1. INTRODUCTION

The vibration and buckling of plates is one of the most important behaviours to be considered in the structural design process. In early days of theoretical research work, the Kirchhoff plate theory was introduced for the analysis of plates. Subsequently thick plate theory was proposed by Reissner (1945) and Mindlin (1951) and transverse shear deformation and the rotatory inertia effect were considered. Since then, Reissner-Mindlin plate theory has been widely used to solve many engineering problems associated with thick plate situation.

Much effort has been paid in the development of numerical analysis techniques to predict natural frequency and buckling load for the thick plates having complex geometries and boundary conditions. Among them, the finite element (FE) analysis is considered as one of the most crucial techniques to predict the thick plate behaviour accurately. Some reviews on the plate FE analysis refers to References (Hughes and Hinton, 1986; Mackerle, 1995). Recently, the isogeometric analysis (IGA) concept (Hughes et al, 2005) has been introduced in engineering analysis. In this new concept, the NURBS (De Boor, 1978) was used to represent both structural geometry and the displacement filed of structures. The NURBS can provide higher continuity of derivatives than that of Lagrange interpolation functions which has been widely used in FE formulation. In addition, its basis functions can be refined without changing the geometry (Cottrell et al., 2009) and the order of the basis function can be elevated without any difficulty.

So far, a few application of isogeometric concept into specific structural problem appear in open literatures (Hughes and Evans, 2010). Therefore, the introduction of the isogeometric concept into the analysis of bar, beam, plate and shell is still one of demanding tasks. In particular, a few works on structural vibration problems using isogeometric concept can be found in References (Cottrell et al., 2006; Shojaeea et al., 2012). Therefore, we introduce the isogeometric concept into the development of new plate element based on Reissner-Mindlin plate theory to determine natural frequency and buckling load. The performance and efficiency of the new plate element is tested with several numerical examples.
where \( n \) is the number of basis functions and \( p \) is the order of the B-spline. A knot vector is said to be uniform if its knots are uniformly spaced and non-uniform otherwise. Moreover, a knot vector is said to be open if its first and last knots are repeated \( p+1 \) times. Basis functions formed from open knot vectors are interpolatory at the ends of the parametric interval \([\xi_i^{n+p+1}]\) but are not, in general, interpolatory at interior knots. It should be noted that we would employ open knot vectors throughout the analysis of elastic bar.

2.2 Basis functions

B-spline basis function is defined recursively starting with \( p=0 \) as:

\[
N_{i,0}(\xi) = \begin{cases} 
1 & \text{if } \xi_i \leq \xi < \xi_{i+1} \\
0 & \text{otherwise}
\end{cases}
\]

For \( p \geq 1 \),

\[
N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi).
\]  

(2)

2.3 B-spline curves

We construct \( n \) basis functions with the order of a B-spline and an appropriately defined knot vector. The piecewise polynomial B-spline curve \( S(\xi) \) of order \( p \) can be obtained by taking a linear combination of basis function and control points:

\[
S(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi) C_i
\]  

(3)

where \( C_i \) is the \( i^{th} \) control point. The piecewise linear interpolation of the control points defines the control net.

2.4 B-spline surface

We construct a B-spline surface using tensor product of spline curve in the direction of \( \xi \) and \( \eta \):

\[
S(\xi, \eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} N_{i,p}(\xi) N_{j,q}(\eta) C_{i,j}
\]

(4)

2.5 NURBS

NURBS can be defined as

\[
S(\xi, \eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} R_{i,j}^{p,q}(\xi, \eta) C_{i,j}
\]

(5)

where \( R_{i,j}^{p,q} \) can be written as

\[
R_{i,j}^{p,q}(\xi, \eta) = \frac{N_{i,p}(\xi) M_{j,q}(\eta) w_{i,j}}{\sum_{i=1}^{n} \sum_{j=1}^{m} N_{i,p}(\xi) M_{j,q}(\eta) w_{i,j}}
\]

(6)

and \( w_{i,j} \) is the weight associated with the control point.

3. REISSNER-MINDLIN ASSUMPTIONS

3.1 Displacement definition

The total domain \( \Omega \) of plate consists of the mid-surface and the thickness as shown in Figure 1 and it can be defined as

\[
\Omega = \{(x_1, x_2, x_3) | (x_1, x_2) \in \Omega_0, x_3 \in \left[-\frac{h}{2}, \frac{h}{2}\right]\}
\]

(7)

where \( \Omega_0 \) is the xy-plane and \( h \) is denoted as thickness of plate.

With transverse shear deformation, the displacement fields can be defined as

\[
\begin{align*}
\mathbf{u}_1(x_1, x_2, x_3) &= x_3 \mathbf{\theta}_2 \\
\mathbf{u}_2(x_1, x_2, x_3) &= -x_2 \mathbf{\theta}_1 \\
\mathbf{u}_3(x_1, x_2, 0) &= \mathbf{\bar{u}}_3
\end{align*}
\]

(8)

where \( \mathbf{\bar{u}}_3 \) is the transverse displacement of a point on the mid-surface, \( \mathbf{\theta}_1 \) is the normal rotation in \( x_2 - x_3 \) plane and \( \mathbf{\theta}_2 \) is the normal rotation in \( x_1 - x_3 \) plane.
3.2 Strain definition
The strains in the plate are defined by linear strain-displacement relationship as follows

\[
\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \frac{1}{2} \left( u_{i,j} + u_{j,i} \right). \tag{9}
\]

Substituting (8) into (9) yields

\[
\varepsilon_P = \begin{bmatrix} \varepsilon_1 \\ \gamma_{12} \end{bmatrix} = x_3 \begin{bmatrix} \theta_{2,1} \\ -\theta_{1,2} \end{bmatrix}, \\
\varepsilon_s = \begin{bmatrix} \gamma_{13} \\ \gamma_{23} \end{bmatrix} = \begin{bmatrix} \theta_2 + \bar{u}_{1,3} \\ -\theta_1 + \bar{u}_{1,2} \end{bmatrix} \tag{10}
\]

in which \(\varepsilon_P\) is the in-plane strain term, \(\varepsilon_s\) denotes the transverse shear strain term respectively.

3.3 Constitutive equation
In this study, the plate is assumed as isotropic material and the normal transverse stress \((\sigma_3)\) is assumed to be negligible. Therefore, the constitutive equation can be written as

\[
\sigma = C \varepsilon \tag{11a}
\]

Or explicitly,

\[
\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \\
\sigma_{13} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & C_{33} & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & C_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{bmatrix} \tag{11b}
\]

in which \(C_{ij}\) are the components of plate rigidity as follows

\[
C_{11} = \frac{E}{1-v^2}, \quad C_{22} = C_{11} = C_{12}, \\
C_{33} = C_{44} = C_{55} = G = \frac{E}{2(1-v)} \tag{12}
\]

where \(E\) is the Young's modulus, \(G\) is the shear modulus and \(v\) are the Poisson ratio.

3.4 Stress resultants
The stress resultants are calculated by integration of the stresses through thickness direction of plate and five stress resultant terms such as \(M = \{M_1, M_2, M_{12}\}\) and \(Q = \{Q_{13}, Q_{23}\}\) can be obtained as follows

\[
M = \begin{bmatrix} M_1 \\ M_2 \\ M_{12} \end{bmatrix} = \int_{-h/2}^{h/2} x_3 \begin{bmatrix} \sigma_1 \\ \sigma_{12} \\ \tau_{12} \end{bmatrix} \, dx_3, \\
Q = \begin{bmatrix} Q_{13} \\ Q_{23} \end{bmatrix} = \int_{-h/2}^{h/2} x_3 \begin{bmatrix} \tau_{13} \\ \tau_{23} \end{bmatrix} \, dx_3. \tag{13}
\]

The above stress resultant terms can be rewritten in the matrix form:

\[
\begin{bmatrix} \varepsilon_P \\ \varepsilon_s \end{bmatrix} = \begin{bmatrix} D & 0 \\ 0 & G \end{bmatrix} \begin{bmatrix} \varepsilon_P \\ \varepsilon_s \end{bmatrix} \tag{14}
\]

where the components of \(D\) and \(G\) are

\[
D_{ij} = \frac{h^3}{12} C_{ij} \quad (i,j = 1,3); \quad G_{ij} = k \rho C_{ij} \quad (i,j = 4,5) \tag{15}
\]

in which \(h\) is the thickness of plate and \(k\) is the shear correction factor (=5/6).

4. ISOGOMETRIC FORMULATION

4.1 Kinematics and displacement field
The geometry and displacement field of the present isogeometric element can be defined in the following form:

\[
x = \sum_{a=1}^{n \times m} R_a x^a, \quad u = \sum_{a=1}^{n \times m} R_a u^a \tag{16}
\]

where \(R_a\) is the NURBS basis function associate with control point \(a\), \(x\) is the position vector of the plate and the displacement vector \(u^a\) associated with control point \(a\) has three components such as

\[
u^a = \{\bar{u}_2, \theta_1^a, \theta_2^a\}. \tag{17}
\]

Note that the one-dimensional index \(a\) as a pointer can be expressed as \(a = m(i-1) + j\) associated with two-dimensional expression and therefore the \(R_a\) of (16) can be interpreted as \(R_{i,j}^a\) of (6).

4.2 Strain-displacement relationship matrix
Using (16), the strains of (10) can be rewritten in the form of the strain-displacement relation matrix \(B\) as follows

\[
\varepsilon_P = \sum_{a=1}^{n \times m} B^a_{\varepsilon} u^a; \quad \varepsilon_s = \sum_{a=1}^{n \times m} B^a_{\varepsilon} u^a \tag{18}
\]

where the sub-matrices of \(B^a\) are

\[
B^a_{\varepsilon} = \begin{bmatrix} 0 & 0 & \frac{\partial R_a}{\partial x_1} \\ 0 & \frac{\partial R_a}{\partial x_2} & 0 \\ \frac{\partial R_a}{\partial x_1} & \frac{\partial R_a}{\partial x_2} & 0 \end{bmatrix}, \quad B^a_{\varepsilon} = \begin{bmatrix} \frac{\partial R_a}{\partial x_1} & 0 & R_a \\ \frac{\partial R_a}{\partial x_2} & 0 & -R_a \\ 0 & -R_a & 0 \end{bmatrix}. \tag{19}
\]
4.3 Free vibration analysis

In the absence of external load and damping effects, the dynamic equilibrium equation based on principle of virtual work can be written as

\[
\int_\Omega \delta \varepsilon_D \mathbf{D} \delta \varepsilon_D \, d\Omega + \int_\Omega \delta \varepsilon_C \mathbf{B} \delta \varepsilon_C \, d\Omega = \int_\Omega \delta \mathbf{u} \mathbf{p} \delta \mathbf{u} \, d\Omega \tag{20}
\]

where \( \mathbf{u} \) is the displacement, \( \dot{\mathbf{u}} \) is the acceleration, \( \rho \) is the density of material and the notation \( \delta \) denotes that the terms are virtual.

The relevant derivation takes place in finite-dimensional subspace to turn the above virtual statement of the problem into a system of algebraic equations. In this study, the subspaces are defined by using the NURBS basis:

\[
\mathbf{u} = \sum_{a=1}^{n \times m} R_a \mathbf{u}_a, \quad \dot{\mathbf{u}} = \sum_{a=1}^{n \times m} R_a \dot{\mathbf{u}}_a \tag{21}
\]

where \( n \times m \) is the total number of the control point in the discretized domain and the virtual terms associated with the displacement and acceleration are

\[
\delta \mathbf{u} = \sum_{a=1}^{n \times m} R_a \delta \mathbf{u}_a, \quad \delta \dot{\mathbf{u}} = \sum_{a=1}^{n \times m} R_a \delta \dot{\mathbf{u}}_a \tag{22}
\]

Substituting (21) and (22) into (20) yields

\[
\delta \mathbf{u}^T [\mathbf{K} - \mathbf{M}] \delta \dot{\mathbf{u}} = 0 . \tag{23}
\]

Since the virtual displacement \( \delta \mathbf{u} \) is arbitrary, the above equation may be written as

\[
\mathbf{K} \mathbf{u} - \mathbf{M} \dot{\mathbf{u}} = 0 \tag{24}
\]

A general solution of (24) may be written

\[
\mathbf{u} = \mathbf{\Phi}_k e^{i\omega_k t} . \tag{25}
\]

Substituting (25) into (24) yields

\[
[\mathbf{K} - \omega_k^2 \mathbf{M}] \mathbf{\Phi}_k = 0 \tag{26}
\]

where \( \mathbf{\Phi}_k \) is a set of displacement-type amplitude at the control points otherwise known as the model vector. \( \omega_k \) is the natural frequency associated with the \( k^{th} \) mode and \( \mathbf{K} \) and \( \mathbf{M} \) are global stiffness and mass matrices which contain contributions from element stiffness and mass matrices.

The structural stiffness and mass matrices in (26) can be written as

\[
\mathbf{K} = \mathbf{K}_{ab} = \mathbf{K}_b^{ab} + \mathbf{K}_s^{ab} = \int_\Omega \mathbf{B}_p^T \mathbf{D}_p \mathbf{B}_p \, d\Omega + \int_\Omega \mathbf{B}_s^T \mathbf{D}_s \mathbf{B}_s \, d\Omega \tag{27}
\]

\[
\mathbf{M} = \mathbf{M}^{ab} = \int_\Omega \mathbf{R}_a^T \rho \mathbf{R}_b \, d\Omega . \tag{28}
\]

Let the span is assumed to be as an isogeometric element and then the above equation can be written in the knot coordinate system as follows

\[
\prod_{e=1}^{n_{el}} [\mathbf{K}^{ab}]^{(e)} = \prod_{e=1}^{n_{el}} \int_{k_{e}}^{k_{e+1}} \left[ \mathbf{B}_p^T \mathbf{D}_p \mathbf{B}_p \right] \mathbf{det}(J) \, d\xi \, d\eta + \prod_{e=1}^{n_{el}} \int_{k_{e}}^{k_{e+1}} \left[ \mathbf{B}_s^T \mathbf{D}_s \mathbf{B}_s \right] \mathbf{det}(J) \, d\xi \, d\eta \tag{29}
\]

\[
\prod_{e=1}^{n_{el}} [\mathbf{M}^{ab}]^{(e)} = \prod_{e=1}^{n_{el}} \int_{k_{e}}^{k_{e+1}} \left[ \mathbf{R}_a^T \rho \mathbf{R}_b \right] \mathbf{det}(J) \, d\xi \, d\eta \tag{30}
\]

where \( n_{el} \) \( = n_{span} \times m_{span} \) is the number of element, \( [k_{e} \ k_{e+1}] \) is knot interval for integration in the \( \xi \) and \( \eta \), \( J \) is Jacobian matrix between \( x \) and \( \xi \), \( a, b \) is the basis function number associated with the target element, \( p \) is the order of basis function and the matrix \( \mathbf{m} \) is

\[
\mathbf{m} = \rho \cdot \begin{bmatrix} h & 0 & 0 \\ 0 & \frac{h^3}{12} & 0 \\ 0 & 0 & \frac{h^3}{12} \end{bmatrix} . \tag{31}
\]

4.4 Linear buckling analysis

For linear buckling analysis, the same type of eigenvalue analysis can be used as follows

\[
[\mathbf{K} - \lambda \mathbf{K}_c] \mathbf{\Phi}_k = 0 \tag{32}
\]

where \( \mathbf{K} \) is the stiffness matrix of (27), \( \lambda \) is the linear buckling load constant and \( \mathbf{K}_c \) is the geometric stiffness such as

\[
\mathbf{K}_c = \int_\Omega \delta \varepsilon^t \mathbf{\sigma}^0 \, d\Omega \tag{33}
\]

where the initial stress \( \mathbf{\sigma}^0 \) and the Green strain \( \varepsilon^t \) are

\[
\mathbf{\sigma}^0 = \begin{bmatrix} \sigma_{11}^0 & \tau_{12}^0 \\ \tau_{12}^0 & \sigma_{22}^0 \end{bmatrix} \tag{34}
\]

\[
\varepsilon^t = \begin{bmatrix} \varepsilon_{11}^t \\ \varepsilon_{12}^t \\ \gamma_{12}^t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} x_3^2 \left( \theta_{1,1}^2 + \theta_{2,2}^2 \right) + \frac{1}{2} \bar{u}_{3,1}^2 \\ \frac{1}{2} x_3^2 \left( \theta_{1,2}^2 + \theta_{2,2}^2 \right) + \frac{1}{2} \bar{u}_{3,2}^2 \\ \theta_{1,1}^0 \theta_{1,2} + \theta_{2,1} \theta_{2,2} + \bar{u}_{3,1} \bar{u}_{3,2} \end{bmatrix} \tag{35}
\]

The geometric stiffness associated with knot span can be derived in the similar manner used in (29).
where the sub-matrices of $G^a$ are

$$G_p^a = \begin{bmatrix} \frac{\partial R_a}{\partial x_1} & 0 & 0 \\ \frac{\partial R_a}{\partial x_2} & 0 \\ 0 & 0 \end{bmatrix}, \quad G_s^a = \begin{bmatrix} 0 & \frac{\partial R_a}{\partial x_1} \\ 0 & \frac{\partial R_a}{\partial x_2} \\ \frac{\partial R_a}{\partial x_1} & 0 \end{bmatrix}, \quad G_{s1}^a = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad G_{s2}^a = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(37)

5. NUMERICAL EXAMPLES

In this section, the efficiency and capability of the present isogeometric plate element is tested by three numerical examples. The lowest frequencies and the buckling load of plates are calculated by using subspace iteration (Bathe, 1996) and the results are compared to the existing reference solutions (Irie et al., 1980; Kitipornchai et al., 1993; Ferreira et al. 2011). Note that all calculations on the stiffness and mass matrices are performed without the reduced integration.

5.1 Circular plate

A circular plate with clamped boundaries is analyzed. The entire plate is used to examine the free vibration behaviour of the plate. The plate is discretized with a mesh of 256, isogeometric elements as shown in Figure 2.

The resulting frequencies are presented in the dimensionless form

$$\Omega_n = \omega_n r^2 \left( \frac{\rho}{E} \right)^{1/2}$$

(38)

where $r$ is the radius of the circular plate, $\rho$ is the density of the material and the $D = Eh^3/12(1 - \nu^2)$ is the flexural rigidity of the plate in which $E$ is the elastic modulus and $\nu = 0.3$ is the Poisson ratio. Two thickness-span ratios $h/2r = 0.1, 0.01$ and four orders of basis function $p = 2, 3, 4, 5$ are used in the analysis.

From numerical test, asymmetric and axisymmetric vibration modes are detected in this example and multiple frequencies are obtained from the axisymmetric modes.

Figure 3. A clamped circular plate: (a) mode 1 – (l) mode 12

<table>
<thead>
<tr>
<th>Mode</th>
<th>Present : order of basis function</th>
<th>Ref</th>
<th>AS</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>17.8129 17.7967 17.7945</td>
<td>20.1761 17.8340</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>27.0761 27.0583 27.0741 27.0583</td>
<td>32.2005 27.1226</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>36.9303 36.8389 36.8838 36.8388</td>
<td>45.7592 37.0790</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>47.1604 47.0658 47.0697 47.0658</td>
<td>60.6319 42.4090</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>57.0665 57.0517 57.0545 57.0517</td>
<td>75.6869 57.9385</td>
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</tr>
<tr>
<td>7</td>
<td>47.1604 47.0798 47.0669 47.0650</td>
<td>60.6319 47.3400</td>
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</tr>
<tr>
<td>8</td>
<td>54.7327 54.6549 54.6549 54.6549</td>
<td>72.2032 54.5570</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>54.9008 54.7105 54.6842 54.6719</td>
<td>75.6869 56.6820</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>57.6989 57.5391 57.5308 57.5308</td>
<td>76.7977 57.9385</td>
<td></td>
</tr>
</tbody>
</table>

Note: AS: Analytical Solutions (Irie et al., 2006); Ref: the collocation with radial basis function (Ferreira et al. 2011).
Table 2. The non-dimensionalized natural frequencies \( \lambda \) of a clamped circular plate with \( h/2r = 0.01. \)

<table>
<thead>
<tr>
<th>Mode</th>
<th>Present: order of basis function</th>
<th>Ref</th>
<th>AS</th>
</tr>
</thead>
<tbody>
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<td></td>
<td>( p=2 )</td>
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<td></td>
</tr>
<tr>
<td>1</td>
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</tr>
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<td></td>
<td></td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>35.4110</td>
<td>34.7821</td>
<td>34.7689</td>
</tr>
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<td></td>
<td>36.7987</td>
<td>34.8457</td>
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<td></td>
<td>-</td>
<td>-</td>
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<tr>
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<td>53.4967</td>
<td>50.9053</td>
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<td>-</td>
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<tr>
<td>10</td>
<td>99.3278</td>
<td>90.5221</td>
<td>90.1106</td>
</tr>
</tbody>
</table>

Note: AS: Analytical Solutions (Irie et al., 2006); Ref: the collocation with radial basis function (Ferreira et al. 2011).

From numerical results, the present results for both thickness-span ratios \( h/2r = 0.1, 0.01 \) have a very good agreement with the analytical solution (Irie et al., 1980) in overall modes. However, it should be noted that there are some discrepancies between analytical solution (Irie et al., 1980) and the reference solution (Ferreira et al., 2011) produces by meshless method denoted as MM. The results are presented in Figure 3 and Tables 1 and 2.

### 5.2 Circular plate with an elliptical hole

A circular plate with an elliptical hole illustrated in Figure 4 is analyzed with the fixed value of \( a/b = 5 \) and four different sizes of hole \( c/b = 1, 2, 3, 4 \). Two different boundary conditions are used: CC and CF where C and F stand for the clamped and free boundary condition respectively. Two thickness-span ratios \( h/a = 0.01 \) and 0.1 are tested. As illustrated in Figure 5, a quarter of plate is discretized with a mesh of 256 isogeometric elements and five orders of basis function \( p = 2, 3, 4, 5, 6 \) are used. From numerical test, we illustrated the first four mode shapes in Figures 6 and 7. The natural frequencies are calculated in the dimensionless form using (38) and provided in Tables 3 and 4. For comparisons, new FE reference solution is also produced by the shell element S8R5 of the ABAQUS since there is no analytical solution for this problem. Note that 353, 351, 389 and 370 FEs are used for the cases \( c/b = 1, 2, 3, 4 \) respectively. The present isogeometric solution has a good agreement with new FE reference solutions denoted by ABAQUS in Tables 3 and 4.
5.3 Square plate

A square plate is adopted to carry out the linear buckling analysis. The plate is subjected to uniaxial load ($N_1 = 1$) as shown in Figure 8. Three orders of basis function ($p = 1, 2, 3$) are used in this example. Five different aspect ratios such as $a/b = 0.5, 1.0, 1.5, 2.0, 2.5$ together with five element meshes $11 \times 22, 16 \times 16, 20 \times 13, 22 \times 11, 25 \times 10$ are used in the analysis.

Three thickness-width ratios of the plate $b/h = 5, 10, 20$ are also employed. The results are presented in the dimensionless form

$$
\bar{N} = N_1 \lambda_n b^2 / \pi^2 D
$$

(39)

where $N_1$ is the intensity of uniaxial load, $\lambda_n$ is the buckling load constant, $b$ is the width of plate and $D$ is the flexural rigidity.
The results are provided in Figure 9 and Table 5. From numerical results, the present isogeometric solution have a good agreement with reference solutions which were produced by using Rayleigh-Ritz method (Kitipornchai et al., 1993) and meshfree method (Liew et al. 2004) respectively. It is found to be that the present isogeometric solutions generally exist between two reference solutions for most cases.

<table>
<thead>
<tr>
<th>b/h</th>
<th>Present : order of basis function</th>
<th>Ref1</th>
<th>Ref2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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Note: Ref1: Meshfree method (Liew et al., 2004); Ref2: Rayleigh-Ritz method (Kitipornchai et al., 1993).

6. CONCLUSIONS

A plate element is developed by using isogeometric concept and used to investigate the behaviours of plate structures under free vibration and linear buckling conditions. The efficiency and accuracy of present isogeometric plate element is demonstrated by using several numerical examples. From numerical tests, the present isogeometric plate element can increase the order of basis function without changing the initial geometry and also can redefine basis function itself without any difficulty. It is also found that the present isogeometric element can produce reliable natural frequencies and buckling loads for plate structures without any locking phenomenon. Finally, the present isogeometric solutions described in this paper are provided as future reference solutions on the vibration and buckling analyses of plates.

REFERENCES


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