A SIMPLY CONNECTED MANIFOLD WITH TWO SYMPLECTIC DEFORMATION EQUIVALENCE CLASSES WITH DISTINCT SIGNS OF SCALAR CURVATURES

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Abstract. We present a smooth simply connected closed eight dimensional manifold with distinct symplectic deformation equivalence classes $[[\omega_i]], i=1,2$ such that the symplectic $Z$ invariant, which is defined in terms of the scalar curvatures of almost Kähler metrics in [5], satisfies $Z(M,[[\omega_1]]) = \infty$ and $Z(M,[[\omega_2]]) < 0$.

1. Introduction

Kazdan and Warner classified closed smooth manifolds of dimension $> 2$ into three classes according to what the scalar curvature functions can be on a manifold [2, Chapter 4].

Recently, we studied an analogous problem on symplectic manifolds with almost Kähler metrics. An almost Kähler metric is a Riemannian metric compatible with a symplectic structure, see the beginning of Section 2. Two symplectic forms $\omega_0$ and $\omega_1$ on $M$ are called deformation equivalent, if there exists a diffeomorphism $\psi$ of $M$ such that $\psi^*\omega_1$ and $\omega_0$ can be joined by a smooth homotopy of symplectic forms, [6]. For a symplectic form $\omega$, its deformation equivalence class shall be denoted by $[[\omega]]$. We denote by $\Omega_{[[\omega]]}$ the set of all almost Kähler metrics compatible with a symplectic form in $[[\omega]]$.

We recall the symplectic $Z$ invariant from [5]. For a smooth closed manifold $M$ of dimension $2n \geq 4$ which admits a symplectic structure, we defined $Z(M,[[\omega]]) = \sup_{g \in \Omega_{[[\omega]]}} \frac{\int_M s_g d\text{vol}_g}{(\text{Vol}_g)^{\frac{n-1}{n}}}$.
where $\text{dvol}_g$, $s_g$, $\text{Vol}_g$ are the volume form, the scalar curvature and the volume of $g$ respectively, and also defined

$$Z(M) = \sup_{[\omega]} Z(M, [[\omega]]).$$

Then we have a basic inequality:

\begin{equation}
Z(M, [[\omega]]) \leq \sup_{\omega \in [[\omega]]} \frac{4\pi c_1(\omega) \left[\left(\frac{\omega^{n-1}}{n-1}\right)^{\frac{1}{n-1}}\right]}{(\frac{\omega_n}{n})^n},
\end{equation}

where $c_1(\omega)$ is the first Chern class of $\omega$.

With $Z$ invariants we have posed the following question:

**Question 1.1.** Let $M$ be a smooth closed manifold of dimension $2n \geq 4$ admitting a symplectic structure.

Is the (necessary and sufficient) condition for a smooth function $f$ on $M$ to be the scalar curvature of some smooth almost-Kähler metric on $M$ as follows?

(a) $f$ is arbitrary, if $0 < Z(M) \leq \infty$,

(b) $f$ is identically zero or somewhere negative, if $Z(M) = 0$ and $M$ admits a scalar-flat almost-Kähler metric,

(c) $f$ is negative somewhere, if otherwise.

Also, is the condition for a smooth function $f$ on $M$ to be the scalar curvature of some smooth almost-Kähler metric in $\Omega_{[[\omega]]}$ as follows?

(a') $f$ is arbitrary, if $0 < Z(M, [[\omega]]) \leq \infty$,

(b') $f$ is identically zero or somewhere negative, if $Z(M, [[\omega]]) = 0$ and $M$ admits a scalar-flat almost-Kähler metric in $\Omega_{[[\omega]]}$,

(c') $f$ is negative somewhere, if otherwise.

This question in turn supplies a motivation to study $Z$ invariants. In previous work [5], we presented a six dimensional non-simply connected closed manifold which admits two symplectic deformation classes $[[\omega_i]]$, $i = 1, 2$, such that their $Z$ values have distinct signs.

The main result in this article is to present a simply connected manifold with two symplectic deformation equivalence classes with similar properties.

### 2. Catanese-LeBrun example

An almost-Kähler metric on a smooth manifold $M^{2n}$ of real dimension $2n$ is a Riemannian metric $g$ compatible with a symplectic structure $\omega$, i.e., $\omega(X, Y) = g(X, JY)$ for an almost complex structure $J$, where $X, Y$ are tangent vectors at a point of the manifold; [3]. We call a Riemannian metric $g$ $\omega$-almost Kähler if $g$ is compatible with $\omega$. An almost-Kähler metric $(g, \omega, J)$ is Kähler if and only if $J$ is integrable. We shall prove the following:

**Theorem 2.1.** There exists a smooth closed simply connected 8-dimensional manifold $N$ with symplectic deformation equivalence classes $[[\omega_i]]$, $i = 1, 2$ such that $Z(N, [[\omega_1]]) = \infty$ and $Z(N, [[\omega_2]]) < 0$. 
The manifold \( N \) in the theorem will be the one studied by Catanese and LeBrun [4]. In fact, \( N \) is (diffeomorphic to) the product of two copies of a complex surface of general type with ample canonical line bundle which is homeomorphic to \( R_8 \), the blow up of the complex projective plane \( \mathbb{CP}_2 \) at 8 points in general position. This general type complex surface is obtained as a small deformation of Barlow’s explicit complex surfaces [1].

In their work, they showed that \( N \) admits two distinct holomorphic deformation classes. But it was not seen whether \( N \) admits two distinct symplectic deformation classes. Examples of smooth manifolds with more than one symplectic deformation class have been an interesting subject to study; refer to [7], [9] or [10]. To prove this theorem, we need the following:

**Proposition 2.2.** Let \( W \) be a complex surface of general type with ample canonical line bundle, homeomorphic to \( R_8 \), the blow up of \( \mathbb{CP}_2 \) at eight points in general position. Consider a Kähler Einstein metric of negative scalar curvature on \( W \) with Kähler form \( \omega_W \) on \( W \). Set \( N := W \times W \).

Then \( Z(N, [[\omega_W + \omega_W]]) = -8 \sqrt{2} \pi \), and it is attained by a Kähler Einstein metric.

**Proof.** The argument here follows the scheme in [5, Section 3]. We recall a few known facts about \( W \) from [9, Section 4]; there is a homeomorphism of \( W \) onto \( R_8 \) which preserves the Chern class \( c_1 \) and there is a diffeomorphism of \( W \times W \) onto \( R_8 \times R_8 \). Note that \( R_8 \) admits a Kähler Einstein metric of positive scalar curvature obtained by Calabi-Yau solution.

Then, the first Chern class of \( W \) can be written as \( c_1(W) = 3E_0 - \sum_{i=1}^{8} E_i \in H^2(W, \mathbb{R}) \cong \mathbb{R}^9 \), where \( E_i, i = 0, \ldots , 8 \), is the Poincare dual of a homology class \( \tilde{E}_i \), \( i = 0, \ldots , 8 \), form a basis of \( H_2(W, \mathbb{Z}) \cong \mathbb{Z}^9 \) and their intersections satisfy \( \tilde{E}_i \cdot \tilde{E}_j = \epsilon_i \delta_{ij} \), where \( \epsilon_0 = 1 \) and \( \epsilon_i = -1 \) for \( i \geq 1 \).

So, in this basis the intersection form becomes

\[
I = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & -1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & -1
\end{bmatrix}
\]

We have the orientation of \( W \) induced by the complex structure and the fundamental class \([W] \in H_4(W, \mathbb{Z}) \cong \mathbb{Z} \). As \( \omega_W \) is Kähler Einstein of negative scalar curvature, we may get \( [\omega_W] = -3E_0 + \sum_{i=1}^{8} E_i \) by scaling if necessary.

With \( N = W \times W \), by Künneth theorem

\[
H^2(N, \mathbb{R}) \cong \pi_1^* H^2(W) \oplus \pi_2^* H^2(W) \cong \mathbb{R}^9 \oplus \mathbb{R}^9,
\]

where \( \pi_i \) are the projection of \( N \) onto the \( i \)-th factor. Then,

\[
c_1(N) = \pi_1^* c_1(W) + \pi_2^* c_1(W) = \pi_1^* (3E_0 - \sum_{i=1}^{8} E_i) + \pi_2^* (3E_0 - \sum_{i=1}^{8} E_i).
\]
Consider any smooth path of symplectic forms \( \omega_t \), \( 0 \leq t \leq \delta \), on \( N \) such that \( \omega_0 = \omega_W + \omega_W \). We may write
\[
[\omega_t] = \sum_{i=0}^{8} \{n_i(t)\pi_i^*E_i + l_i(t)\pi_i^*E_i\} \in H^2(N, \mathbb{R})
\]
for some smooth functions \( n_i(t), l_i(t), i = 0, \ldots, 8 \). As they are connected, their first Chern class \( c_1(\omega_t) = c_1(N) \). Using the intersection form we compute;
\[
(2) \quad [\omega_t]^4([W \times W]) = \left[ \sum_{i=0}^{8} \{n_i(t)\pi_i^*E_i + l_i(t)\pi_i^*E_i\} \right]^4([W \times W])
\]
\[
= 6\{n_0^2(t) - \sum_{i=1}^{8} n_i^2(t)\}\{l_0^2(t) - \sum_{i=1}^{8} l_i^2(t)\} > 0.
\]
As \( n_0(0) = -3 \) and \( n_i(0) = 1, i = 1, \ldots, 8 \), so \( n_0^2(t) > \sum_{i=1}^{8} n_i^2(t) \). We get \( n_0(t) < 0 \). Similarly we also have \( l_0(0) = -3, l_i(0) = 1, i = 1, \ldots, 8 \), \( l_0^2(t) > \sum_{i=1}^{8} l_i^2(t) \) and \( l_0(t) < 0 \).
\[
\sum_{i=1}^{8} n_i(t) \leq \sqrt{8} \sqrt{\sum_{i=1}^{8} n_i^2(t)},
\]
\[
(3) \quad 3n_0(t) + \sum_{i=1}^{8} n_i(t) \leq 3n_0(t) + 2\sqrt{2} \sqrt{\sum_{i=1}^{8} n_i^2(t)}
\]
\[
< 3n_0(t) + 2\sqrt{2} \sqrt{n_0^2(t)} = (3 - 2\sqrt{2})n_0(t) < 0.
\]
So, \( c_1 : [\omega_t]^3([W \times W]) < 0 \). Set \( A_n = n_0^2(t) - \sum_{i=1}^{8} n_i^2(t) \), \( A_l = l_0^2(t) - \sum_{i=1}^{8} l_i^2(t) \), \( B_n = 3n_0(t) + \sum_{i=1}^{8} n_i(t) \) and \( B_l = 3l_0(t) + \sum_{i=1}^{8} l_i(t) \). From above, \( A_n, A_l > 0 \) and \( B_n, B_l < 0 \). By the inequality of arithmetic and geometric means we have
\[
\frac{c_1}{[\omega_t]^{3/4}} = \frac{3}{6^{1/4}} \left\{ \frac{A_nB_l + A_lB_n}{A_n^{3/4}A_l^{3/4}} \right\} = \frac{3}{6^{1/4}} \left\{ \left( \frac{A_n}{A_l} \right)^{3/4} \frac{B_l}{\sqrt{A_l}} + \left( \frac{A_l}{A_n} \right)^{3/4} \frac{B_n}{\sqrt{A_n}} \right\}
\]
\[
\leq -6^{1/4} \sqrt{B_lB_n}. \sqrt{A_lA_n}.
\]
From (3),

$$\frac{B_n^2}{A_n^2} \geq \frac{(3n_0(t) + 2\sqrt{2 \sum_{i=1}^8 n_i^2(t)})^2}{n_0^2(t) - \sum_{i=1}^8 n_i^2(t)} = \frac{(3 - 2\sqrt{2/(y - 1)})^2}{1 - y}$$

where \( y = \sum_{i=1}^8 n_i^2(t) \). By calculus, \( \frac{(3 - 2\sqrt{2/(y - 1)})^2}{1 - y} \geq 1 \) for \( y \in [0, 1) \) with equality at \( y = \frac{8}{27} \). So, we get \( \frac{B_n^2}{A_n^2} \geq 1 \) and similarly \( \frac{b_n^2}{a_n^2} \geq 1 \).

We have \( c_1([\omega_i])^3 \leq -6\pi \); the equality is achieved exactly when \( n_0(t) = -3, n_i(t) = 1, i = 1, \ldots, 8 \) modulo scaling, i.e., when \([\omega_i]\) is a positive multiple of \(-c_1(N)\). The Kähler form of a product Kähler Einstein metric of negative scalar curvature on \( N = W \times W' \) belongs to this class.

As the expression \( \frac{4\pi c_1([\omega])}{\|\omega\|^3} \) is invariant under a change \( \omega \mapsto \phi^{*}(\omega) \) by any diffeomorphism \( \phi \), so from (1) the above inequality gives

$$Z(N, [[\omega_0]]) \leq \sup_{\omega \in [[\omega_0]]} \frac{4\pi}{6} \cdot 24^{1/4} C_1 \cdot [\omega]^3 \leq -8\sqrt{2\pi}.$$ 

As the equality is attained by a Kähler Einstein metric, \( Z(N, [[\omega_0]]) = -8\sqrt{2\pi} \).

□

Proof of Theorem 2.1. Consider the positive Kähler Einstein metric on \( R_8 \) and let \( \omega_1 \) be the Kähler form of the product positive Kähler Einstein metric on \( R_8 \times R_8 \), which is diffeomorphic to \( N \). We have \( Z(N, [[\omega_1]]) = \infty \) (scaling by different constants on each factor gives \( \infty \)). And let \( \omega_2 \) be \( \omega_W + \omega_W \). Then \( Z(N, [[\omega_2]]) < 0 \) from Proposition 2.2. From the fact that these values are different, we conclude that \([\omega_1]\) and \([\omega_2]\) are distinct symplectic deformation equivalence classes. This proves Theorem 2.1.

□

In contrast to \( Z(N, [[\omega_2]]) < 0 \), for dimension \( n \geq 5 \) there are no examples known to have negative Yamabe invariant and Petean proved that the Yamabe invariant of any simply connected smooth closed manifold is nonnegative; [8]. Of course the Yamabe invariant \( Y(N) \) is positive.

Remark 2.3. We get \( Z(N, [[\omega_1]]) = \infty \), \( Z(N, [[\omega_2]]) < 0 \) and \( Z(N) = \infty \) from Theorem 2.1. As led by Question 1.1, we therefore expect for \( N \) that a smooth function is the scalar curvature of some almost-Kähler metrics in \([\omega]\) if and only if it is somewhere negative, and that any smooth function is the scalar curvature of some almost-Kähler metrics.

In fact, we may need certain surjectivity of the derivative of a scalar curvature map at the Kähler negative Einstein metric as well as the Kähler positive Einstein metric. This kind of argument is already outlined in [5, Section 4].

Question 2.4. Does there exist a simply connected closed 6-dimensional smooth manifold with two symplectic deformation classes with distinct signs of \( Z(\cdot, [[\omega]]) \)?
Question 2.5. Does there exist a closed 4-dimensional smooth manifold with two symplectic deformation classes \([\omega_i], \ i = 1, 2\) such that \(Z(\cdot, [[\omega_1]]) > 0\) (or \(Z(\cdot, [[\omega_1]]) = 0\)) and \(Z(\cdot, [[\omega_2]]) < 0\)?

Using further products, one may obtain, for each \(n \geq 3\), examples of closed symplectic \(2n\)-dimensional manifolds admitting two symplectic deformation equivalence classes with distinct signs of \(Z(\cdot, [\cdot])\) invariants.

References


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