DUALITY OF $Q_K$-TYPE SPACES

Mujun Zhan and Guangfu Cao

Abstract. For $BMO$, it is well known that $VMO^{**} = BMO$. In this paper such duality results of $Q_K$-type spaces are obtained which generalize the results by M. Pavlović and J. Xiao.

1. Introduction

Let $D = \{ z : |z| < 1 \}$ be the unit disk of complex plane $\mathbb{C}$ and $H(D)$ denote the class of functions analytic in $D$. For $a \in D$, $\varphi_a(z) = \frac{z-a}{1-\overline{a}z}$ is the Möbius map of $D$. Let $K : [0, \infty) \to [0, \infty)$ be a right-continuous and nondecreasing function. For $0 < p < \infty$ and $-2 < q < \infty$, the space $Q_K(p, q)$ consists of all functions $g \in H(D)$ such that

$$
\|g\|^p_{Q_K(p, q)} = \sup_{a \in D} \int_D |g'(z)|^p (1 - |z|^2)^q K(1 - |\varphi_a(z)|^2) dA(z) < \infty,
$$

where $dA(z)$ is the Euclidean area element on $D$. For $p \geq 1$, under the norm $\|g\| = |g(0)| + \|g\|_{Q_K(p, q)}$, $Q_K(p, q)$ is a Banach space. A $g \in H(D)$ is said to belong to $Q_{K,0}(p, q)$ space if $g \in Q_K(p, q)$ satisfying

$$
\lim_{|a| \to 1} \int_D |g'(z)|^p (1 - |z|^2)^q K(1 - |\varphi_a(z)|^2) dA(z) = 0.
$$

Spaces $Q_K(p, q)$ and $Q_{K,0}(p, q)$ are first introduced in [7], and it was proved that $Q_K(p, q) \subset B_\infty^{p, q}$. Setting $K(t) = t^s$, $s \geq 0$, $Q_K(p, q) = F(p, q, s)$. Hence, with different parameters, $Q_K(p, q)$ coincides with many classical function spaces such as $BMOA, Q_p$ and the Hardy space $H^2$. Note that $Q_K(p, q)$ generalizes the space $Q_K = Q_K(2, 0)$ (see [1]).

An important tool in the study of $Q_K(p, q)$ spaces is the auxiliary function $\varphi_K$ which is defined by

$$
\varphi_K(s) = \sup_{0 < t \leq 1} \frac{K(st)}{K(t)}, \quad 0 < s < \infty.
$$

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It is clear to see that $\varphi_K(s)$ is nondecreasing and right-continuous on $(0, \infty)$. If $0 < s < 1$, $\varphi_K(s) < 1$, and if $s \geq 1$, $\varphi_K(s) \geq 1$ by the definition of $K$. We further assume that

$$\int_0^1 \varphi_K(t) \frac{dt}{t} < \infty$$

and

$$\int_1^\infty \varphi_K(t) \frac{dt}{t^2} < \infty.$$  

The conditions (1) and (2) appeared firstly in [2]. We write that $A \approx B$ if there is a constant $c > 0$ such that $A \leq cB$. We write $A \preceq B$ whenever $A \preceq B \preceq A$. We know that (2) implies that $K(2t) \approx K(t)$. We also know that $Q_K(p, q) = Q_{K_1}(p, q)$ for $K_1 = \inf(K(r), K(1))$ (see Theorem 3.1 in [7]) and so the function $K$ can be assumed to be bounded.

2. Preliminaries

In the section, we will now state some preliminary results about the $Q_K(p, q)$ spaces that we will use later.

**Lemma 2.1.** Let $B_X$ denote the unit ball of the given Banach space $(X, \| \cdot \|_X)$ and co will denote the compact-open topology. Then $(B_{Q_K(p, q)}, \text{co})$ is compact.

**Proof.** By [6], for $g \in Q_K(p, q)$ and all $z \in D$, we have

$$|g(z)| \preceq C(z)\|g\|_{Q_K(p, q)} \leq C(z)\|g\|_{Q_K(p, q)},$$

where

$$C(z) = \begin{cases} 1 & \text{for } p > q + 2, \\ -\log(1 - |z|) & \text{for } p = q + 2, \\ (1 - |z|)^{1 - \frac{q+2}{p}} & \text{for } p < q + 2. \end{cases}$$

Thus $B_{Q_K(p, q)}$ is relatively compact with respect to the compact-open topology by Montel’s theorem. If $\{g_n\}$ is a sequence in $B_{Q_K(p, q)}$, we obtain

$$\sup_{a \in D} \int_D \lim_{n \to \infty} |g_n'(z)|^p (1 - |z|^2)^q K(1 - |\varphi_a(z)|^2) dA(z) \leq \inf_{n \to \infty} \|g_n\|_{Q_K(p, q)}^p \leq 1$$

by Fatou’s Lemma. It follows that $(B_{Q_K(p, q)}, \text{co})$ is co-closed and thus also co-compact. □

**Lemma 2.2** ([9]). Let $p \geq 1$ and $g \in Q_K(p, q)$, $g_r(z) = g(rz)$, $K$ satisfies (2). Then $\|g - g_r\|_{Q_K(p, q)} \to 0$ as $r \to 1$ if and only if $g \in Q_{K,0}(p, q)$.

**Proof.** See [9] Proposition 2.3.3. □
Lemma 2.3. For $a \in D$ and $p > 1$, we have

$$\int_{D} (1 - |z|^2)^{-\frac{p}{p-1}} [K(1 - |\varphi_{a}(z)|^2)]^{-\frac{1}{p-1}} dA(z) \lesssim \int_{2}^{+\infty} \frac{[\varphi_{K}(s)]^{\frac{1}{p-1}}}{s^{2 - \frac{p}{p-1}}} ds.$$  

Proof. Since $1 - |\varphi_{a}(z)|^2 = \left(\frac{1 - |z|^2(1 - |a|^2)}{|1 - za|}\right)^{2}$, it follows from the definition of $\varphi_{K}(s) = \sup_{0 < t \leq 1} \frac{K(st)}{K(t)}$ that $K(t) \geq \frac{K(st)}{\varphi_{K}(s)}$ and thus $[K(t)]^{-\frac{1}{p-1}} \leq \frac{[\varphi_{K}(s)]^{\frac{1}{p-1}}}{s^{2 - \frac{p}{p-1}}}$. Therefore

$$\int_{D} (1 - |z|^2)^{-\frac{p}{p-1}} [K(1 - |\varphi_{a}(z)|^2)]^{-\frac{1}{p-1}} dA(z) \leq \int_{D} (1 - |z|^2)^{-\frac{p}{p-1}} \frac{\varphi_{K}((1 - |z|^2)^{2})}{K(1 - |a|^2)} dA(z)$$

$$= [K(1 - |a|^2)]^{-\frac{1}{p-1}} \int_{D} (1 - |z|^2)^{-\frac{p}{p-1}} [\varphi_{K}((1 - |z|^2)^{2})] dA(z)$$

$$\lesssim \int_{D} (1 - |z|^2)^{-\frac{p}{p-1}} \frac{2}{1 - |z|^2} dA(z)$$

$$\lesssim \int_{2}^{+\infty} \frac{[\varphi_{K}(s)]^{\frac{1}{p-1}}}{s^{2 - \frac{p}{p-1}}} ds. \quad \Box$$

Lemma 2.4 ([8]). Let $K$ satisfy (1) and $1 < p < \infty$. Then $g \in Q_{K}(p, g)$ if and only if $\int_{D} |g^{(n)}(z)|^{p}(1 - |z|^2)^{np-p+q} K(1 - |\varphi_{a}(z)|^2) dA(z) \lesssim \infty$ and $g \in Q_{K,0}(p, g)$ if and only if $\lim_{|a| \rightarrow 1} \int_{D} |g^{(n)}(z)|^{p}(1 - |z|^2)^{np-p+q} K(1 - |\varphi_{a}(z)|^2) dA(z) = 0$, respectively.

Lemma 2.5 ([10]). For all $z \in D$ and $t < 1$,

$$\int_{0}^{2\pi} \frac{d\theta}{|1 - ze^{-i\theta}|^{t}} \lesssim 1.$$  

Proof. See [10] Theorem 1.7. \hfill \Box

Lemma 2.6. If $f(z) = \sum_{k=0}^{\infty} a_{k} z^{k}$ and $g(z) = \sum_{k=1}^{\infty} b_{k} z^{k}$, define the invertible linear operator $D^{n} : (H(D), co) \rightarrow (H(D), co)$ by

$$D^{n} g(z) = \frac{1}{(n-1)!} \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} b_{k+1} z^{k}, \quad n \in \mathbb{N},$$

then

$$\int_{D} f(z) g^{(n)}(z) dA(z) = \int_{D} f(z) D^{n} g(z)(1 - |z|^2)^{n-1} dA(z).$$

Lemma 2.7 ([10]). For $\alpha > -1$, every analytic function in $L^{1}(D, (1 - |z|^2)^{\alpha} dA(z))$.
has the formula

\[ f(z) = (\alpha + 1) \int_D f(\omega) \frac{(1 - |\omega|^2)^\alpha}{(1 - z\pi^{\frac{\alpha}{2}})^{\alpha+2}} dA(\omega). \]

**Proof.** See [10] Corollary 1.5. \( \square \)

**Lemma 2.8** (Riesz-Thorin convexity theorem). Assume \( T \) is a bounded linear operator from \( L_p \) to \( L_p \) and at the same time from \( L_q \) to \( L_q \). Then it is also a bounded operator from \( L_r \) to \( L_r \) for any \( r \) between \( p \) and \( q \).

Now we introduce \( R(p, q, K) \) spaces. Let \( E_{k,j} \) be the pairwise disjoint sets given by

\[ E_{k,j} = \left\{ z \in D : 1 - \frac{1}{2^k} \leq |z| \leq 1 - \frac{1}{2^{k+1}}, \pi j 2^{k+1} \leq \arg z \leq \pi (j + 1) \frac{2^{k+2}}{2^{k+1}} \right\}, \]

where \( k = 0, 1, 2, \ldots \) and \( j = 0, 1, 2, \ldots, 2^{k+2} - 1 \), so that

\[ \bigcup_{k=0}^\infty \bigcup_{j=0}^{2^{k+2} - 1} E_{k,j} = D. \]

We denote \( m = j - 1 + \sum_{i=0}^{k} 2^{i+1} \) so that

\[ E_1 = E_{0,0}, \ldots, E_4 = E_{0,3}, E_5 = E_{1,0}, \ldots, E_{12} = E_{1,7}, E_{13} = E_{2,0}, \ldots. \]

Let \( a_m \) denote the center of \( E_m \). The \( R(p, q, K) \) consists of those functions \( f \in H(D) \) for which \( f(z) = \sum_{m=1}^\infty f_m(z) \), where each \( f_m \in H(D) \) and

\[ \sum_{m=1}^\infty \left( \int_D |f_m(z)|^{\frac{p}{r}} (1 - |z|^2)^{\frac{p}{r} - \frac{n}{r}} |K(1 - |\varphi_{a_m}(z)|^2)|^{-\frac{1}{r}} dA(z) \right)^{\frac{r}{p}} < \infty. \]

The norm of \( R(p, q, K) \) is given by

\[ \|f\|_{R(p,q,K)} = \inf \sum_{m=1}^\infty \left( \int_D |f_m(z)|^{\frac{p}{r}} (1 - |z|^2)^{\frac{p}{r} - \frac{n}{r}} |K(1 - |\varphi_{a_m}(z)|^2)|^{-\frac{1}{r}} dA(z) \right)^{\frac{r}{p}}, \]

where the infimum is taken over all such representations of \( f \).

**Remark 2.9.** It is easy to check

\[ 1 \lesssim \frac{1 - |\varphi_a(z)|^2}{1 - |\varphi_{a_m}(z)|^2} \lesssim 1, \quad a \in E_m, \quad z \in D. \]

**Remark 2.10.** For \( K(t) = t^s, 0 < s < \infty, \)

\[ R(p, q, K) = R(p, q, s) \]

\[ = \inf \sum_{m=1}^\infty \left( \int_D |f_m(z)|^{\frac{p}{r}} (1 - |z|^2)^{\frac{p}{r} - \frac{n}{r}} (1 - |\varphi_{a_m}(z)|^2)^{-\frac{1}{r}} dA(z) \right)^{\frac{r}{p}}, \]
were introduced in [3]. They show \( R(p, q, s) \) is the dual of \( F_0(p, q, s) \) as well as the predual of \( F(p, q, s) \).

**Remark 2.11.** For \( p > 1 \) and \( \int_{2}^{\infty} \frac{\varphi_K(s)}{s^{p-1}} ds < \infty \), it is easy to see
\[
\| \cdot \|_{R(p, q, K)} \lesssim \| \cdot \|_{H_{\infty}}.
\]
Indeed, let \( f \in H_{\infty} \), then the representation of \( f \) can be chosen to be \( f \) itself, by Lemma 2.3, we get
\[
\left( \int_{D} |f(z)|^{\frac{q}{p-1}} (1 - |z|^2)^{-\frac{q}{p-1}} [K(1 - |a_m(z)|^2)]^{-\frac{1}{p}} dA(z) \right)^{\frac{1}{q}}
\]
\[
\leq \| f \|_{H_{\infty}} \left( \int_{D} (1 - |z|^2)^{-\frac{q}{p-1}} [K(1 - |a_m(z)|^2)]^{-\frac{1}{p}} dA(z) \right)^{\frac{1}{p}}
\]
\[
\lesssim \| f \|_{H_{\infty}} \left( \int_{2}^{\infty} \frac{\varphi_K(s)}{s^{p-1}} ds \right)^{\frac{1}{p}} < \infty
\]
for any center point \( a_m \).

**Proposition 2.12.** For \( p > 1 \), if \( f \in R(p, q, K) \), then
\[
|f(z)| \leq (1 - |z|)^{\frac{d q}{p-1} - 2} \| f \|_{R(p, q, K)}.
\]

**Proof.** Fix \( z \in D \). Using the inequality on page 39 in [10], which states that for \( f \in H(D) \), \( s \in R \) and \( t \in (0, \infty) \),
\[
(1 - |z|^2)^s |f(z)|^t \lesssim \int_{D} (1 - |\omega|^2)^{s - 2} |f(\omega)|^t dA(\omega).
\]
Let \( s = 2 - \frac{q}{p-1} \), \( t = \frac{p}{p-1} \), we obtain
\[
|f_m(z)|^{\frac{q}{p-1}} \leq (1 - |z|)^{\frac{d q}{p-1} - 2} \int_{D} |f_m(\omega)|^{\frac{q}{p-1}} (1 - |\omega|^2)^{-\frac{1}{p-1}} dA(\omega)
\]
and hence
\[
(1 - |z|)^{2 - \frac{d q}{p-1}} |f(z)|
\]
\[
\leq (1 - |z|)^{2 - \frac{d q}{p-1}} \sum_{m=1}^{\infty} (|f_m(z)|^{\frac{q}{p-1}})^{\frac{1}{p-1}}
\]
\[
\lesssim (1 - |z|)^{2 - \frac{d q}{p-1}} \sum_{m=1}^{\infty} (1 - |z|)^{\frac{d q}{p-1} - 2} \int_{D} |f_m(\omega)|^{\frac{q}{p-1}} (1 - |\omega|^2)^{-\frac{1}{p-1}} dA(\omega)
\]
\[
= \sum_{m=1}^{\infty} \left( \int_{D} |f_m(\omega)|^{\frac{q}{p-1}} (1 - |\omega|^2)^{-\frac{1}{p-1}} dA(\omega) \right)^{\frac{1}{p-1}}
\]
\[
\lesssim \sum_{m=1}^{\infty} \left( \int_{D} |f_m(\omega)|^{\frac{q}{p-1}} (1 - |\omega|^2)^{-\frac{1}{p-1}} [K(1 - |a_m(z)|^2)]^{-\frac{1}{p}} dA(\omega) \right)^{\frac{1}{p-1}}.
\]
Taking infimum over all the representations of \( f \), we finish the proof. \( \square \)
Remark 2.13. It is easy to see the norm-topology of \( R(p, q, K) \) is finer than the compact-open topology by Proposition 2.12. Furthermore we can verify that the normed space \( R(p, q, K) \) is complete using the completeness criterion.

3. The \( Q_{K,0}(p, q) - R(p, q, K) \) duality

To give the main theorem of this section, we need the following two lemmas and Theorem 3.1.

Lemma 3.1. Let \( g \in H(D) \) be given by \( g(z) = \sum_{k=1}^{\infty} b_k z^k \), \( K \) satisfy (1) and \( 1 < p < \infty \). Then \( f \in Q_K(p, q) \) if and only if

\[
\sup_{a \in D} \int_D |D^n g(z)|^p (1 - |z|^2)^{np-p+q} K(1 - |\varphi_a(z)|^2) dA(z) \frac{1}{z} < \infty,
\]

\( f \in Q_K(p, q) \) if and only if

\[
\lim_{|\alpha| \to 1} \int_D |D^n g(z)|^p (1 - |z|^2)^{np-p+q} K(1 - |\varphi_a(z)|^2) dA(z) \frac{1}{z} = 0.
\]

Proof. By Lemma 2.4 and the fact \( D^1 g(z) = g'(z) \)

\[
D^n g(z) = \frac{1}{(n - 1)!} (z^{n-1} g^{(n)}(z) + \sum_{j=1}^{n-2} c_{n,j} z^{n-1-j} g^{(n-j)}(z) + nD^{n-1}(z))
\]

the result follows by induction. \( \square \)

Lemma 3.2. For \( p > \max\{1, 1 + q\} \) and \( n \in N \) with \( n > 1 + \frac{q-1}{p} \), we define the linear operator \( S \) on the set of Borel measurable function \( H \) on \( D \) by

\[
S(H)(\omega) = (1 - |\omega|^2)^{\gamma} \int_D H(z) \left( \frac{1 - |z|^2}{|1 - z\omega|^n+1} \right)^{\frac{n-1-\gamma}{p-\gamma}} dA(z),
\]

where \( \omega \in D \) and \( \gamma \in (\max\{0, -q-(n-1)(p-1)\}, \min\{n, p-1-q\}) \). If the auxiliary function \( \varphi_K \) satisfies

\[
\int_{\mathbb{D}} \left( \frac{|\varphi_K(\omega)|}{|1 - |\omega||^{n+1}} \right) d\omega < \infty,
\]

then \( S \) maps \( L^\infty(D, d\mu_a) \) and \( L^1(D, d\mu_a) \) into \( L^\infty(D, d\mu_a) \) and \( L^1(D, d\mu_a) \), respectively, where

\[
d\mu_a = \frac{(1 - |\omega|^2)^{-\frac{\gamma + n-1}{p-\gamma}}}{|K(1 - |\varphi_a(\omega)|^2)|^{\frac{1}{p-\gamma}}} dA(z), \quad a \in D.
\]

Proof. Since \( \gamma < n \) and \( \gamma > 0 \), then by Theorem 1.7 in [10], we get

\[
\|S(H)\|_{L^\infty(D, d\mu_a)} \leq \|H\|_{L^\infty(D, d\mu_a)} (1 - |\omega|^2)^{\gamma} \int_D \left( \frac{1 - |z|^2}{|1 - z\omega|^n+1} \right)^{\frac{n-1-\gamma}{p-\gamma}} dA(z)
\]

\[
\leq \|H\|_{L^\infty(D, d\mu_a)} (1 - |\omega|^2)^{\gamma} (1 - |\omega|^2)^{-\gamma}
\]

\[
= \|H\|_{L^\infty(D, d\mu_a)} < \infty.
\]

\[
\|S(H)\|_{L^1(D, d\mu_a)} \leq \int_D (1 - |\omega|^2)^{\gamma} \int_D |H(z)| \left( \frac{1 - |z|^2}{|1 - z\omega|^n+1} \right)^{\frac{n-1-\gamma}{p-\gamma}} dA(z) \frac{(1 - |\omega|^2)^{-\frac{\gamma+n-1}{p-\gamma}}}{|K(1 - |\varphi_a(\omega)|^2)|^{\frac{1}{p-\gamma}}} dA(\omega)
\]

\[
\leq \int_D \frac{1 - |\omega|^2}{|1 - z\omega|^n+1} dA(\omega) \frac{(1 - |\omega|^2)^{-\frac{\gamma+n-1}{p-\gamma}}}{|K(1 - |\varphi_a(\omega)|^2)|^{\frac{1}{p-\gamma}}} dA(\omega).
\]
\[
\begin{align*}
&= \int_D |H(z)|(1 - |z|^2)^{n-1-\gamma} \int_D \frac{(1 - |\omega|^2)^{-\frac{\gamma+\gamma}{\gamma+\gamma} + \gamma}}{[1 - |\omega|^{n+1}]^{\frac{1}{\gamma+\gamma}} dA(\omega) dA(z).
\end{align*}
\]

It suffices to show that
\[
M(a) = \int_D \frac{(1 - |\omega|^2)^{-\frac{\gamma+\gamma}{\gamma+\gamma} + \gamma}}{[1 - |\omega|^{n+1}]^{\frac{1}{\gamma+\gamma}} dA(\omega) \lesssim \frac{(1 - |z|^2)^{-\frac{\gamma+\gamma}{\gamma+\gamma} + \gamma - n+1}}{[K(1 - |\varphi_a(\omega)|^2)]^{\frac{1}{\gamma+\gamma}}},
\]

Fix a, z ∈ D, let λ = \varphi_z(a), both \varphi_a(\omega) and \varphi_\lambda(\varphi_z(\omega)) are Möbius transformations of D mapping a to zero. Therefore there is a unimodular constant e^{i\theta} (which is \frac{1 - |u|^2}{1 - |\lambda u|^2}) such that
\[
\varphi_a(\omega) = e^{i\theta} \varphi_\lambda \circ \varphi_z(\omega).
\]

Note \(|\lambda|^2 = |\varphi_z(a)|^2 = |\varphi_a(z)|^2\), direct computation yields
\[
M(a) = \int_D \frac{(1 - |\omega|^2)^{-\frac{\gamma+\gamma}{\gamma+\gamma} + \gamma}}{[1 - |\omega|^{n+1}]^{\frac{1}{\gamma+\gamma}} dA(\omega) = \int_D \frac{(1 - |\varphi_z(u)|^2)^{-\frac{\gamma+\gamma}{\gamma+\gamma} + \gamma}}{[1 - |\varphi_z(u)|^{n+1}]^{\frac{1}{\gamma+\gamma}} dA(u)}
\]

\[
\frac{(1 - |\omega|^2)^{-\frac{\gamma+\gamma}{\gamma+\gamma} + \gamma}}{[1 - |\omega|^{n+1}]^{\frac{1}{\gamma+\gamma}}} \lesssim \int_D \frac{(1 - |u|^2)^{-\frac{\gamma+\gamma}{\gamma+\gamma} + \gamma}}{[1 - |u|^2]^{\frac{1}{\gamma+\gamma}} dA(u)}
\]

\[
\frac{(1 - |\varphi_a(z)|^2)^{-\frac{\gamma+\gamma}{\gamma+\gamma} + \gamma}}{[K(1 - |\varphi_a(\omega)|^2)]^{\frac{1}{\gamma+\gamma}}} \lesssim \int_D \frac{(1 - |u|^2)^{-\frac{\gamma+\gamma}{\gamma+\gamma} + \gamma}}{[1 - |u|^2]^{\frac{1}{\gamma+\gamma}} dA(u)}
\]

\[
\frac{(1 - |\omega|^2)^{-\frac{\gamma+\gamma}{\gamma+\gamma} + \gamma}}{[1 - |\omega|^{n+1}]^{\frac{1}{\gamma+\gamma}}} \lesssim \int_D \frac{(1 - |u|^2)^{-\frac{\gamma+\gamma}{\gamma+\gamma} + \gamma}}{[1 - |u|^2]^{\frac{1}{\gamma+\gamma}} dA(u)}
\]

\[
\frac{(1 - |\varphi_a(z)|^2)^{-\frac{\gamma+\gamma}{\gamma+\gamma} + \gamma}}{[K(1 - |\varphi_a(\omega)|^2)]^{\frac{1}{\gamma+\gamma}}} \lesssim \int_D \frac{(1 - |u|^2)^{-\frac{\gamma+\gamma}{\gamma+\gamma} + \gamma}}{[1 - |u|^2]^{\frac{1}{\gamma+\gamma}} dA(u)}
\]

\[
\frac{(1 - |\omega|^2)^{-\frac{\gamma+\gamma}{\gamma+\gamma} + \gamma}}{[1 - |\omega|^{n+1}]^{\frac{1}{\gamma+\gamma}}} \lesssim \int_D \frac{(1 - |u|^2)^{-\frac{\gamma+\gamma}{\gamma+\gamma} + \gamma}}{[1 - |u|^2]^{\frac{1}{\gamma+\gamma}} dA(u)}
\]

\[
\frac{(1 - |\varphi_a(z)|^2)^{-\frac{\gamma+\gamma}{\gamma+\gamma} + \gamma}}{[K(1 - |\varphi_a(\omega)|^2)]^{\frac{1}{\gamma+\gamma}}} \lesssim \int_D \frac{(1 - |u|^2)^{-\frac{\gamma+\gamma}{\gamma+\gamma} + \gamma}}{[1 - |u|^2]^{\frac{1}{\gamma+\gamma}} dA(u)}
\]

\[
\frac{(1 - |\omega|^2)^{-\frac{\gamma+\gamma}{\gamma+\gamma} + \gamma}}{[1 - |\omega|^{n+1}]^{\frac{1}{\gamma+\gamma}}} \lesssim \int_D \frac{(1 - |u|^2)^{-\frac{\gamma+\gamma}{\gamma+\gamma} + \gamma}}{[1 - |u|^2]^{\frac{1}{\gamma+\gamma}} dA(u)}
\]

\[
\frac{(1 - |\varphi_a(z)|^2)^{-\frac{\gamma+\gamma}{\gamma+\gamma} + \gamma}}{[K(1 - |\varphi_a(\omega)|^2)]^{\frac{1}{\gamma+\gamma}}} \lesssim \int_D \frac{(1 - |u|^2)^{-\frac{\gamma+\gamma}{\gamma+\gamma} + \gamma}}{[1 - |u|^2]^{\frac{1}{\gamma+\gamma}} dA(u)}
\]

\[
\frac{(1 - |\omega|^2)^{-\frac{\gamma+\gamma}{\gamma+\gamma} + \gamma}}{[1 - |\omega|^{n+1}]^{\frac{1}{\gamma+\gamma}}} \lesssim \int_D \frac{(1 - |u|^2)^{-\frac{\gamma+\gamma}{\gamma+\gamma} + \gamma}}{[1 - |u|^2]^{\frac{1}{\gamma+\gamma}} dA(u)}
\]

\[
\frac{(1 - |\varphi_a(z)|^2)^{-\frac{\gamma+\gamma}{\gamma+\gamma} + \gamma}}{[K(1 - |\varphi_a(\omega)|^2)]^{\frac{1}{\gamma+\gamma}}} \lesssim \int_D \frac{(1 - |u|^2)^{-\frac{\gamma+\gamma}{\gamma+\gamma} + \gamma}}{[1 - |u|^2]^{\frac{1}{\gamma+\gamma}} dA(u)}
\]

\[
\frac{(1 - |\omega|^2)^{-\frac{\gamma+\gamma}{\gamma+\gamma} + \gamma}}{[1 - |\omega|^{n+1}]^{\frac{1}{\gamma+\gamma}}} \lesssim \int_D \frac{(1 - |u|^2)^{-\frac{\gamma+\gamma}{\gamma+\gamma} + \gamma}}{[1 - |u|^2]^{\frac{1}{\gamma+\gamma}} dA(u)}
\]
we assume to the fact $-\frac{q-1}{p-1} + \gamma + 1 > 0$ by $\gamma < p - 1 - q$ and also use Lemma 2.5 due to the fact $-\frac{q-1}{p-1} + \gamma - n + 2 < 1$ by $\gamma > -q - (n-1)(p-1)$. □

**Theorem 3.3.** For $p > \max\{1, 1 + q\}$ and $n \in N$ with $n > 1 + \frac{q-1}{p-1}$ and $\varphi_K$ satisfies $\int_2^\infty \frac{|\varphi_K(s)|}{s} ds < \infty$, we define the Banach spaces $X_m \subseteq H(D)$ and $Y_m \subseteq H(D)$ by

$$
\|g\|_{X_m} = (\int_D |D^n g(z)|^p (1 - |z|^2)^{(n-1)p+q} K(1 - |\varphi_{a_m}(z)|^2) dA(z))^{\frac{1}{p}} < \infty,
$$

$$
\|f\|_{Y_m} = (\int_D |f(z)|^{\frac{q}{p-\gamma}} (1 - |z|^2)^{-\frac{q}{p-\gamma}} [K(1 - |\varphi_{a_m}(z)|^2)]^{-\frac{p-\gamma}{p-1}} dA(z))^{\frac{p-1}{p}} < \infty,
$$

we assume $g(0) = 0$, then $(X_m)^* \cong Y_m$ under the pairing

$$
\langle f, g \rangle = \int_D \overline{f(z)} g(z) dA(z).
$$

Moreover, for every $f \in Y_m$, $\|f\|_{Y_m} \approx \sup_{g \in B_{X_m}} |\langle f, g \rangle|$ where the constants do not depend on $m$.

**Proof.** For $f(z) = \sum_{k=0}^\infty a_k z^k \in Y_m$ and $g(z) = \sum_{k=1}^\infty b_k z^k \in X_m$, by Lemma 2.6, $\int_D \overline{f(z)} g(z) dA(z) = \int_D \overline{f(z)} D^n g(z) (1 - |z|^2)^{n-1} dA(z)$. Using Hölder’s inequality we obtain

$$
|\langle f, g \rangle| = |\int_D \overline{f(z)} D^n g(z) (1 - |z|^2)^{n-1} dA(z) |
$$

$$
\leq (\int_D |D^n g(z)|^p (1 - |z|^2)^{(n-1)p+q} K(1 - |\varphi_{a_m}(z)|^2) dA(z))^{\frac{1}{p}}
$$

$$
\times (\int_D |f(z)|^{\frac{q}{p-\gamma}} (1 - |z|^2)^{-\frac{q}{p-\gamma}} [K(1 - |\varphi_{a_m}(z)|^2)]^{-\frac{p-\gamma}{p-1}} dA(z))^{\frac{p-1}{p}}
$$

$$
= \|g\|_{X_m} \|f\|_{Y_m}.
$$

It follows $Y_m \subseteq (X_m)^*$. Conversely, let $L \in (X_m)^*$ and consider $T : X_m \to L^p$ given by

$$
T(g) = D^n g(z) (1 - |z|^2)^{(n-1)+\frac{q}{p}} [K(1 - |\varphi_{a_m}(z)|^2)]^{\frac{p}{p-1}}.
$$

Let $G = T(X_m)$. By Hahn-Banach theorem, $L \circ T^{-1} : G \to C$ can be extended (preserving the norm) to a bounded linear functional on $L^p$, denoted here by $L \circ T^{-1}$. Therefore we can find $h_0 \in L^{\frac{p}{p-1}}$ such that

$$
(L \circ T^{-1})(f) = \int_D f(z) h_0(z) dA(z) \quad \text{for all } f \in L^p
$$
and \(\|L \circ T^{-1}\| = (\int_D |h_0(z)|^{\frac{p}{m}} dA(z))^{\frac{m}{n}}\). Especially for all \(g \in X_m\) taking \(f = T(g) \in L^p\), we have

\[
L(g) = \int_D T(g|h_0(z)|^{\frac{p}{m}}) dA(z)
\]

\[
= \int_D D^n g(z)(1 - |z|^2)^{(n-1)+\frac{p}{m}} [K(1 - |\varphi_{am}(z)|^2)]^{\frac{m}{n}} h_0(z) dA(z)
\]

\[
= \int_D D^n g(z)(1 - |z|^2)^{n-1} h(z) dA(z),
\]

where \(h(z) = h_0(z)(1 - |z|^2)^{\frac{p}{m}} [K(1 - |\varphi_{am}(z)|^2)]^{\frac{m}{n}}\) and

\[
\|L\| = \left( \int_D |h_0(z)|^{\frac{m}{n}} dA(z) \right)^{\frac{m}{n}} .
\]

We now claim that \(D^n g \in L^q(D, (1 - |z|^2)^{n-1} dA(z))\) whenever \(g \in X_m\).

\[
\int_D |D^n g(z)|(1 - |z|^2)^{n-1} dA(z)
\]

\[
= \int_D \int_D |D^n g(z)| (1 - |z|^2)^{(n-1)+\frac{p}{m}} [K(1 - |\varphi_{am}(z)|^2)]^{\frac{m}{n}} (1 - |z|^2)^{-\frac{p}{m}} K(1 - |\varphi_{am}(z)|^2)\]

\[
\leq \|g\|_{X_m} \left( \int_D (1 - |z|^2)^{-\frac{p}{m}} [K(1 - |\varphi_{am}(z)|^2)]^{-\frac{m}{n}} dA(z) \right)^{\frac{m}{n}}
\]

\[
\lesssim \|g\|_{X_m} \left( \int_2^{+\infty} \frac{|\varphi_K(s)|^{\frac{m}{n}}}{s^2} \frac{ds}{s^{\frac{m}{n}}} \right)^{\frac{n-1}{m}} < \infty.
\]

Thus by Lemma 2.7

\[
D^n g(z) = n \int_D D^n g(\omega) \frac{(1 - |\omega|^2)^{n-1}}{(1 - z\omega)^n} dA(\omega).
\]

If \(g \in X_m\), then

\[
L_0 = \int_D \frac{D^n g(\omega)}{(1 - z\omega)^n} dA(\omega).
\]

Let \(f_0 = n \int_D \frac{D^n g(\omega)}{(1 - z\omega)^n} dA(\omega)\), \(f_0\) is analytic, thus it remains to show \(\|f_0\|_{Y_m} \lesssim \|L\|\) where the constant does not depend on \(m\). It suffices to show

\[
\|f_0\|_{Y_m} \lesssim \int_D |h(z)|^{\frac{m}{n}} (1 - |z|^2)^{-\frac{p}{m}} [K(1 - |\varphi_{am}(z)|^2)]^{-\frac{m}{n}} dA(z).
\]
Fix $\gamma \in (\max\{0, -q - (n-1)(p-1)\}, \min\{n, p-1-q\})$, the interval is non-empty by $n > 1 + \frac{q-1}{p}$. Define the functions $F$ and $H$ by

$$F(\omega) = f_0(\omega)(1 - |\omega|^2)^\gamma, \quad H(z) = h(z)(1 - |z|^2)^\gamma.$$  

Then it reduces to show

$$\int_D |F(\omega)|^{\frac{p}{p-1}} (1 - |\omega|^2)^{-\frac{2q}{p-1}} [K(1 - |\varphi_{am}(\omega)|^2)]^{-\frac{1}{p-1}} dA(\omega) \leq \int_D |H(z)|^{\frac{p}{p-1}} (1 - |z|^2)^{-\frac{2q}{p-1}} [K(1 - |\varphi_{am}(z)|^2)]^{-\frac{1}{p-1}} dA(z),$$  

where

$$F(\omega) = (1 - |\omega|^2)^\gamma n \int_D h(z) \frac{(1 - |z|^2)^n}{(1 - |z|^2)^{n+1}} dA(z) = n(1 - |\omega|^2)^\gamma \int_D H(z) (1 - |z|^2)^{n-1-\gamma} dA(z).$$

Since $2 - \frac{q}{p} > 1$, $\int_2^{+\infty} \frac{|\varphi_K(s)|^{\frac{1}{p-1}}}{s^{\frac{2q}{p-1}}} ds \leq \int_2^{+\infty} \frac{|\varphi_K(s)|^{\frac{1}{p-1}}}{s^{\frac{2q}{p-1}}} ds$. Thus we can use Lemma 2.8 and Lemma 3.2 to prove the above inequality and note that the constants obtained are independent of $m$. \hfill \Box

**Theorem 3.4.** Let $p > \max\{1, 1+q\}$, $K$ satisfy (1) and $\varphi_K$ satisfies

$$\int_2^{+\infty} \frac{|\varphi_K(s)|^{\frac{1}{p-1}}}{s^{\frac{2q}{p-1}}} ds < \infty.$$  

Then $R(p, q, K) \equiv Q_{K,0}(p, q)^*$ under the pairing

$$\langle f, g \rangle = \int_D \overline{f(z)} g(z) dA(z).$$

That is, every $f \in R(p, q, K)$ induces a bounded linear functional $\langle f, \cdot \rangle : Q_{K,0}(p, q) \to C$. Conversely, if $f \in Q_{K,0}(p, q)^*$, then there exists $f \in R(p, q, K)$ such that $L(g) = \langle f, g \rangle$ for all $g \in Q_{K,0}(p, q)$. Moreover, for every $f \in R(p, q, K)$, $\|f\|_{R(p, q, K)} \approx \sup_{g \in B_{Q_{K,0}(p, q)}} |\langle f, g \rangle|$.  

**Proof.** Let $f \in R(p, q, K)$ and $g \in Q_{K,0}(p, q)$. Given $\epsilon > 0$, there exists a representation of $f$ such that

$$\sum_{m=1}^{\infty} \int_D |f_m(z)| g(z) dA(z) \leq \|g\|_{Q_{K,0}(p, q)} \sum_{m=1}^{\infty} \left( \int_D |f_m(z)|^{\frac{p}{p-1}} (1 - |z|^2)^{-\frac{2q}{p-1}} [K(1 - |\varphi_{am}(z)|^2)]^{-\frac{1}{p-1}} dA(z) \right)^{\frac{p-1}{p}} \leq \|g\|_{Q_{K,0}(p, q)} \|f\|_{R(p, q, K)} + \epsilon.$$  

It follows $|\langle f, g \rangle| \leq \|g\|_{Q_{K,0}(p, q)} \|f\|_{R(p, q, K)}$ and thus $f \in Q_{K,0}(p, q)^*$.  


To prove that every \( L \in Q_{K,0}(p,q)^* \), there is an \( f \in R(p,q,K) \) such that \( L(g) = \langle f, g \rangle \), \( g \in Q_{K,0}(p,q) \), we consider \( A = \{ a_m : m \geq 1 \} \).

Noticing

\[
1 \lesssim \frac{1 - |\varphi_a(z)|^2}{1 - |\varphi_{a_m}(z)|^2} \lesssim 1, \quad a \in E_m, \quad z \in D,
\]

we get that the supremum in the definition of the \( Q_{K}(p,q) \)-norm can be taken over \( A \). Suppose now that each \( X_m \) consists of those functions \( g \) holomorphic on \( D \) which \( g(0) = 0 \) and

\[
\| g \|_{X_m} = \left( \int_D |D^n g(z)|^p (1 - |z|^2)^{(n-1)p+q} K(1 - |\varphi_{a_m}(z)|^2) dA(z) \right)^{\frac{1}{p}} < \infty.
\]

Denote by \( X \) the direct \( c_0 \)-sum of \( X_m \), that is, the space of holomorphic functions \( \{ g_m \}_{m=1}^{\infty} \) on \( D \) such that \( g_m \in X_m \) for every \( m = 1, 2, 3, \ldots \) and \( \lim_{m \to \infty} \| g_m \|_{X_m} = 0 \). The norm in \( X \) is given by

\[
\| \{ g_m \}_{m=1}^{\infty} \|_X = \sup_{m=1,2,3,\ldots} \| g_m \|_{X_m}.
\]

It is clear that the space \( Q_{K,0} \) with this new norm is a normed subspace of \( X \). Define \( Y \) to be the \( l^1 \)-sum of the spaces \( Y_m \), the dual of \( X \) is isometrically isomorphic to the \( l^1 \)-sum of the spaces \( X_m^* \). By Theorem 3.3, we can replace \( X_m^* \) by \( Y_m \) and so the dual of \( X \) is equal to \( Y : X^* \cong Y \), with the paring

\[
\sum_{m=1}^{\infty} \langle f_m, g_m \rangle, \quad f_m \in Y_m, \quad g_m \in X_m.
\]

Let \( L \in Q_{K,0}(p,q)^* \). Due to the fact that \( X^* \cong Y \) and \( Q_{K,0}(p,q) \subset X \), using a Hahn-Banach extension of \( L \) to \( X \) we obtain \( f_m \in Y_m \) such that

\[
L(g) = \sum_{m=1}^{\infty} \langle f_m, g \rangle, \quad g \in Q_{K,0}(p,q)
\]

holds and the norm \( \| L \| \) of \( L \) obeys

\[
\sum_{m=1}^{\infty} \| f_m \|_{Y_m} \lesssim \| L \|.
\]

Finally, if \( f(z) = \sum_{m=1}^{\infty} f_m(z) \), then this series converges uniformly on compact sets of \( D \) by Proposition 2.12. Thus

\[
f \in H(D), \quad \| f \|_Y \leq \sum_{m=1}^{\infty} \| f_m \|_{Y_m} \text{ and } L(f) = \langle f, g \rangle.
\]

Thus we have proved \( Q_{K,0}(p,q)^* \subset R(p,q,K) \). \( \square \)
4. The $E(p,q,K) - Q_K(p,q)$ duality

Lemma 4.1. Let $p > \max\{1,1+q\}$ and $\varphi_K$ satisfies \( \int_2^{+\infty} \frac{[\varphi_K(s)]^{p-1}}{s^{2+q-p}} \) \( ds < \infty \). Then the polynomials are dense in $R(p,q,K)$.

Proof. Combining Theorem 3.3 and the proof of Lemma 4.2 in [5], the result follows directly. \qed

The following result is an immediate consequence of Lemma 4.1, Lemma 2.2 and Theorem 3.4.

Proposition 4.2. Let $q \geq 0$, $p > 1+q$, $f \in R(p,q,K)$, $K$ satisfies (1), (2) and $\varphi_K$ satisfies \( \int_2^{+\infty} \frac{[\varphi_K(s)]^{p-1}}{s^{2+q-p}} \) \( ds < \infty \). Then $\| f - f_r \|_{R(p,q,K)} \to 0$ as $r \to 1$.

Remark. If $q \geq 0$, $1+q < p < 2$, then $\frac{\varphi_K(s)}{s^2} \leq \frac{[\varphi_K(s)]^{p-1}}{s^{2+q-p}}$. It follows that the condition \( \int_2^{+\infty} \frac{[\varphi_K(s)]^{p-1}}{s^{2+q-p}} \) \( ds < \infty \) implies condition (2).

Let $E(p,q,K) = \{ L \in Q_K(p,q)^\ast : L|_{(R,Q,K)^\ast} \text{ is continuous} \}$ be the subspace of $Q_K(p,q)^\ast$. Since $(R,Q,K)^\ast$ is compact by Lemma 2.1. Hence, by the Dixmier-Ng theorem in [4], we have:

Theorem 4.3. $E(p,q,K)$ space is a Banach space and $J : Q_K(p,q) \to E(p,q,K)^\ast$ is an isometric isomorphism.

Theorem 4.4. For $q \geq 0$, $p > 1+q$, $K$ satisfies (1), (2) and $\varphi_K$ satisfies

\( \int_2^{+\infty} \frac{[\varphi_K(s)]^{p-1}}{s^{2+q-p}} \) \( ds < \infty \),

the restriction map $E(p,q,K) \to Q_K,0(p,q)^\ast$ is an isometric isomorphism.

Proof. Combining Proposition 4.2, Theorem 3.4 and the proof of Theorem 4.2 in [3], we can obtain the result. \qed

Corollary 4.5. Let $q \geq 0$, $p > 1+q$, $K$ satisfies (1), (2) and $\varphi_K$ satisfies

\( \int_2^{+\infty} \frac{[\varphi_K(s)]^{p-1}}{s^{2+q-p}} \) \( ds < \infty \),

then $Q_K,0(p,q)^{**}$ is isometrically isomorphic to $Q_K(p,q)$ and $R(p,q,K)$ is isomorphic to $E(p,q,K)$.

Proof. Combining Theorem 4.3 and Theorem 4.4, we can get $Q_K,0(p,q)^{**}$ is isometrically isomorphic to $Q_K(p,q)$. $R(p,q,K)$ is isomorphic to $E(p,q,K)$ by Theorem 3.4 and Theorem 4.4. \qed
DUALITY OF $Q_K$-TYPE SPACES

References


Mujun Zhan
Department of Mathematics
Guangzhou University
Guangzhou 510006, P. R. China
and
Department of Mathematics
GuangDong Pharmaceutical College
Guangzhou 510006, P. R. China
E-mail address: rzhan@163.com

Guangfu Cao
Department of Mathematics
GuangDong Pharmaceutical College
Guangzhou 510006, P. R. China
E-mail address: guangfucao@163.com