) (5). Let \( V \) be an IFS in \( Y \). Then \( V^c \) is an IFS in \( Y \). Since \( f \) is an intuitionistic fuzzy \( \theta \)-continuous function, by the hypothesis, \( \text{cl}_\theta(f^{-1}(V^c)) \leq f^{-1}(\text{cl}(V^c)) \). Thus
\[
f^{-1}(\text{int}_\theta(V)) = f^{-1}((\text{cl}_\theta(V^c))^c) = (f^{-1}((\text{cl}_\theta(V^c))^c))^c \leq (\text{cl}_\theta(f^{-1}(V^c))^c) = (\text{cl}_\theta(f^{-1}(V)))^c = \text{int}_\theta(f^{-1}(V)).
\]

(5) \( \Rightarrow \) (3). Let \( U \) be an IFS in \( Y \). Then \( V^c \) is an IFS in \( Y \). By the hypothesis,
\[
 f^{-1}(\text{int}_\theta(V^c)) \leq \text{int}_\theta(f^{-1}(V^c)).
\]
Thus
\[
 \text{cl}_\theta(f^{-1}(V)) = (\text{int}_\theta((f^{-1}(V))^c))^c = (\text{int}_\theta(f^{-1}(V)))^c \leq (f^{-1}(\text{int}_\theta(V^c))^c) = f^{-1}(\text{int}(V)). \quad \Box
\]

**Theorem 4.5.** Let \( f : (X, T) \to (Y, T') \) be a bijection. Then the following statements are equivalent:

1. \( f \) is an intuitionistic fuzzy \( \theta \)-continuous function.
(2) \( f^{-1}(\text{int}_\theta(V)) \leq \text{int}_\theta(f^{-1}(V)) \) for each IFS \( V \) of \( Y \).

(3) \( \text{int}_\theta(f(U)) \leq f(\text{int}_\theta(U)) \) for each IFS \( U \) in \( X \).

Proof. By Theorem 4.4, it suffices to show that (2) is equivalent to (3).

(2) \( \Rightarrow \) (3). Let \( U \) be an IFS in \( X \). Then \( f(U) \) is an IFS in \( Y \). By the hypothesis, \( f^{-1}(\text{int}_\theta(f(U))) \leq \text{int}_\theta(f^{-1}(f(U))) \). Since \( f \) is one-to-one,

\[
f^{-1}(\text{int}_\theta(f(U))) \leq \text{int}_\theta(f^{-1}(f(U))) = \text{int}_\theta(U).
\]

Since \( f \) is onto,

\[
\text{int}_\theta(f(U)) = f(f^{-1}(\text{int}_\theta(f(U)))) \leq f(\text{int}_\theta(U)).
\]

(3) \( \Rightarrow \) (2). Let \( V \) be an IFS in \( Y \). Then \( f(\text{int}_\theta(V)) \leq \text{int}_\theta(f(\text{int}_\theta(V))) \). Since \( f \) is onto,

\[
\text{int}_\theta(V) = \text{int}_\theta(f(f^{-1}(V))) \leq f(\text{int}_\theta(f^{-1}(V))).
\]

Since \( f \) is one-to-one,

\[
f^{-1}(\text{int}_\theta(V)) \leq f^{-1}(f(\text{int}_\theta(f^{-1}(V)))) = \text{int}_\theta(f^{-1}(V)). \quad \blacksquare
\]

Recall that function \( f : (X, \mathcal{T}) \to (Y, \mathcal{T}') \) is said to be an intuitionistic fuzzy weakly continuous if and only if for each IFOS \( V \) in \( Y \), \( f^{-1}(V) \leq \text{int}(f^{-1}((\text{cl}_\theta(V))) \) (See [6]).

**Theorem 4.6 ([6]).** Let \( f : (X, \mathcal{T}) \to (Y, \mathcal{T}') \) be a function. Then the following statements are equivalent:

1. \( f \) is an intuitionistic fuzzy weakly continuous function.
2. \( f(\text{cl}_\theta(U)) \leq \text{cl}_\theta(f(U)) \) for each IFS \( U \) in \( X \).
3. \( \text{cl}_\theta(f^{-1}(V)) \leq f^{-1}(\text{cl}_\theta(V)) \) for each IFS \( V \) in \( Y \).
4. \( f^{-1}(\text{int}_\theta(V)) \leq \text{int}_\theta(f^{-1}(V)) \) for each IFS \( V \) of \( Y \).
5. \( f^{-1}(\text{int}_\theta(V)) \leq \text{int}(f^{-1}(V)) \) for each IFS \( V \) of \( Y \).

Proof. (1) \( \Leftrightarrow \) (2) \( \Leftrightarrow \) (3) \( \Leftrightarrow \) (4). See [6].

(3) \( \Rightarrow \) (5). Let \( V \) be an IFS in \( Y \). Then \( V^c \) is an IFS in \( Y \). Since \( f \) is an intuitionistic fuzzy weakly continuous function, by the hypothesis, \( \text{cl}(f^{-1}(V^c)) \leq f^{-1}(\text{cl}_\theta(V^c)) \). Thus

\[
f^{-1}(\text{int}_\theta(V)) = f^{-1}((\text{cl}_\theta(V^c))^c) = (f^{-1}(\text{cl}_\theta((V^c))))^c \leq (\text{cl}(f^{-1}(V^c)))^c = (\text{cl}((f^{-1}(V))^c))^c = \text{int}(f^{-1}(V)).
\]

(5) \( \Rightarrow \) (3). Let \( V \) be an IFS in \( Y \). Then \( V^c \) is an IFS in \( Y \). By the hypothesis,

\[
f^{-1}(\text{int}_\theta(V^c)) \leq \text{int}(f^{-1}(V^c)).
\]

Thus

\[
\text{cl}(f^{-1}(V)) = (\text{int}((f^{-1}(V^c))^c) = (\text{int}(f^{-1}(V))^c)^c \leq (\text{int}(f^{-1}(V^c))^c)^c = f^{-1}((\text{int}_\theta(V^c)))^c = f^{-1}(\text{cl}_\theta(V)). \quad \blacksquare
\]

**Theorem 4.7.** Let \( f : (X, \mathcal{T}) \to (Y, \mathcal{T}') \) be a bijection. Then the following statements are equivalent:

1. \( f \) is an intuitionistic fuzzy weakly continuous function.
(2) $f^{-1}(\text{int}_\theta(V)) \leq \text{int}(f^{-1}(V))$ for each IFS $V$ of $Y$.
(3) $\text{int}_\theta(f(U)) \leq f(\text{int}(U))$ for each IFS $U$ in $X$.

Proof. By Theorem 4.6, it suffices to show that (2) is equivalent to (3).
(2) $\Rightarrow$ (3). Let $U$ be an IFS in $X$. Then $f(U)$ is an IFS in $Y$. By the hypothesis, $
abla^{-1}(\text{int}(f(U))) \leq \text{int}(f^{-1}(f(U)))$. Since $f$ is one-to-one,

$$f^{-1}(\text{int}_\theta(f(U))) \leq \text{int}(f^{-1}(f(U))) = \text{int}(U).$$

Since $f$ is onto,

$$\text{int}_\theta(f(U)) = f(f^{-1}(\text{int}(U))) \leq f(\text{int}(U)).$$

(3) $\Rightarrow$ (2). Let $V$ be an IFS in $Y$. Then $f^{-1}(V)$ is an IFS in $X$. By the hypothesis, $\text{int}_\theta(f^{-1}(V)) \leq f(\text{int}(f^{-1}(V)))$. Since $f$ is onto,

$$\text{int}_\theta(V) \leq f(\text{int}(f^{-1}(V))).$$

Since $f$ is one-to-one,

$$f^{-1}(\text{int}_\theta(V)) \leq f^{-1}(f(\text{int}(f^{-1}(V)))) = \text{int}(f^{-1}(V)).$$

□

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