Behavior of Solutions of a Fourth Order Difference Equation

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ABSTRACT. In this paper, we introduce an explicit formula for the solutions and discuss the global behavior of solutions of the difference equation

\[ x_{n+1} = \frac{a x_{n-3}}{b - c x_{n-1} x_{n-3}}, \quad n = 0, 1, \ldots \]

where \(a, b, c\) are positive real numbers and the initial conditions \(x_{-3}, x_{-2}, x_{-1}, x_0\) are real numbers.

1. Introduction

Difference equations have played an important role in analysis of mathematical models of biology, physics and engineering. Recently, there has been a great interest in studying properties of nonlinear and rational difference equations. One can see [3, 7, 9, 10, 11, 12, 13, 14, 16, 17] and the references therein.

In [8], E.M. Elsayed determined the solutions to some difference equations. He obtained the solution to the difference equation

\[ x_{n+1} = \frac{x_{n-3}}{1 - x_{n-1} x_{n-3}}, \quad n = 0, 1, \ldots \tag{1.1} \]

where the initial conditions are arbitrary nonzero positive real numbers. But he didn’t point to any constraints on the initial conditions.

In fact, if we start with initial conditions \(x_0 = 2, x_{-1} = 1, x_{-2} = 1, x_{-3} = 0.5\) in equation (1.1), then undefined value for \(x_3\) will be obtained. Therefore, additional information about the initial conditions must be given for any solution of equation (1.1) to be well-defined.

In [4], M. Aloqeili discussed the stability properties and semicycle behavior of the solutions of the difference equation

\[ x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}, \quad n = 0, 1, \ldots \]
with real initial conditions and positive real number \( a \).

In [1], we have discussed the oscillation, boundedness and the global behavior of all admissible solutions of the difference equation

\[ x_{n+1} = \frac{Ax_{n-1}}{B - Cx_nx_{n-2}}, \quad n = 0, 1, \ldots \]

where \( A, B, C \) are positive real numbers.

In [2], we have also discussed the oscillation, periodicity, boundedness and the global behavior of all admissible solutions of the difference equation

\[ x_{n+1} = \frac{Ax_{n-2r-1}}{B - C \prod_{i=1}^{k} x_{n-2i}}, \quad n = 0, 1, \ldots \]

where \( A, B, C \) are positive real numbers.

In [5], the authors investigated the asymptotic behavior of solutions of the equation

\[ x_{n+1} = \frac{ax_{n-1}}{b + cx_nx_{n-1}}, \quad n = 0, 1, \ldots \]

with positive parameters \( a \) and \( c \), negative parameter \( b \) and nonnegative initial conditions.

In [6], they also used the explicit formula for the solutions of the equation

\[ x_{n+1} = \frac{ax_{n-1}}{b + cx_nx_{n-1}}, \quad n = 0, 1, \ldots \]

with positive parameters and nonnegative initial conditions in investigating their behavior.

In [15], H. Sedaghat determined the global behavior of all solutions of the rational difference equations

\[ x_{n+1} = \frac{ax_{n-1}}{x_nx_{n-1} + b}, \quad x_{n+1} = \frac{ax_nx_{n-1}}{x_n + bx_{n-2}}, \quad n = 0, 1, \ldots \]

where \( a, b > 0 \).

In this paper, we introduce an explicit formula and discuss the global behavior of solutions of the difference equation

\[ (1.2) \quad x_{n+1} = \frac{ax_{n-3}}{b - cx_nx_{n-3}}, \quad n = 0, 1, \ldots \]

where \( a, b, c \) are positive real numbers and the initial conditions \( x_{-3}, x_{-2}, x_{-1}, x_0 \) are real numbers.

2. Solution of Equation (1.2)

We define \( \alpha_i = x_{-2+i}x_{-4+i} \), \( i = 1, 2 \).
Theorem 2.1. Let $x_{-3}, x_{-2}, x_{-1}$ and $x_0$ be real numbers such that for any $i \in \{1, 2\}$, $\alpha_i \neq \frac{b}{\sum_{k=i}^{m} r_{j,k}}$ for all $n \in \mathbb{N}$. If $a \neq b$, then the solution $\{x_n\}_{n=-3}^{\infty}$ of equation (1.2) is

$$
(2.1) \quad x_n = \begin{cases} 
  x_{-3} \prod_{j=0}^{n-3} \frac{(\frac{b}{n})^{2j+1}\theta_j - c}{(\frac{b}{n})^{2j+1}\theta_j - c}, & n = 1, 5, 9, 
  x_{-2} \prod_{j=0}^{n-2} \frac{(\frac{b}{n})^{2j+1}\theta_j - c}{(\frac{b}{n})^{2j+1}\theta_j - c}, & n = 2, 6, 10, 
  x_{-1} \prod_{j=0}^{n-1} \frac{(\frac{b}{n})^{2j+1}\theta_j - c}{(\frac{b}{n})^{2j+1}\theta_j - c}, & n = 3, 7, 11, 
  x_0 \prod_{j=0}^{n} \frac{(\frac{b}{n})^{2j+1}\theta_j - c}{(\frac{b}{n})^{2j+1}\theta_j - c}, & n = 4, 8, 12, 
\end{cases}
$$

where $\theta_i = \frac{a - b + c\alpha_i}{\alpha_i}$, $\alpha_i = x_{-2+i}, x_{-4+i}$, and $i = 1, 2$.

Proof. We can write the given solution as

$$
x_{4m+1} = x_{-3} \prod_{j=0}^{m} \frac{(\frac{b}{n})^{2j+1}\theta_j - c}{(\frac{b}{n})^{2j+1}\theta_j - c}, \quad x_{4m+2} = x_{-2} \prod_{j=0}^{m} \frac{(\frac{b}{n})^{2j+1}\theta_j - c}{(\frac{b}{n})^{2j+1}\theta_j - c},
$$

$$
x_{4m+3} = x_{-1} \prod_{j=0}^{m} \frac{(\frac{b}{n})^{2j+1}\theta_j - c}{(\frac{b}{n})^{2j+1}\theta_j - c}, \quad x_{4m+4} = x_{0} \prod_{j=0}^{m} \frac{(\frac{b}{n})^{2j+2}\theta_j - c}{(\frac{b}{n})^{2j+2}\theta_j - c}, \quad m = 0, 1, \ldots
$$

It is easy to check the result when $m = 0$. Suppose that the result is true for $m > 0$. Then

$$
x_{4(m+1)+1} = \frac{ax_{4m+1}}{b - cx_{4m+1}x_{4m+3}} = \frac{ax_{-3} \prod_{j=0}^{m} \frac{(\frac{b}{n})^{2j+1}\theta_j - c}{(\frac{b}{n})^{2j+1}\theta_j - c}}{b - cx_{-3} \prod_{j=0}^{m} \frac{(\frac{b}{n})^{2j+1}\theta_j - c}{(\frac{b}{n})^{2j+1}\theta_j - c} x_{-1} \prod_{j=0}^{m} \frac{(\frac{b}{n})^{2j+2}\theta_j - c}{(\frac{b}{n})^{2j+2}\theta_j - c}}
$$

$$
= \frac{ax_{-3} \prod_{j=0}^{m} \frac{(\frac{b}{n})^{2j+1}\theta_j - c}{(\frac{b}{n})^{2j+1}\theta_j - c}}{b - cx_{-3}(\prod_{j=0}^{m} \frac{(\frac{b}{n})^{2j+1}\theta_j - c)}{x_{-1} \prod_{j=0}^{m} \frac{(\frac{b}{n})^{2j+2}\theta_j - c}{(\frac{b}{n})^{2j+2}\theta_j - c}}
$$

$$
= \frac{ax_{-3} \prod_{j=0}^{m} \frac{(\frac{b}{n})^{2j+2}\theta_j - c}{(\frac{b}{n})^{2j+2}\theta_j - c}}{b - cx_{-1}x_{-3}(\theta_1 - c)(\frac{1}{(\frac{b}{n})^{2m+2}\theta_1 - c})}
$$

$$
= \frac{ax_{-3}(\frac{b}{n})^{2m+2}\theta_1 - c)}{b((\frac{b}{n})^{2m+2}\theta_1 - c) - c\alpha_1(\theta_1 - c)}
$$

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where \( \zeta \) completes the proof.

Therefore, using the explicit formula of its solution.

3. Global Behavior of Equation (1.2)

In this section, we investigate the global behavior of equation (1.2) with \( a \neq b \), using the explicit formula of its solution.

We can write the solution of equation (1.2) as

\[
x_{4(m+1)+2} = x_{-2} \prod_{j=0}^{m+1} \left( \frac{b}{a} \right)^{2j} \theta_2 - c, \quad x_{4(m+1)+3} = x_{-1} \prod_{j=0}^{m+1} \left( \frac{b}{a} \right)^{2j+1} \theta_1 - c
\]

and

\[
x_{4(m+1)+4} = x_0 \prod_{j=0}^{m+1} \left( \frac{b}{a} \right)^{2j+2} \theta_2 - c.
\]

This completes the proof. \( \square \)

Theorem 3.1. Let \( \{x_n\}_{n=-3}^{\infty} \) be a solution of equation (1.2) such that for any \( i \in \{1, 2\} \), \( \alpha_i \neq -\frac{b}{c} \sum_{k=0}^{i-1} \left( \frac{b}{a} \right)^k \) for all \( n \in \mathbb{N} \). If \( \alpha_i = \frac{b-a}{c} \) for all \( i \in \{1, 2\} \), then \( \{x_n\}_{n=-3}^{\infty} \) is periodic with prime period 4.

Proof. Assume that \( \alpha_i = \frac{b-a}{c} \) for all \( i \in \{1, 2\} \). Then \( \theta_i = 0 \) for all \( i \in \{1, 2\} \).

Therefore,

\[
x_{4m+2t+i} = x_{-4+2t+i} \prod_{j=0}^{m} \zeta(j, t, i) = x_{-4+2t+i}, \quad m = 0, 1, \ldots
\]

This completes the proof. \( \square \)
In the following Theorem, suppose that \( \alpha_i \neq \frac{k-a}{c} \) for all \( i \in \{1, 2\} \).

**Theorem 3.2.** Let \( \{x_n\}_{n=-3}^{\infty} \) be a solution of equation (1.2) such that for any \( i \in \{1, 2\} \), \( \alpha_i \neq \frac{b}{n+1} \) for all \( n \in \mathbb{N} \). Then the following statements are true.

1. If \( a < b \), then \( \{x_n\}_{n=-3}^{\infty} \) converges to 0.
2. If \( a > b \), then \( \{x_n\}_{n=-3}^{\infty} \) converges to a period-4 solution.

**Proof.**

1. If \( a < b \), then \( \zeta(j, t, i) \) converges to \( \frac{\alpha}{b} < 1 \) as \( j \to \infty \), for all \( t \in \{0, 1\} \) and \( i \in \{1, 2\} \). So, for every pair \((t, i) \in \{0, 1\} \times \{1, 2\}\) we have for a given \( 0 < \epsilon < 1 \) that, there exists \( j_0(t, i) \in \mathbb{N} \) such that, \( |\zeta(j, t, i)| < \epsilon \) for all \( j \geq j_0(t, i) \). If we set \( j_0 = \max_{0 \leq t \leq 1, 1 \leq i \leq 2} j_0(t, i) \), then for all \( t \in \{0, 1\} \) and \( i \in \{1, 2\} \) we get

\[
|x_{4m+2t+i}| = |x_{-4+2t+i}| \left| \prod_{j=0}^{m} \zeta(j, t, i) \right| \\
= |x_{-4+2t+i}| \left| \prod_{j=0}^{j_0-1} \zeta(j, t, i) \right| \left| \prod_{j=j_0}^{m} \zeta(j, t, i) \right| \\
< |x_{-4+2t+i}| \left| \prod_{j=0}^{j_0-1} \zeta(j, t, i) \right| \epsilon^{m-j_0+1}.
\]

As \( m \) tends to infinity, the solution \( \{x_n\}_{n=-3}^{\infty} \) converges to 0.

2. If \( a > b \), then \( \zeta(j, t, i) \to 1 \) as \( j \to \infty \), \( t \in \{0, 1\} \) and \( i \in \{1, 2\} \). This implies that, for every pair \((t, i) \in \{0, 1\} \times \{1, 2\}\), there exists \( j_1(t, i) \in \mathbb{N} \) such that \( \zeta(j, t, i) > 0 \) for all \( t \in \{0, 1\} \) and \( i \in \{1, 2\} \). If we set \( j_1 = \max_{0 \leq t \leq 1, 1 \leq i \leq 2} j_1(t, i) \), then for all \( t \in \{0, 1\} \) and \( i \in \{1, 2\} \) we get

\[
x_{4m+2t+i} = x_{-4+2t+i} \prod_{j=0}^{m} \zeta(j, t, i) \\
= x_{-4+2t+i} \prod_{j=0}^{j_1-1} \zeta(j, t, i) \exp \left( \sum_{j=j_1}^{m} \ln(\zeta(j, t, i)) \right).
\]

We shall test the convergence of the series \( \sum_{j=j_1}^{\infty} |\ln(\zeta(j, t, i))| \).

Since for all \( t \in \{0, 1\} \) and \( i \in \{1, 2\} \) we have \( \lim_{j \to \infty} \left| \frac{\ln(\zeta(j+1, t, i))}{\ln(\zeta(j, t, i))} \right| = 0 \), using L’Hospital’s rule we obtain

\[
\lim_{j \to \infty} \left| \frac{\ln(\zeta(j+1, t, i))}{\ln(\zeta(j, t, i))} \right| = \left( \frac{b}{a} \right)^2 < 1.
\]
It follows from the ratio test that the series \( \sum_{j=0}^{\infty} | \ln(\zeta(j,t,i)) | \) is convergent. This ensures that there are four positive real numbers \( \mu_{ti} \), \( t \in \{0,1\} \) and \( i \in \{1,2\} \) such that

\[
\lim_{m \to \infty} x_{4m+2t+i} = \mu_{ti}, \quad t \in \{0,1\} \text{ and } i \in \{1,2\}
\]

where

\[
\mu_{ti} = x_{-4+2t+i} \prod_{j=0}^{\infty} \left( \frac{b}{a} \right)^{2j+t} \left( \frac{b}{a} \right)^{2j+t+1} \theta_i - c, \quad t \in \{0,1\} \text{ and } i \in \{1,2\}.
\]

This completes the proof. \( \square \)

**Example (1)** Figure 1. shows that if \( a = 2, b = 3, c = 1 \) \((a < b)\), then the solution \( \{x_n\}_{n=-3}^{\infty} \) of equation (1.2) with initial conditions \( x_{-3} = 0.2, x_{-2} = 2, x_{-1} = -2 \) and \( x_0 = 0.4 \) converges to 0.

**Example (2)** Figure 2. shows that if \( a = 3, b = 1, c = 0.8 \) \((a > b)\), then the solution \( \{x_n\}_{n=-3}^{\infty} \) of equation (1.2) with initial conditions \( x_{-3} = 0.2, x_{-2} = 2, x_{-1} = -2 \) and \( x_0 = 0.4 \) converges to a period-4 solution.

**4. Case \( a = b \)**

In this section, we investigate the behavior of the solution of the difference equation

\[
x_{n+1} = \frac{ax_{n-3}}{a-cx_{n-1}x_{n-3}}, \quad n = 0,1,\ldots
\]
Theorem 4.1. Let \( x_{-3}, x_{-2}, x_{-1} \) and \( x_0 \) be real numbers such that for any \( i \in \{1, 2\}, \alpha_i \neq \frac{a}{c(n+1)} \) for all \( n \in \mathbb{N} \). Then the solution \( \{x_n\}_{n=-3}^\infty \) of equation (3.1) is

\[
x_n = \begin{cases} 
  x_{-3} \prod_{j=0}^{n-1} \frac{a-(2\alpha_{i+j}+1)c_1}{a-(2\alpha_{i+j}+1)c_1}, & n = 1, 5, 9, \ldots \\
  x_{-2} \prod_{j=0}^{n-2} \frac{a-(2\alpha_{i+j})c_2}{a-(2\alpha_{i+j})c_2}, & n = 2, 6, 10, \ldots \\
  x_{-1} \prod_{j=0}^{n-1} \frac{a-(2\alpha_{i+j})c_1}{a-(2\alpha_{i+j})c_1}, & n = 3, 7, 11, \ldots \\
  x_0 \prod_{j=0}^{n-2} \frac{a-(2\alpha_{i+j})c_2}{a-(2\alpha_{i+j})c_2}, & n = 4, 8, 12, \ldots
\end{cases}
\]

(3.2)

Proof. The proof is similar to that of Theorem (2.1) and will be omitted. \( \square \)

We can write the solution of equation (3.1) as

\[ x_{4m+2t+i} = x_{-4+2t+i} \prod_{j=0}^{m} \gamma(j, t, i), \]

where \( \gamma(j, t, i) = \frac{a-(2\alpha_{i+j}+1)c_1}{a-(2\alpha_{i+j}+1)c_1}, t \in \{0, 1\} \) and \( i \in \{1, 2\} \).

Theorem 4.2. Let \( \{x_n\}_{n=-3}^\infty \) be a nontrivial solution of equation (3.1) such that for any \( i \in \{1, 2\}, \alpha_i \neq \frac{a}{c(n+1)} \) for all \( n \in \mathbb{N} \). If \( \alpha_i = 0 \) for all \( i \in \{1, 2\} \), then \( \{x_n\}_{n=-3}^\infty \) is periodic with prime period 4.

Proof. Assume that \( \alpha_i = 0 \) for all \( i \in \{1, 2\} \). Then \( \gamma(j, t, i) = 1 \) for all \( t \in \{0, 1\} \) and \( i \in \{1, 2\} \). Therefore,

\[ x_{4m+2t+i} = x_{-4+2t+i} \prod_{j=0}^{m} \gamma(j, t, i) = x_{-4+2t+i}, \quad m = 0, 1, \ldots \]

This completes the proof. \( \square \)

In the following Theorem, suppose that \( \alpha_i \neq 0 \) for all \( i \in \{1, 2\} \).

Theorem 4.3. Let \( \{x_n\}_{n=-3}^\infty \) be a solution of equation (3.1) such that for any \( i \in \{1, 2\}, \alpha_i \neq \frac{a}{c(n+1)} \) for all \( n \in \mathbb{N} \). Then \( \{x_n\}_{n=-3}^\infty \) converges to 0.

Proof. It is clear that \( \gamma(j, t, i) \to 1 \) as \( j \to \infty \), \( t \in \{0, 1\} \) and \( i \in \{1, 2\} \). This implies that, for every pair \((t, i) \in \{0, 1\} \times \{1, 2\}\), there exists \( j_2(t, i) \in \mathbb{N} \) such that, \( \gamma(j, t, i) > 0 \) for all \( t \in \{0, 1\} \) and \( i \in \{1, 2\} \). If we set \( j_2 = \max_{0 \leq t \leq 1, 1 \leq i \leq 2} j_2(t, i) \), then for all \( t \in \{0, 1\} \) and \( i \in \{1, 2\} \) we get

\begin{align*}
  x_{4m+2t+i} &= x_{-4+2t+i} \prod_{j=0}^{m} \gamma(j, t, i) \\
  &= x_{-4+2t+i} \prod_{j=0}^{j_2-1} \gamma(j, t, i) \exp \left( - \sum_{j=j_2}^{m} \ln \frac{1}{\gamma(j, t, i)} \right).
\end{align*}
We shall show that \( \sum_{j=j_0}^{\infty} \frac{1}{\gamma(j,t,i)} = \sum_{j=j_0}^{\infty} \frac{a-(2j+t+1)c_\alpha}{a-(2j+t)c_\alpha} = \infty \), by considering the series \( \sum_{j=j_0}^{\infty} \frac{1}{\gamma(j,t,i)} \). As
\[
\lim_{j \to \infty} \frac{\ln(1/\gamma(j,t,i))}{-c\alpha_i/(a-(2j+t)c_\alpha)} = \lim_{j \to \infty} \frac{\ln((a-(2j+t+1)c_\alpha)/(a-(2j+t)c_\alpha))}{-c\alpha_i/(a-(2j+t)c_\alpha)} = 1,
\]
using the limit comparison test, we get \( \sum_{j=j_0}^{\infty} \frac{1}{\gamma(j,t,i)} = \infty \). Then
\[
x_{4m+2t+i} = x_{-4+2t+i} \prod_{j=0}^{j_2-1} \gamma(j,t,i) \exp \left( - \sum_{j=j_2}^{m} \ln \frac{1}{\gamma(j,t,i)} \right)
\]
converges to 0 as \( m \to \infty \). Therefore, \( \{x_n\}_{n=-3}^{\infty} \) converges to 0. \( \square \)

5. Case \( a = b = c \)

In this section, we investigate the behavior of the solution of the difference equation
\[
x_{n+1} = \frac{x_{n-3}}{1-x_{n-1}x_{n-3}}, \quad n = 0, 1, \ldots
\]

**Theorem 5.1.** Let \( x_{-3}, x_{-2}, x_{-1} \) and \( x_0 \) be real numbers such that for any \( i \in \{1, 2\}, \alpha_i \neq \frac{1}{n+1} \) for all \( n \in \mathbb{N} \). Then the solution \( \{x_n\}_{n=-3}^{\infty} \) of equation (3.3) is
\[
x_n = \begin{cases} 
  x_{-3} \prod_{j=0}^{n-2} \frac{1-(2j+1)c_\alpha}{1-(2j+2)c_\alpha}, & n = 1, 5, 9, \ldots \\
  x_{-2} \prod_{j=0}^{n-2} \frac{1-(2j+2)c_\alpha}{1-(2j+3)c_\alpha}, & n = 2, 6, 10, \ldots \\
  x_{-1} \prod_{j=0}^{n-2} \frac{1-(2j+1)c_\alpha}{1-(2j+2)c_\alpha}, & n = 3, 7, 11, \ldots \\
  x_0 \prod_{j=0}^{n-2} \frac{1-(2j+2)c_\alpha}{1-(2j+3)c_\alpha}, & n = 4, 8, 12, \ldots 
\end{cases}
\]

**Proof.** The proof is similar to that of Theorem (2.1) and will be omitted. \( \square \)

**Theorem 5.2.** Let \( \{x_n\}_{n=-3}^{\infty} \) be a nontrivial solution of equation (3.3) such that for any \( i \in \{1, 2\}, \alpha_i \neq \frac{1}{n+1} \) for all \( n \in \mathbb{N} \). If \( \alpha_i = 0 \) for all \( i \in \{1, 2\} \), then \( \{x_n\}_{n=-3}^{\infty} \) is periodic with prime period 4.

**Proof.** Assume that \( \alpha_i = 0 \) for all \( i \in \{1, 2\} \). Then
\[
x_{4m+2t+i} = x_{-4+2t+i}, \quad m = 0, 1, \ldots
\]
This completes the proof. \( \square \)

In the following Theorem, suppose that \( \alpha_i \neq 0 \) for all \( i \in \{1, 2\} \).

**Theorem 5.3.** Let \( \{x_n\}_{n=-3}^{\infty} \) be a solution of equation (3.3) such that for any \( i \in \{1, 2\}, \alpha_i \neq \frac{1}{n+1} \) for all \( n \in \mathbb{N} \). Then \( \{x_n\}_{n=-3}^{\infty} \) converges to 0.
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Figure 3: $x_{n+1} = \frac{x_{n-3}}{1 - 1.5x_{n-1}x_{n-3}}$

Example (3) Figure 3 shows that if $a = b = 1, c = 1.5$, then the solution $\{x_n\}_{n=-3}^{\infty}$ of equation (3.1) with initial conditions $x_{-3} = 5, x_{-2} = -1, x_{-1} = 1.3$ and $x_0 = -1.1$ converges to 0.

Example (4) Figure 4 shows that if $a = b = c$, then the solution $\{x_n\}_{n=-3}^{\infty}$ of equation (3.3) with initial conditions $x_{-3} = 5, x_{-2} = 1, x_{-1} = 1.3$ and $x_0 = -1.1$ converges to 0.

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