1. INTRODUCTION

Bernoulli discovered that the curvature of an elastic beam is proportional to the bending moment. It is generally eligible for thin beam situation and has been widely adopted in the analysis of beams. Later, Timoshenko (1921) proposed a beam theory which includes the transverse shear deformation effect. Because of its new aspect, Timoshenko beam theory has been intensively used to investigate the behaviours of thick beams and it has been also adopted to formulate various finite element (FE) models by many investigators (Kapur, 1966; Nickel and Secor, 1972; Dawe, 1978; Levinson, 1981; Heyliger and Reddy, 1988). However, it was found that Timoshenko beam FEs can exhibit shear locking phenomenon. In order to overcome shear locking, the reduced integration rule was firstly used (Prathap and Bhashyam, 1982). Other attempts have been subsequently introduced to eliminate shear locking. Assumed strains were introduced to formulate two-node beam element based on Timoshenko beam theory (Lee and Ma 1993). Different order of shape function was used for deflection and shear terms respectively: two-node element (Kosmatka, 1994) and three-node element (Reddy, 1997). From previous works, it is quite obvious that the selection of a relevant shape function is crucial to obtain accurate Timoshenko beam solutions.

Apart from the selection of shape function, it has been also important to produce the geometry of the structure in accurate manner. Therefore, there were enormous amount of works (Rogers, 2000) to develop the precise definitions of computer-aided geometric design (CAGD). Generally, the CAGD has been used in the development of FE mesh generation technique (Lee, 1998) to produce the subdivisions in FE analysis. The FE mesh generation based on CAGD has some drawbacks. Particularly, the FE mesh obtained by the FE mesh generator based on CAGD does not exactly represent the real geometry. In other words, there is the gap between the geometric model and FE model. Therefore, Timoshenko beam FE solutions have an inherited limitation in its accuracy. The integration of geometric model and analysis model is highly desirable to carry out the numerical analysis in efficient way.

In this context, isogeometric analysis (Hughes et al., 2005, 2008, 2010; Cottrell et al., 2009) was introduced to unify the separate process of geometric modelling and FE analysis. The isogeometric element formulation uses B-splines instead of the interpolation function normally used in the FE formulation. The use of B-spline function gives more freedom to choose the basis functions with different order and to use the refinement schemes some of which is not available in FE analysis (Zienkiewicz and Taylor, 1989).

Therefore, the isogeometric approach is introduced to carry out the static analysis of Timoshenko beams as a sequel of recent study (Lee and Park, 2013). For this purpose, in this paper, a brief description of B-splines and an isogeometric formulation for a thick beam element are first provided. Then the performance of this isogeometric beam element is investigated through static analyses of beam structures. In particular, there have been some investigations on the effect of h-, p- and k-refinement to the accuracy of beam deflections. The shear locking phenomenon of
Timoshenko beam element is also examined and numerical results on the locking in isogeometric beam analysis are provided. In the tests, Gauss quadrature rule is used for numerical integration and all the numerical results are provided as new isogeometric reference solutions for static analysis of Timoshenko beams.

2. B-SPLINES

2.1. Knot vector

A knot vector is a set of coordinates in parametric space and has the form $\Xi = \{\xi_1, \xi_2, \ldots, \xi_{n+p+1}\}$, where $\xi_i \in \mathbb{R}^1$ is the $i^{th}$ knot, $n$ is number of basis functions and $p$ is the order of the basis function. A knot vector is said to be uniform if its knots are uniformly spaced and non-uniform otherwise. Moreover, a knot vector is said to be open if its first and last knots are repeated $p + 1$ times. Basis functions formed from open knot vectors are interpolatory at the ends of the parametric interval $[\xi_1, \xi_{n+p+1}]$ but are not, in general, interpolatory at interior knots. It should be noted that open knot vectors are employed throughout this study.

2.2. Basis function

$B$-spline basis functions are defined recursively (De Boor, 1978), beginning with order $p = 0$ as

$$N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \leq \xi < \xi_{i+1} \\ 0 & \text{otherwise} \end{cases}$$ (1)

and following with higher orders $p = 1,2,3,\cdots$, as

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi).$$ (2)

As an example, quadratic basis functions having control points $n = 8$ generated from the open knot vector $\Xi = \{0,0,0,0.2,0.4,0.6,0.8,0.8,1,1,1\}$ is illustrated in Fig. 1.

![Figure 1. Basis function for a B-spline](image1)

It is well known that they are $C^{p-1}$-continuous if internal knots are not repeated. However, if a knot has multiplicity, the function is $C^{p-k}$-continuous at the particular knot. For example, when a knot has multiplicity $p$, the basis function is and is $C^0$ interpolatory at that location.

2.3. Curve definition

For one-dimensional problems, a $B$-spline can be defined in the following form:

$$S(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi)C_i$$ (3)

where $n$ is the number of control points $C_i$ is the order of the function, and $N_{i,p}(\xi)$ is a univariate $B$-spline basis function of order $p$ corresponding to knot vector $\Xi = \{\xi_1, \xi_2, \ldots, \xi_{n+p+1}\}$.

Fig. 2 presents a quadratic two-dimensional $B$-spline curve generated from the basis functions in Fig. 1 with its control point net.

![Figure 2. B-spline curve](image2)

A $B$-spline curve has continuous derivatives of order $p - 1$, which can be decreased by $k$ if a knot or a control point has multiplicity $k + 1$. An important property of $B$-spline curves is their affine covariance; that is, an affine transformation of the curve is obtained by applying the transformation to its control points.

2.4. Refinements

The $h$-refinement strategy inserts knots without changing the curve geometrically or parametrically. This subdivision strategy changes the control points but maintains the order of basis functions. The $p$-refinement strategy increases the polynomial order of basis functions without changing the geometry and parameterization of a curve. This strategy changes both the control points and the order of basis functions but not the knot span. The $k$-refinement strategy first elevates the order of the original curve and then only inserts unique knot values, unlike the $h$- and $p$-refinement strategies. This enhances continuity between the elements and provides higher regularity at new element interfaces than the $h$- and $p$-refinement strategies.

3. ISOGEOMETRIC FORMULATION

3.1. Kinematics

In the present formulation, Timoshenko beam theory is adopted and the beam is assumed to be linear elastic and isotropic material properties. Therefore, the deformation of the beam can be illustrated as shown in Fig. 3.
If translational displacement is \( u = 0 \), the kinematic assumptions can be written as

\[
\begin{align*}
\mathbf{u}(x, z) &= -z \theta \\
\mathbf{w}(x, z) &= w
\end{align*}
\]  

(4)

where \( \theta \) is the rotation of the beam cross section normal to the mid-axis \( x \), and \( w \) is the transverse displacement of the beam mid-axis.

3.2. Strains

Normal and transverse shear strain of the beam can be defined by the derivatives of Eq. (4) as follows:

\[
\begin{align*}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} = -\frac{\partial \theta}{\partial x} = -z \kappa \\
\gamma_{zx} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} - \theta = \gamma.
\end{align*}
\]  

(5)

where \( \kappa \) is the curvature.

3.3. Stress

Using Hooke law, normal stress can be written as

\[
\sigma_{xx} = E \varepsilon_{xx} = -z E \kappa
\]  

(6)

where \( E \) is Young modulus.

Similarly, the transverse shear stress is

\[
\tau_{zx} = k_s G \gamma_{zx} = k_s G \gamma
\]  

(7)

where \( k_s \) is the shear correction factor and \( G \) is the shear modulus.

Stress resultants can then be evaluated by integrating the stresses defined in Eqs. (6) and (7) through the thickness direction:

\[
\begin{align*}
M &= - \int_A z \sigma_{xx} \, dA = -E \kappa \int_A z^2 \, dA = EI \kappa \\
Q &= - \int_A \tau_{zx} \, dA = -k_s G \gamma \int_A 1 \, dA = -k_s G A \gamma
\end{align*}
\]  

(8)

where \( M \) is the moment, \( Q \) is the transverse shear force, \( I \) is the moment of inertia and \( A \) is the cross sectional area.

3.4. Weak form

The weak form for thick beam problems can be defined as follows:

\[
a(\delta \mathbf{u}, \mathbf{u}) = \mathcal{L}(\delta \mathbf{u}, f)
\]  

(9)

where

\[
\begin{align*}
a(\delta \mathbf{u}, \mathbf{u}) &= \int_0^L \delta \mathbf{u}^T \mathbf{v} \, d\mathbf{v} \, d\mathbf{x} \\
\mathcal{L}(\delta \mathbf{u}, f) &= - \int_0^L \delta \mathbf{u} \cdot \mathbf{f} \, dx - \delta \mathbf{u} \cdot \mathbf{f}
\end{align*}
\]

\[
\mathbf{u} = \begin{bmatrix} w \\ \theta \end{bmatrix}; \quad \mathbf{v} = \begin{bmatrix} 0 & -\nabla_x \\ \nabla_x & -1 \end{bmatrix}; \quad \mathbf{D} = \begin{bmatrix} EI & 0 \\ 0 & k_s G A \end{bmatrix}
\]

in which \( \mathbf{u} \) is the displacement, \( L \) is the length of the beam, \( \mathbf{f} \) is the uniform load vector and \( \mathbf{f} \) is the point load vector. The notation denotes that the terms are virtual.

3.5. Galerkin method

The Galerkin method is used to turn a weak statement of the problem into a system of algebraic equations. The relevant derivations take place in a finite-dimensional subspace. In this study, the subspaces are defined by using the isoparametric NURBS basis:

\[
\delta \mathbf{u}^h = \mathbf{w}^h = \sum_{a=1}^{n_{eq}} N_a \mathbf{d}_a.
\]  

(10)

Furthermore, the function \( g^h \) is given similarly by quantities \( g_{a}, \ a = 1, \ldots, n_{be} \)

\[
g^h = \sum_{a=n_{eq}+1}^{n_{eq}+n_{be}} N_a g_a.
\]  

(11)

Therefore, trial functions can be expressed as

\[
\mathbf{u}^h = \sum_{a=1}^{n_{eq}} N_a \mathbf{d}_a + \sum_{a=n_{eq}+1}^{n_{eq}+n_{be}} N_a g_a.
\]  

(12)

where \( n_{eq} \) is the number of equations and \( n_{be} \) is the number of boundary conditions.

Substitution Eq. (10) and Eq. (12) into the weak form of Eq. (9) yields

\[
\sum_{a=1}^{n_{eq}} \mathbf{d}_a \left( \sum_{b=1}^{n_{be}} a(N_a, N_b) \mathbf{d}_b - \mathcal{L}(N_a, f) - a(N_a, g^h) \right) = 0.
\]  

(13)

Since \( \mathbf{d}_a \) can be arbitrary, Eq. (13) reduces to

\[
\sum_{b=1}^{n_{be}} a(N_a, N_b) \mathbf{d}_b = \mathcal{L}(N_a, f) - a(N_a, g^h), \quad a = 1, \ldots, n_{eq}.
\]  

(14)
Now the terms $K_{ab}$ and $F_a$ can be defined by using Eq. (14) as follows:

$$K_{ab} = a(N_a, N_b)$$

$$F_a = L(N_a, f) - a(N_a, B^b).$$

For $a, b = 1, \ldots, n_{eq}$, an algebraic equation can be finally provided as

$$Kd = F \tag{16}$$

where the stiffness matrix $K$, force vector $F$ and displacement vector $d$. Eq. (16) can be written in the following form

$$K_{ab}d_b = F_a \tag{17}$$

where the stiffness matrix and force vector can be written explicitly as follows:

$$K_{ab} = \int_0^L \begin{bmatrix} 0 & -\partial_x N_a, p \\ -\partial_x N_a, p & 0 \end{bmatrix}^T \begin{bmatrix} EI \\ GA \end{bmatrix} \begin{bmatrix} 0 & -\partial_x N_b, p \\ -\partial_x N_b, p & 0 \end{bmatrix} \partial_x \begin{bmatrix} 0 \\ 0 \end{bmatrix} dx$$

$$= K_{ap}^b + K_{bp}^a$$

$$= \int_0^L \begin{bmatrix} 0 & \partial_x N_a, p \\ \partial_x N_a, p & 0 \end{bmatrix} EI \begin{bmatrix} 0 \\ 0 \end{bmatrix} \partial_x N_b, p dx$$

$$+ \int_0^L \begin{bmatrix} \partial_x N_a, p \partial_x N_b, p & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\partial_x N_a, p \\ \partial_x N_b, p \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \partial_x \begin{bmatrix} 0 \\ 0 \end{bmatrix} dx$$

$$F_a = -\int_0^L \begin{bmatrix} N_a, p \\ 0 \end{bmatrix} \partial_x \begin{bmatrix} \bar{q} \\ \bar{M} \end{bmatrix} dx$$

where $\partial_x = \partial / \partial x$ is the differential operator. $N_a, p$ is the basis function of the order $p$ associated with control point used in the formulation, $\bar{q}$ is a uniform load, $\bar{p}$ is a point load and $\bar{M}$ is moment at a certain point.

4. NUMERICAL EXAMPLES

Numerical tests are performed to explore the important aspects of isogeometric Timoshenko beam analysis. Three $p$-, $h$- and $k$-refinement tests are carried out and knot vectors associated with the number of isogeometric element for three refinements are summarized in Table 1. The existence of the shear locking phenomenon is also investigated. In this test, three beams are considered: cantilever, simple beam and fixed-fixed beam. The following material properties are used: $E = 10^9$, $v = 0.0$, $L = 2$, $b = 1$, $h = 0.1$. All units are assumed to be consistent.

4.1. Cantilever beam

Cantilever beam is used for the $p$- and $h$-refinement tests. The geometry of the cantilever beam is illustrated in Fig. 4.

4.1.1. The $p$-refinement test

The $p$-refinement test is firstly carried out. A single isogeometric element¹ and four orders of basis functions ($p = 1, 2, 3, 4$) are used (See Fig. 5) with the fixed value of thickness-to-length ratio ($h/L = 0.01$). In addition, three Gauss integration points are employed: Case A: full integration (FI) for all stiffness terms; Case B: full reduced integration (RI) for all stiffness terms; and Case C: selectively reduced integration (SRI): reduced integration for the transverse shear stiffness term $K_{pi}$ and full integration for the bending stiffness term $K_{pi}$. Note that the number of Gauss points for full integration is assumed to be $(p + 1)$ when a basis function of order $p$ is used.

![Figure 5. Basis functions used in p-refinement test](image)

<table>
<thead>
<tr>
<th>Table 2. Deflection of Cantilever using FI, RI and SRI for ()</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gauss Points</td>
</tr>
<tr>
<td>Order</td>
</tr>
<tr>
<td>P=1</td>
</tr>
<tr>
<td>P=2</td>
</tr>
<tr>
<td>P=3</td>
</tr>
<tr>
<td>P=4</td>
</tr>
</tbody>
</table>

¹: The meaning of the single element in isogeometric analysis is the subdivision created by two components of knot vector which should be different values since multiple components exist.

<table>
<thead>
<tr>
<th>Table 1. Knot vectors for three refinement tests</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of element</td>
</tr>
<tr>
<td>------------------------------------------------</td>
</tr>
<tr>
<td>$h$-refinement</td>
</tr>
<tr>
<td>$p$-refinement</td>
</tr>
<tr>
<td>$k$-refinement</td>
</tr>
</tbody>
</table>

Note: $(A)_m$: Knot A has multiplicity $m$. $k = 1, (n_{et} - 1)$. $<A>$ is a fixed value.
From numerical results in Table 2, the RI produces the same maximum deflection as those obtained by SRI. On the other hand, the FI produces very poor solution when the first order \((p = 1)\) basis function is used. In addition, the FI also produces poor results when the basis function with the order \(p = 2\) is used. From this result, the beam element based on the isogeometric approach also can suffer from shear locking when lower-order basis functions \((p = 1\) and 2\) are used. However, it should be noted that the use of a linear basis function with single element is definitely a severe test condition. As an example, the deformed shapes of the beam obtained by using SRI and single element are illustrated in Fig. 6. The present isogeometric element produces a poor deformed shape with a linear basis function \((p = 1)\). But, the present isogeometric element can obtain the exact solutions using only one element with higher orders basis function \((p = 3\) and 4\).

4.1.2. The \(h\)-refinement test

For \(h\)-refinement test, it begins with a single isogeometric element and knots will be gradually inserted to increase the number of the elements. Three slenderness ratios \((h/L = 0.01, 0.1, 0.2)\) and six different isogeometric element meshes \((n_{\text{el}} = 1, 2, 4, 8, 16\) and 32\) is used. Fig. 7 illustrates the way how to increase the number of isogeometric elements for the fixed order of basis function \((p = 2)\). The number of elements increases from 1 to 2, 4 and 8 with knot insertion.
solution using only one element. Moreover, SRI can effectively eliminate locking for the case with lower order basis function.

4.2. Simply supported beam

The geometry of the simply supported beam is illustrated in Fig. 9.

![Figure 9. Geometry of simply supported beam](image)

The same six isogeometric element meshes used in Section 4.1.2 are employed here and the present isogeometric solution is normalized by the analytical solution (see Appendix) and the normalized solution is illustrated in Fig. 10. From numerical results shown in Fig. 10, it is found to be that there is some discrepancy between the isogeometric solution with basis functions of order less than $p = 3$ and the exact solution in thin-beam cases. But, this discrepancy disappears with a larger number of elements and higher-order basis functions.

The deformed shapes are illustrated in Fig. 11 for four cases: $p = 1, 2, 3$ and 4. Each case uses 6 meshes with a different number of elements ($e = 1, 2, 4, 8, 16$ and 32) together with FI and SRI. Fig. 12 shows that the number of control point is crucial when a lower-order basis function is used, especially for the case with a single element. In addition, it is presumed that the position of the control points also affects the isogeometric solution. In particular, it is identified that the deformed shape obtained with a basis function of order $p = 3$ deviates from the exact solution.

4.3. Fixed–fixed beam

For the fixed–fixed beam (Fig. 12), the same four basis functions are again used to investigate the effect of the order of the basis function on the deformed shape.

![Figure 12. Geometry of fixed–fixed beam](image)
As illustrated in Fig. 13, for thin beams, severe locking occurs when using full integration and a basis function of order $p = 1$; furthermore, no notable improvement is achieved by increasing the number of elements.

With SRI, poor convergence is detected with the basis function of order $p = 3$. However, much improvement is achieved with the appropriate number of elements.

As illustrated in Fig. 14, near the fixed boundaries, this beam shows more complex deformation shapes than those of two previous beams. The exact deformed shape is obtained with a basis function of order $p = 4$ regardless of the number of element, integration scheme and the slenderness ratio of the beam. Shear locking is completely eliminated with a basis function of order $p = 4$ in this beam even with a single element. However, with a basis function of order $p = 3$ a good deformed shape cannot be obtained with SRI and a single element.

Finally, the $k$-refinement is used to improve the deformed shape of the fixed–fixed beam. Here, three isogeometric elements is used and the order of the basis functions (Fig. 15) is increased from $p = 2$ to 5. For comparison, the $p$-refinement test is performed with the basis function illustrated in Fig. 16. However, it should be noted that the results of these two refinement strategies cannot be simply compared on the basis of the number of isogeometric elements used in the analysis since the number of control points is also crucial. In this test, three slenderness ratios ($h/L = 0.01, 0.1, 0.2$) are used with selective reduced integration.

From the numerical results in Fig. 17, the deformed shape of the beam can be improved through the $k$-refinement with only three elements and a maximum of eight control points. In particular, if the basis functions of order greater than $p = 4$ is used, almost the exact deformed shape is produced with only six control points. However, with $p$-refinement, even with seven control points, the solution differs from the exact solution. At this point, recall that higher-order FE analyses usually require more computation time. However, with isogeometric elements, the order of the basis function can be increased without a significant increase in the number of control points which means that there is no much increase of computation time. This makes it possible to produce accurate numerical solutions with minimal computation time. The
reference solutions obtained with $p$-refinement with basis functions of order $p = 2$ to $5$ resemble those obtained with $k$-refinement but require twice as many control points to reach the exact solution.

5. CONCLUSIONS

Timoshenko beam element is developed by using isogeometric approach. The performance of the present Timoshenko beam element is demonstrated by using numerical examples. The following specific conclusions are drawn from the numerical results:

1. The present element can choose very wide range of $B$-spline basis function without any difficulty for displacement field.

2. The complete integration between structural geometry and analysis model is achieved with the use of the same $B$-spline basis function for the structural geometry and displacement field.

3. The present element provides a very easy manipulation on the structural geometry and no further effort is required to perform $h$- and $p$- refinements during the analysis.

4. Lower-continuity basis functions can produce some discontinuity in the solution field with a coarse mesh, and therefore an appropriate size of control point net is necessary required for isogeometric analysis.

5. Higher-orders of basis functions ($p \geq 3$) can provide accurate solutions and it eliminate shear locking in thin beams situation. In particular, no locking phenomenon appears in the present isogeometric beam solutions with full Gauss integration rule and quartic order of basis function.

6. The Gauss quadrature rule is successfully used in beam isogeometric analysis, but the different type of numerical integration rule might be required in isogeometric analysis for more complex problems and further research is therefore expected.
7. The SRI can improve isogeometric beam solution in great manner because it effectively mitigates locking phenomenon when lower order of basis function is especially used in the analysis. It should be noted that the isogeometric solutions with SRI show some discrepancy with exact solution if small number of isogeometric element is used so that an appropriate number of isogeometric element is required to produce the solution close to the exact solution.

8. The effectiveness of $k$-refinement is clearly demonstrated from the fixed-fixed beam example since only half number of control point is required to achieve the exact solution against $p$-refinement when the basis function of cubic order is used in the analysis.

It is found to be that the present isogeometric beam element does not have any restrictions on the choice of the order of basis function in analysis and so it can ultimately act as a powerful and versatile tool for both accurate modelling and analysis of thick beam structures. Finally, the isogeometric solutions presented in this paper are provided as future reference solutions based on isogeometric approach.

REFERENCES


Proceedings of Annual Symposium of Computational Structural Engineering Institute of Korea.


APPENDIX

In this study, three analytical solutions (Timoshenko and Goodier, 1970) is used as reference solution and described in Appendix Table 1.

<table>
<thead>
<tr>
<th>Beam type</th>
<th>Analytical solution</th>
<th>Thin beam ($w_{thin}$)</th>
<th>Thick beam ($w_{thick}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cantilever</td>
<td>$PL^3/3EI$</td>
<td>$PL^3/3EI$</td>
<td>$PL/ksGA$</td>
</tr>
<tr>
<td>Simply supported beam</td>
<td>$5qL^4/384EI$</td>
<td>$5qL^4/384EI + qL^2/4GAks$</td>
<td></td>
</tr>
<tr>
<td>Fixed-Fixed beam</td>
<td>$qL/384EI$</td>
<td>$qL/384EI + qL^2h^2/806GIk_s$</td>
<td></td>
</tr>
</tbody>
</table>

Note. P is a point load, L is the length of the beam, E is the elastic modulus, G is the shear modulus, I is the moment of inertia, A is the cross-sectional area and $k_s$ is a shear correction factor. $q_{0}$ is a uniform load.

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