THE PROPERTIES OF NONOSCILLATION AND FINITE VALENCE

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1. Introduction.

In this paper we shall consider the differential equation

\[ y''(z) + p(z)y(z) = 0, \quad |z| < 1, \]

where \( p(z) \) is a regular function in the open unit circle \( E \). The ratio \( f(z) = u(z)/v(z) \) of any two independent solution \( u(z) \) and \( v(z) \) of (1.1) will be a function \( f(z) \), meromorphic in \( E \) with only simple poles, and such that \( f'(z) \neq 0 \). The Schwarzian derivative of \( f(z) \),

\[ S_f(z) = \varphi_f'(z) - \frac{1}{2}\varphi_f^2(z), \quad \varphi_f(z) = f''(z)/f'(z) \]

is connected with \( p(z) \) by

\[ S_f(z) = 2p(z). \]

If no solution of (1.1) (except the solution \( y(z) = 0 \)) has more than one zero in \( E \) then \( f(z) \) is univalent in \( E \). Conversely, every univalent function \( f(z) \) in \( E \) can be written as the ratio of two independent solutions of the equation (1.1). These connections were first stated by Z.Nehari ([1] Theorem 1). In this paper we give that the connections of nonoscillation and finite valence. In Section 2, Theorem 2.1 may be state us a criteria of nonossilation. In Section 3, we obtain a simpler criteria for the finite valent of single valent meromorphic function.

2. A criteria of nonoscillation.

(1.1) is called nonoscillation in \( E \) if none of its solutions (except \( y(z) = 0 \)) has infinite many zeros in \( E \). Correspondingly we call a single valued meromorhic function finite valent in a domain \( D \) if for each \( a \) the equation \( f(z) = a \) has only a finite number of solutions \( z \) in \( D \).

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Theorem 2.1. Let $p(z)$ be regular in $|z| < 1$ and assume there exists $x_0, 0 < x_0 < 1$, such that for all $z$ with $x_0 < |z| < 1$

(2.1) \[(1 - |z|^2)^2 |p(z)| \leq 1.\]

Then (1.1) is nonoscillation in $|z| < 1$.

Proof. Let $z_1, z_2 (z_1 \neq z_2)$ be any two points inside $E$. $z_1, z_2$ determine uniquely a circle $C$ passing through them and orthogonal to $|z| = 1$. Let us call the part of $C$ which lies between $z_1$ and $z_2$ and inside $E$, the orthogonal arc between $z_1$ and $z_2$, and let us denote it by $[z_1 z_2]$. Let $x_0$ be fixed and denote the ring $x_0 < |z| < 1$ by $R$.

Assume now that there exists a nontrivial solution $y(z)$ of (1.1) with infinitely many zeros in $E$. From this infinity of zeros we choose a sequence converging to a point $\alpha$ on $|z| = 1$. It follow that we can choose two zeros $z_1$ and $z_2$ of $y(z)$, belonging to this sequence, such that they, together with the orthogonal arc between them, lie in $R$.

There exists a linear transformation from $|z| < 1$ onto $|\zeta| < 1$ given by

(2.2) \[z = e^{i\theta} \frac{\zeta - \alpha}{1 - \bar{\alpha} \zeta}, \quad |\alpha| < 1,\]

which carries $z_1$ and $z_2$ into $\zeta = \rho$ and $\zeta = -\rho$ respectively $(0 < \rho < 1)$. (2.2) transforms $[z_1 z_2]$ into the segment $(-\rho, \rho)$. Define for $|\zeta| < 1$ by

(2.3) \[g(\zeta) = f \left( e^{i\theta} \frac{\zeta - \alpha}{1 - \bar{\alpha} \zeta} \right).\]

The substitution (2.2) transforms (1.1) into

(2.4) \[y_1''(\zeta) + p_1(\zeta)y_1(\zeta) = 0,\]

where

(2.5) \[S_g(\zeta) = 2p_1(\zeta)\]

and

(2.6) \[y \left( e^{i\theta} \frac{\zeta - \alpha}{1 - \bar{\alpha} \zeta} \right) = y_1(\zeta)\sigma(\zeta).\]
Here \( \sigma(\zeta) \) is regular and nonzero in \( |\zeta| < 1 \). It follows that there exists a solution \( y_1(\zeta) \neq 0 \) of (2.4) such that \( y_1(\rho) = y_1(-\rho) = 0 \). Setting \( \zeta = x + iy \), multiplying (2.4) on the segment \((-\rho, \rho)\) by \( \bar{y}_1 \) and integrating from \(-\rho\) to \( \rho \), we obtain

\[
\int_{-\rho}^{\rho} |y_1'|^2 dx = \int_{-\rho}^{\rho} p_1 |y_1|^2 dx.
\]

Writing \( y_1 = u + iv \) we have

(2.7) \[
\int_{-\rho}^{\rho} (u_x^2 + v_x^2) dx = \int_{-\rho}^{\rho} p_1 (u^2 + v^2) dx.
\]

It can be shown that (2.2) and (2.3) imply

\[
|S_f(z)|(1 - |z|^2)^2 = |S_\varphi(\zeta)|(1 - |\zeta|^2)^2.
\]

It follows therefore by (1.2), (2.1) and (2.5) that

\[
(1 - x^2)^2 |p_1(x)| \leq 1, \quad -\rho \leq x \leq \rho.
\]

Hence,

\[
\left| \int_{-\rho}^{\rho} p_1 (u^2 + v^2) dx \right| \leq \int_{-\rho}^{\rho} \frac{u^2 + v^2}{(1 - x^2)^2} dx < \rho^2 \int_{-\rho}^{\rho} \frac{u^2 + v^2}{(\rho^2 - x^2)^2} dx.
\]

Now the inequality

\[
\rho^2 \int_{-\rho}^{\rho} \frac{u^2}{(\rho^2 - x^2)^2} dx < \int_{-\rho}^{\rho} u'^2 dx
\]

holds for continuously differentiable real functions \( u(x), \ -\rho \leq x \leq \rho \), which have at \( \pm \rho \) zeros of the first order [1]. Then we have

\[
\left| \int_{-\rho}^{\rho} p_1 (u^2 + v^2) dx \right| < \int_{-\rho}^{\rho} (u_x^2 + v_x^2) dx,
\]

which gives the desired contradiction to (2.7) and we have therefore proved Theorem 2.1. □

This nonoscillation Theorem may now be stated as a criteria of finite valent for meromorphic functions.
COROLLARY 2.1.1. Let \( f(z) \) be meromorphic in \( |z| < 1 \) and assume that
\[
(1 - |z|^2)^2 |S_f(z)| \leq 2 \text{ for } x_0 < |z| < 1, \ 0 < x_0 < 1
\]
Then \( f(z) \) is finite valent in \( |z| < 1 \).

Proof. Assume that there exists a complex number \( \omega \) (which may be \( \infty \)) such that \( f(z) - \omega = 0 \) has an infinity of roots in \( E \), then there exist \( z_1, z_2, (z_1 \neq z_2) \) such that \( f(z_1) = f(z_2) = \omega \), and that \( z_1, z_2 \) and the orthogonal arc between them lie in \( R \).

Consider now \( f(z) \) and the corresponding (1.1) not in \( E \), but only in any simply connected domain \( D \) containing the arc \([z_1z_2]\) and contained in \( R \). We obtain therefore a solution \( y(z) \) of (1.1), analytic and therefore single valued in \( D \), such that \( y(z_1) = y(z_2) = 0 \), while \( p(z) \) satisfies (2.1) in \( D \) (and especially on \([z_1z_2]\)). But only this used in the proof of Theorem 2.1. \( \Box \)

NEHARI RESULT ([2]) : For the unit circle he proved that if \( p(z) \) is regular in \( |z| < 1 \) and if
\[
(2.8) \quad \int_0^{2\pi} |p(e^{i\theta})| d\theta < \infty,
\]
then (1.1) is nonoscillation.

The integral on the left hand side of (2.8) is defined as the limit, for \( \rho \to 1 \), of the nondecreasing function
\[
\int_0^{2\pi} |p(e^{i\theta})| d\theta
\]
and (2.8) is therefore equivalent to
\[
(2.9) \quad \int_0^{2\pi} |p(\rho e^{i\theta})| d\theta < c, \ c < \infty, \ 0 < \rho < 1.
\]

Nehari Result may be deduced from Theorem 2.1. Indeed, setting
\[
p(z) = \sum_{n=0}^{\infty} a_n z^n,
\]
(2.9) implies
\[ |a_n| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|p(\rho e^{i\theta})|}{\rho^{n+1}} \rho d\theta < \frac{c}{2\pi \rho^n}, \quad n = 0, 1, \ldots. \]

Letting \( \rho \rightarrow 1 \), we obtain \( |a_n| \leq c/2\pi \) and therefore
\[ |p(z)| \leq \sum_{n=0}^{\infty} |a_n||z_n|^n \leq \frac{c}{2\pi (1 - |z|)}. \]

This implies now the existence of at \( x_0 \), such (2.1) holds for \( x_0 < |z| < 1 \), i.e., the assumption of Theorem 2.1 is satisfied. So (1.1) is nonoscillation.

3. A finite valent theorem for a domain.

Let \( D \) be a simply connected domain in the \( z \) plane, having at least two boundary points and let \( w = \psi(z) \) be a function mapping \( D \) onto \( |w| < 1 \). Let \( D' \) be any closed domain lying in the interior of \( D \) and denote by \( R' \) the domain \( D - D' \). The map of \( R' \) under the transformation \( w = \psi(z) \) covers a circular ring \( R, x_0 < |w| < 1 \), with \( 0 < x_0 < 1 \) and \( x_0 \) near enough to 1. Let \( f(z) \) be a meromorphic function in \( D \) and define \( g(w) \) in \( |w| < 1 \) by
\[ g(w) = f(\psi^{-1}(w)). \]

\( f(z) \) is finite valent in \( D \) if and only if \( g(w) \) is so in \( |w| < 1 \). The transformation formula \( S_f(z) \) under the conformal mapping \( w = \psi(z) \) is
\[ (3.1) \quad [S_f(z) - S_\psi(z)] = S_g(w) \left( \frac{dw}{dz} \right)^2. \]

Applying now Corollary 2.1.1 to \( g(w) \) it follows that \( f(z) \) will be finite valent in \( D \) if the condition
\[ (3.2) \quad |S_f(z) - S_\psi(z)| \leq \frac{2}{(1 - |\psi(z)|^2)^2} \left| \frac{d\psi}{dz} \right|^2 \]
holds for all \( z \in D - D' \). Similarly it follows that if \( p(z) \) is regular in \( D \) and if
\[ (3.3) \quad |p(z) - \frac{1}{2} S_\psi(z)| \leq \frac{1}{(1 - |\psi(z)|^2)^2} \left| \frac{d\psi}{dz} \right|^2 \]
holds for all \( z \) in \( D - D' \), then (1.1) is nonoscillation in \( D \).
REMARK. (3.2) and (3.3) are independent of the normalization of the Riemann mapping function \( w = \psi(z) \) mapping \( D \) onto \(|w| < 1\). Let \( w_1 = \psi_1(z) \) be another such function mapping \( D \) onto \(|w_1| < 1\). The function \( w_1(w) = \psi_1(\psi^{-1}(w)) \) is a linear mapping of \(|w| < 1\) onto \(|w_1| < 1\) and it follows by the invariance of the Schwarzian derivative with respect to all linear transformation, that \( S_{w_1}(z) = S_w(z) \), i.e,

\[
(3.4) \quad S_{\psi_1}(z) = S_{\psi}(z).
\]

Moreover, for a linear mapping of the unit circle onto itself, the relation

\[
\frac{1 - |w_1(w)|^2}{1 - |w|^2} = \left| \frac{dw_1}{dw} \right|
\]

holds, which implies

\[
(3.5) \quad \frac{1}{(1 - |\psi_1(z)|^2)^2} \left| \frac{d\psi_1}{dz} \right|^2 = \frac{1}{(1 - |\psi(z)|^2)^2} \left| \frac{d\psi}{dz} \right|^2.
\]

(3.4) and (3.5) show clearly that condition (3.2) and (3.3) are independent of the normalization of the mapping \( \psi(z) \).

Restricting ourselves to domains bounded by a finite number of Jordan curves, we have the following property:

Let \( D \) be a multiply connected domain in the \( z \) plane, bounded by a finite number of Jordan curves. Let \( S \) be its universal covering surface. Let \( w = \psi(z) \) map \( S \) onto \(|w| < 1\) and let \( D' \) be any closed domain in \( D \). A function \( f(z) \), meromorphic and single valued in \( D \), will be finitely valent there if condition (3.3) holds for all \( z \) in \( D - D' \).

This property enable us now to obtain a simpler criterion for the finite valence of single valued meromorphic functions in the case in which the \( n \)-boundaries of the domain are analytic Jordan curves.

THEOREM 3.1. Let \( D \) be a domain in the \( z \)-plane such that its boundary \( B \) consists of a finite number of analytic Jordan curves. Let \( S \) be its universal covering surface. Let \( z_0 \in D \) and denote by \( B_\varepsilon \) the level curve \( g(z, z_0, D) = \varepsilon, \ \varepsilon > 0, \) of the harmonic Green's function \( g(z, z_0, D) \) with pole at \( z_0 \). Let \( w = \psi(z) \) map \( S \) onto \(|w| < 1\) and \( f(z) \) be meromorphic and single valued in \( D \) and set

\[
M(\varepsilon) = \max_{z \in B_\varepsilon} |S_f(z)|.
\]
If

\[ \lim_{\varepsilon \to 0} \varepsilon^2 M(\varepsilon) = 0, \]

then \( f(z) \) is finite valent in \( D \).

**Proof.** Suppose \( D \) is not simply connected. Choose \( \psi(z) \) on \( S \) so that \( \psi(z_0) = 0 \). Let \( z \) be the coordinate in \( D \) and not on \( S \), so that \( \psi(z) \) is a many valued function. By the connecting the \( n \) - boundary curves \( B_1, \ldots, B_n \) of \( D \) by \( n - 1 \) cuts \( \nu_1, \ldots, \nu_{n-1} \), we obtain a simply connected domain \( D^* \). \( D^* \) allows us to fix uniquely a branch \( \psi_1(z) \) of \( \psi(z) \). Assume that none of the cuts \( \nu_1, \ldots, \nu_{n-1} \) go through \( z_0 \). Let the branch \( \psi_1(z) \) be defined \( \psi_1(z_0) = 0 \), and consider the behavior of this branch in \( D \) and on \( B \). From the analyticity of the bounary curves it follows that \( \psi_1(z) \) and its derivatives are piecewise analytic on \( B \). Moreover, \( \frac{d\psi_1(z)}{dz} \neq 0 \) in \( \overline{D} = D \cup B \) and it follows that, for all \( z \) in \( D \),

\[ |S_{\psi_1}(z)| = |S_{\psi}(z)| \leq M, \ 0 < M < \infty, \]

\[ \left| \frac{d\psi_1(z)}{dz} \right| \geq m, \ 0 < m < \infty. \]

For every \( \varepsilon > 0 \) let us now consider the following two closed region in \( D \):

\[ D_1(\varepsilon) = \{ z : g(z, z_0, D) \geq \varepsilon \} \]

and

\[ D_2(\varepsilon) = \{ z | z = \psi^{-1}(w), \ |w| \leq e^{-\varepsilon} \}. \]

Then we have [5, pp. 50-51]

\[ D_2(\varepsilon) \subset D_1(\varepsilon). \]

(3.9) implies now that \( |\psi(z)| \geq e^{-\varepsilon} \) for the level curve \( B_\varepsilon(g(z, z_0, D) = \varepsilon) \) and in particular

\[ |\psi_1(z)| \geq e^{-\varepsilon}, \ z \in B_\varepsilon, \varepsilon > 0. \]
We have therefore for each \( z \in B_\epsilon \)

\[
1 - |\psi_1(z)|^2 = (1 + |\psi_1(z)||1 - |\psi_1(z)|| \leq 2(1 - e^{-\epsilon}) < 2\epsilon
\]

which implies

\[
(3.10) \quad M(\epsilon)(1 - |\psi_1(z)|^2)^2 < 4M(\epsilon)\epsilon^2.
\]

Using now our assumption (3.6), it follows from (3.1), (3.5), (3.7) (3.8) and (3.10) that there exists \( \epsilon_0 > 0 \) such that

\[
|S_f(z) - S_{\psi_1}(z)| \leq \frac{2}{(1 - |\psi_1(z)|^2)^2} \left| \frac{d\psi_1}{dz} \right|^2.
\]

for all \( z \) with \( 0 < g(z, z_0, D) < \epsilon_0 \), i.e., for all \( z \in D - D_1(\epsilon_0) \). So we have proved Theorem 3.1 for a multiply connected domain.

If \( D \) is simply connected domain we use condition (3.2). Relation (3.7) and (3.8) hold now for the single valued function \( \psi(z) \) and in this case, clearly, \( D_1(\epsilon) = D_2(\epsilon) \). Therefore Theorem 3.1 is established. \( \square \)

References


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