RELATIONS BETWEEN DECOMPOSITION SERIES AND TOPOLOGICAL SERIES OF CONVERGENCE SPACES

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Abstract. In this paper, we will show some relations between decomposition series \( \{ \pi^\alpha q : \alpha \text{ is an ordinal} \} \) and topological series \( \{ \tau_\alpha q : \alpha \text{ is an ordinal} \} \) for a convergence structure \( q \) and the formula \( \pi^\beta(\tau_\alpha q) = \pi^{\omega^\alpha \beta} q \), where \( \omega \) is the first limit ordinal and \( \alpha \) and \( \beta(\geq 1) \) are ordinals.

I. Introduction and Preliminaries

A convergence structure \( q \) on a set \( X \) defined by [1] in 1964 is a function from the set \( F(X) \) of all filters on \( X \) into the set \( P(X) \) of all subsets of \( X \), satisfying the following conditions:

1. \( x \in q(\dot{x}) \) for all \( x \in X \);
2. \( F \leq G \) implies \( q(F) \subseteq q(G) \);
3. \( x \in q(F) \) implies \( x \in q(F \cap \dot{x}) \),

where \( \dot{x} \) denotes the principal ultrafilter containing \( \{ x \} \); \( F \) and \( G \) are in \( F(X) \). Then the pair \( (X, q) \) is called a convergence space. If \( x \in q(F) \), then we say that \( F \) \( q \)-converges to \( x \). The filter \( \mathcal{V}_q(x) \) obtained by intersecting all filters which \( q \)-converge to \( x \) is called the \( q \)-neighborhood filter at \( x \). If \( \mathcal{V}_q(x) \) \( q \)-converges to \( x \) for each \( x \in X \), then \( q \) is said to be pretopological and the pair \( (X, q) \) is called a pretopological convergence space.

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Let \( C(X) \) be the set of all convergence structures on \( X \), partially ordered as follows:

\[
q_1 \leq q_2 \iff q_2(\mathcal{F}) \subseteq q_1(\mathcal{F}) \text{ for all } \mathcal{F} \in \mathcal{F}(X).
\]

If \( q_1 \leq q_2 \), then we say that \( q_1 \) is coarser than \( q_2 \), and \( q_2 \) is finer than \( q_1 \). By [2], we know that if \( q_1 \) is pretopological, then

\[
q_1 \leq q_2 \iff \mathcal{V}_{q_1}(x) \subseteq \mathcal{V}_{q_2}(x) \text{ for all } x \in X.
\]

For any \( q \in C(X) \), we define a related convergence structure \( \pi(q) \), as follows:

\[
x \in \pi(q)(\mathcal{F}) \iff \mathcal{V}_q(x) \subseteq \mathcal{F}.
\]

In this case, \( \pi(q) \) is called the pretopological modification of \( q \).

In 1973, Kent and Richardson [3] introduced the associated decomposition series \( \{ \pi^\alpha q : \alpha \text{ is an ordinal} \} \) defined by

\[
\pi^\alpha q(\mathcal{F}) \xrightarrow{q} x \iff \mathcal{V}_q^\alpha(x) \subseteq \mathcal{F}, \text{ for each } \mathcal{F} \in \mathcal{F}(X),
\]

where

\[
A \in \mathcal{V}_q^\alpha(x) \iff x \in I_q^\alpha(A), \text{ and}
\]

\[
I_q^\alpha(A) = \begin{cases} \quad I_q(I_q^{\alpha-1}(A)), & \text{if } \alpha - 1 \text{ exists,} \\ \cap_{\beta < \alpha} I_q^\beta(A), & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}
\]

In 1996, Park [4] studied the \( n \)-th pretopological modification \( \pi^n q \) and quotient map for a convergence space \( q \).

In 1999, for a convergence space \((X, q)\) with a second convergence structure \( p \), Wilde [5] introduced that \((X, q)\) is “\( p \)-topological” iff \( \mathcal{F} \xrightarrow{q} x \) implies \( \mathcal{V}_p(\mathcal{F}) \xrightarrow{q} x \). Also they showed that there is a finest \( p \)-topological convergence structure \( \tau_p q \) on \( X \) coarser than \( q \) and \( \mathcal{F} \xrightarrow{\tau_p q} x \) iff there exist \( \mathcal{G} \xrightarrow{q} x \), such that \( \mathcal{F} \geq \mathcal{V}_p^n(\mathcal{G}) \), for some \( n \in \mathbb{N} \). Furthermore, they induced the topological series for \( q \), the descending ordinal sequence \( \{ \tau_\alpha q : \alpha \text{ is an ordinal} \} \) defined recursively on \( X \) as follows:
Decomposition Series and Topological Series

\[ \tau_0 q = q \]
\[ \tau_1 q : \mathcal{F} \xrightarrow{\tau_1 q} x \iff \exists \mathcal{G} \xrightarrow{q} x \text{ and } n \in N \text{ such that } \mathcal{F} \geq \mathcal{V}^n_q(\mathcal{G}) \]
\[ \tau_2 q : \mathcal{F} \xrightarrow{\tau_2 q} x \iff \exists \mathcal{G} \xrightarrow{q} x \text{ and } n \in N \text{ such that } \mathcal{F} \geq \mathcal{V}^n_{\tau_1 q}(\mathcal{G}) \]
\[ \tau_3 q : \mathcal{F} \xrightarrow{\tau_3 q} x \iff \exists \mathcal{G} \xrightarrow{q} x \text{ and } n \in N \text{ such that } \mathcal{F} \geq \mathcal{V}^n_{\tau_2 q}(\mathcal{G}) \]
\[ \vdots \]
\[ \tau_\alpha q : \mathcal{F} \xrightarrow{\tau_\alpha q} x \iff \exists \mathcal{G} \xrightarrow{q} x, \text{ } n \in N \text{ and } \beta < \alpha \text{ such that } \mathcal{F} \geq \mathcal{V}^n_{\tau_\beta q}(\mathcal{G}). \]

In this paper, we will show some relations between decomposition series \( \{\pi^\alpha q : \alpha \text{ is an ordinal} \} \) and topological series \( \{\tau_\alpha q : \alpha \text{ is an ordinal} \} \) for a convergence structure \( q \) and the formula \( \pi^\beta(\tau_\alpha q) = \pi^{\omega^{\alpha \beta}} q \), where \( \omega \) is the first limit ordinal and \( \alpha \) and \( \beta (\geq 1) \) are ordinals.

2. Decomposition Series, the Neighborhood and Interior Filter of a Filter

We shall summarize some results from [3] and other sources using more modern notation and terminology. We are mainly interested in comparing properties of decomposition series with those of the topological series, which will be introduced in [5].

Let \( (X, q) \) be a convergence space. For \( A \subseteq X \), we recall that \( I^0_q(A) = A, I^1_q = I_q(A) = \{ x : A \in V_q(x) \} \)

Given an ordinal number \( \alpha \geq 1 \), let \( I^\alpha_q \) and \( cl^\alpha_q \) denote the \( \alpha \)th iterations of interior operator and closure operator for \( q \), respectively. For \( A \subseteq X \), we inductively define:

\[ I^\alpha_q(A) = \begin{cases} 
I_q(I_q^{\alpha-1}(A)), & \text{if } \alpha - 1 \text{ exists}, \\
\cap_{\beta < \alpha} I_q^\beta(A), & \text{if } \alpha \text{ is a limit ordinal.} 
\end{cases} \]

\[ cl^\alpha_q(A) = \begin{cases} 
cl_q(cl_q^{\alpha-1}(A)), & \text{if } \alpha - 1 \text{ exists}, \\
\cup_{\beta < \alpha} (cl_q^\beta(A)), & \text{if } \alpha \text{ is a limit ordinal.} 
\end{cases} \]
PROPOSITION 2.1. ([5]). For every ordinal \( \alpha \) and \( A \subseteq X \), \( X \setminus cl^\alpha_q(A) = I^\alpha_q(X \setminus A) \).

If \((X, q)\) is a convergence space and \( \alpha \geq 1 \), let \( \pi^\alpha q \) be the pre-topology on \( X \) whose neighborhood filter is \( \mathcal{V}^\alpha_q(x) \), that is, \( \mathcal{V}_{\pi^\alpha q}(x) = \mathcal{V}^\alpha_q(x) \), where \( A \in \mathcal{V}^\alpha_q(x) \iff x \in I^\alpha_q(A) \). Since \( \beta < \alpha \) implies \( I^\alpha_q(A) \subseteq I^\beta_q(A) \), it follows that \( \mathcal{V}^\alpha_q(x) \subseteq \mathcal{V}^\beta_q(x) \), and consequently \( \pi^\alpha q \leq \pi^\beta q \).

**Definition 2.2.** ([3], [5]). The descending chain \( \{ \pi^\alpha q : \alpha \geq 1 \} \) of pretopologies on \( X \) is called the decomposition series of \((X, q)\).

Clearly \( \pi^1 q = \pi q \) is the pretopological modification of \( q \), which is the finest pretopological convergence structure on \( X \) coarser than \( q \).

**Definition 2.3.** ([5]). For any ordinal \( \alpha, p \in C(X) \) and \( G \in F(X) \), we define the neighborhood filter \( \mathcal{V}_p(G) \) and the interior filter \( I_p(G) \) of \( G \), respectively, as follows:

\[
\mathcal{V}_p^1(G) = \mathcal{V}_p(G), \quad \mathcal{V}_p^\alpha(G) = \{ A \subseteq X : I^\alpha_p(A) \in G \}.
\]

\[
I_p^1(G) = I_p(G), \quad I_p^\alpha(G) = \{ I^\alpha_p(G) : G \in \mathcal{V}_p(G) \} \quad \text{if} \quad I_p(G) \neq \emptyset, \quad \forall G \in \mathcal{G},
\]

where \([\mathcal{B}] \) means the filter generated by \( \mathcal{B} \) if \( \mathcal{B} \) is a filter base.

Then we know that if \( \alpha < \beta \), then \( \mathcal{V}_p^\beta(G) \leq \mathcal{V}_p^\alpha(G) \leq G \leq I_p^\alpha(G) \leq I_p^\beta(G) \).

**PROPOSITION 2.4.** For any ordinals \( \alpha, \beta, x \in X \) and \( A \subseteq X \),

1. \( I^\alpha q^+\beta(A) = I^\beta_q(I^\alpha q(A)) \).
2. \( \mathcal{V}^\alpha q^+\beta(x) = \mathcal{V}^\beta q(I^\alpha q(x)) \).

**Proof.** (1) Let \( \alpha \) be a fixed ordinal. We use transfinite induction on \( \beta \). If \( \beta = 1 \), \( I^\alpha q^+1 = I_q(I^\alpha q(A)) \) follows by definition. Next, let \( \beta \) be any arbitrary ordinal.

Case 1. Assume that there exists \( \bar{\beta} \) such that \( \bar{\beta} + 1 = \beta \). By the induction hypothesis, \( I^\alpha q^+\bar{\beta}(A) = I^\bar{\beta}_q(I^\alpha q(A)) \), and so \( I^\alpha q^+\beta(A) = I^\alpha q^+\bar{\beta}+1(A) = I_q(I^\alpha q^+\bar{\beta}(A)) = I_q(I^\beta_q(I^\alpha q(A))) = I^\beta_q(I^\alpha q(A)) \).

Case 2. Assume that \( \beta \) is a limit ordinal. \( I^\alpha q^+\beta(A) = \cap_{\gamma < \beta} I^\alpha q^+\gamma(A) = \cap_{\gamma < \beta} I^\gamma_q(I^\alpha q(A)) = I^\beta_q(I^\alpha q(A)) \).
(2) \( A \in \mathcal{V}_q^{\alpha+\beta}(x) \iff x \in I_q^{\alpha+\beta}(A) \iff x \in I_q^\beta(I_q^\alpha(A)) \iff I_q^\alpha(A) \in \mathcal{V}_q^\beta(x) \iff A \in \mathcal{V}_q^\alpha(\mathcal{V}_q^\beta(x)). \) 

**Corollary 2.5.** For any ordinals \( \alpha, \beta, \) and \( F \in F(X), \)

1. \( I_q^{\alpha+\beta}(F) = I_q^\beta(I_q^\alpha(F)) \) if these are filters.
2. \( \mathcal{V}_q^{\alpha+\beta}(F) = \mathcal{V}_q^\alpha(\mathcal{V}_q^\beta(F)). \)

### 3. \( p \)-Topological Convergence Spaces

In this section, we will summation some propositions about \( p \)-topological convergence space of [5] and [6], and change two propositions, which are the following Theorem 3.4 and 3.7.

Henceforth \((X, q)\) means a convergence space equipped with a second convergence structure \( p. \)

**Definition 3.1.** ([5]). A convergence space \((X, q)\) is \( p \)-topological iff \( F \overset{q}{\rightarrow} x \) implies that there exists a \( G \overset{q}{\rightarrow} x \) such that \( F \geq I_p(G). \)

**Proposition 3.2.** ([5]). \((X, q)\) is \( p \)-topological, iff \( F \overset{q}{\rightarrow} x = \Rightarrow \mathcal{V}_p(F) \overset{q}{\rightarrow} x. \)

**Proposition 3.3.** ([5]). Let \((X, q)\) be a pretopological convergence. Then \((X, q)\) is \( p \)-topological iff \( \mathcal{V}_q(x) = I_p(\mathcal{V}_q(x)) \).

**Proof.** \(( \Rightarrow )\) Since \( \mathcal{V}_q(x) \overset{q}{\rightarrow} x \) and \((X, q)\) is \( p \)-topological, there exists \( G \overset{q}{\rightarrow} x \) such that \( \mathcal{V}_q(x) \geq I_p(G). \) Then \( G \geq \mathcal{V}_q(x) \), so \( G \geq I_p(G) \). This implies \( G = \mathcal{V}_q(x) = I_p(G) = I_p(\mathcal{V}_q(x)) \)

\(( \Leftarrow )\) Let \( F \overset{q}{\rightarrow} x. \) Then \( F \geq \mathcal{V}_q(x) = I_p(\mathcal{V}_q(x)). \) Thus, \((X, q)\) is \( p \)-topological, since \( \mathcal{V}_q(x) \overset{q}{\rightarrow} x. \)

**Theorem 3.4.** If \((X, q)\) is a pretopological and \( p \)-topological, then \( q \leq \pi^\omega p. \)
Proof. Since $(X, q)$ is a pretopological and $p$-topological, $V_q(x) = I_p(V_q(x))$.

Claim: $V_q(x) \leq V_\omega^p(x)$. Let $V \in V_q(x)$. Then $I_p(V) \subseteq I_p(V_q(x)) = V_q(x)$. By Induction, $I_p^n(V) \subseteq V_q(x)$ for all $n \in \mathbb{N}$, so $x \in I_p^n(V)$ for all $n \in \mathbb{N}$. Thus $x \in \cap_{n<\omega} I_p^n(V) = I_\omega^p(V)$, and hence $V \subseteq V_\omega^p(x)$. Thus the Claim is proved.

From $V_\omega^p(x) = V_\pi^\omega(x)$, we obtain $q \leq \pi^\omega p$. □

Proposition 3.5. ([5]). Let $p$ and $q$ be topological. Then $(X, q)$ is $p$-topological iff $q \leq \pi^\omega p$.

Proof. Since $q$ is topological, $V_q(x)$ has a filter base of $q$-open sets.

($\implies$) Since $(X, q)$ is $p$-topological and topological, by Theorem 3.4, $q \leq \pi^\omega p = p$.

($\impliedby$) Let $q \leq p$. Then $I_q(A) \subseteq I_p(A) \subseteq A$. This implies that each $q$-open set is $p$-open, so $I_p(V_q(x)) = V_q(x)$, by Proposition 3.3. $(X, q)$ is $p$-topological. □

Proposition 3.6. ([5]). If $(X, q)$ is $p$-topological and $p < p'$, then $(X, q)$ is $p'$-topological.

Proof. It follows from $p < p'$ implies $I_p(\mathcal{G}) \supseteq I_p'(\mathcal{G})$. □

Note that for $q \in C(X)$, $\tau_q = \{ A \subseteq X : I_q(A) = A \}$ is a topology on $X$ and $\tau_q$ is the convergence structure defined by

$$\tau_q(\mathcal{F}) \overset{q}{\rightsquigarrow} x \iff V_{\tau_q}(x) \subseteq \mathcal{F}, \text{ for each } \mathcal{F} \in F(X),$$

where $V_{\tau_q}(x)$ is the $\tau_q$-neighborhood filter at $x \in X$. Then $\tau_q$ is the finest topological convergence structure on $X$ coarser than $q$.([5]).

Now, we obtain the following theorem, which is different from Corollary 2.4 of [6].

Theorem 3.7. If $(X, q)$ is $p$-topological, then:

1. $(X, \pi q)$ is $p$-topological and $\tau_q \leq \pi q \leq \pi^\omega p$.
2. $(X, \tau q)$ is $p$-topological.
Proof. (1) Let \( F \rightarrow x \); then there exists a \( G \rightarrow x \) such that \( F \geq I_p(G) \geq I_p(V_q(x)) \). This holds for every \( F \rightarrow x \), so

\[
V_{\pi q}(x) = V_q(x) = \bigcap \{ F \in F(X) : F \rightarrow x \} \geq I_p(V_q(x)) = I_p(V_{\pi q}(x)).
\]

Thus \((X, \pi q)\) is \( p \)-topological, so the first part is proved.

It is clear that \( \tau q \leq \pi q \). Since \((X, \pi q)\) is \( p \)-topological and pre-topological, by Theorem 3.4, \( \pi q \leq \pi^\omega p \).

(2) Since \((X, \tau q)\) is \( \tau q \)-topological and \( \tau q \leq \pi q \leq \pi^\omega p \leq p \), by Proposition 3.6, \((X, \tau q)\) is \( p \)-topological. \( \square \)

Definition 3.8 For \( q, p \in C(X) \), \( \tau_p q \) is defined by:

\[
F \rightarrow x \iff \exists G \rightarrow x \text{ and } n \in N \text{ such that } F \geq V^n_p(G).
\]

Proposition 3.9. For \( q, p \in C(X) \), \((X, \tau_p q)\) is \( p \)-topological.

Proof. Let \( F \rightarrow x \). Then there exists \( G \rightarrow x \) and \( n \in N \) such that \( F \geq V^n_p(G) \), so \( V_p(F) \geq V_p(V^n_p(G)) = V^n+1_p(G) \), [5]. Thus \( V_p(F) \rightarrow x \). This means \((X, \tau_p q)\) is \( p \)-topological. \( \square \)

4. Relations between Decomposition Series and Topological Series of Convergence Spaces

In this section, we will remind “topological series” defined by [5] and show relations between decomposition series and supratopological series, the formul\( \pi^\beta(\tau_\alpha q) = \pi^{\omega^\alpha \beta} q \), where \( \omega \) is the first limit ordinal and \( \alpha \) and \( \beta(\geq 1) \) are ordinals.

Let \( q \in C(X) \) and \( \alpha \geq 0 \) ordinal number. The topological series for \( q \) is the descending ordinal sequence \( \{\tau_\alpha q\} \) defined recursively on \( X \) as follows:

\[
\begin{align*}
\tau_0 q &= q \\
\tau_1 q : F \rightarrow x &\iff \exists G \rightarrow x \text{ and } n \in N \text{ such that } F \geq V^n_q(G)
\end{align*}
\]
τ_2q : F \xrightarrow{\tau_2q} x \iff \exists G \xrightarrow{q} x \text{ and } n \in N \text{ such that } F \geq V_n(\tau_1q)(G)

τ_3q : F \xrightarrow{\tau_3q} x \iff \exists G \xrightarrow{q} x \text{ and } n \in N \text{ such that } F \geq V_n(\tau_2q)(G)

\vdots

τ_\alpha q : F \xrightarrow{\tau_\alpha q} x \iff \exists G \xrightarrow{q} x \text{ and } n \in N \text{ and } \beta < \alpha \text{ such that } F \geq V_n(\tau_\beta q)(G),

where we know that \( \tau_1q = \tau_q q, \tau_2q = \tau_{\tau_1q}q = \tau_{\tau_qq}q, \ldots \), etc.

Also, we know that if there exists \( \alpha' \) such that \( \alpha = \alpha' + 1 \), then

F \xrightarrow{\tau_{\alpha'}q} x \iff \exists G \xrightarrow{q} x \text{ and } n \in N \text{ such that } F \geq V_n(\tau_{\alpha}q)(G),

**Proposition 4.1.** ([5]). For \( q \in C(X) \), there exists \( \tilde{q} \) which is the finest \( q \)-topological convergence structure on \( X \), and \( F \xrightarrow{\tilde{q}} x \iff F \geq V^n_q(x) \) for some \( n \in N \).

**Lemma 4.2.** If \( G \xrightarrow{q} x \), then \( V^{n+1}_q(x) \leq V^n_q(G) \).

**Proof.** A \( \in V^{n+1}_q(x) \implies x \in I^{n+1}_q(A) \implies x \in I_q(I^n_q(A)) \implies I^n_q(A) \in V_q(x) \implies I^n_q(A) \in G \), since \( G \xrightarrow{q} x \implies G \geq V_q(x) \). Thus \( A \in V^n_q(G) \). \( \square \)

**Proposition 4.3.** \( \tilde{q} = \tau_1q \).

**Proof.** We have already known \( \tilde{q} \geq \tau_1q \), so it remain to show \( \tau_1q \geq \tilde{q} \).

Let \( F \xrightarrow{\tau_1q} x \). Then there exists \( G \xrightarrow{q} x \) and \( n \in N \) such that \( F \geq V^n_q(G) \).

By the above Lemma, \( F \geq V^n_q(G) \geq V^{n+1}_q(x) \), so \( F \xrightarrow{\tilde{q}} x \). \( \square \)

**Proposition 4.4.** (1) \( q \geq \pi^n q \geq \tilde{q} \geq \pi^\omega q \). (2) \( \pi(\tau_1q) = \pi^\omega q \).

**Proof.** (1) It is clear that \( q \geq \pi^n q \). Let \( n \in N \) and \( F \in F(X) \). Then \( F \xrightarrow{\pi^n q} x \iff F \geq V^n_q(x) \implies F \xrightarrow{\tilde{q}} x \), since \( x \xrightarrow{\tilde{q}} x \). Thus, \( \pi^n q \geq \tilde{q} \) for each \( n \in N \).
Also, $\mathcal{F} \xrightarrow{\bar{q}} x \iff \exists n \in N$ such that $\mathcal{F} \geq V^m_q(x) \geq \cap_{m<\omega} V^m_q(x) = V^\omega_q(x) = V^\omega_{\pi_q}(x) = V^\omega_{\pi(\tau_1 q)}(x) \iff \mathcal{F} \xrightarrow{\pi_q} x$.

(2) Since $\bar{q} = \tau_1 q$, by (1), $\pi(\tau_1 q) \geq \pi(\pi_q) = \pi_q$. While, by Theorem 3.7, $\pi(\tau_1 q) \leq \pi_q$, since $\tau_1 q$ is a $q$-topological. Thus, $\pi(\tau_1 q) = \pi_q$.

We know that for $q \in C(X)$, the first term in the topological series for $q$ is $\tau_1 q = \bar{q}$. $\tau_1 q$ is the finest topological convergence structure on $X$ and also the lower $q$-topological modification of $q$, since $\tau_1 q = \bar{q} \leq \pi_q \leq q$. Note that $q$ has no upper $q$-topological modification unless $q$ is a topology. We next show that that $\tau_2 q$ is related to $\tau_1 q$ exactly as $\tau_1 q$ is related to $q$. Note that the lower $\tau_1 q$-topological modification of $\tau_1 q$ is $\tilde{\tau}_1 q$ defined by:

$\mathcal{F} \xrightarrow{\tilde{\tau}_1 q} x \iff \exists G \xrightarrow{\tau_1 q} x$ and $n \in N$ such that $\mathcal{F} \geq V^n_{\tau_1 q}(G)$.

**Proposition 4.5.** For any $q \in C(X)$, $\tau_2 q = \tilde{\tau}_1 q$.

**Proof.** $\mathcal{F} \xrightarrow{\tau_2 q} x \implies \exists G \xrightarrow{q} x$ and $n \in N$ such that $\mathcal{F} \geq V^n_{\tau_1 q}(G)$. But $G \xrightarrow{\tau_1 q} x$ since $\tau_1 q \leq q$. Thus $\mathcal{F} \xrightarrow{\tilde{\tau}_1 q} x$.

Conversely, $\mathcal{F} \xrightarrow{\tilde{\tau}_1 q} x \implies \exists G \xrightarrow{\tau_1 q} x$ and $n \in N$ such that $\mathcal{F} \geq V^n_{\tau_1 q}(G)$. Also, $G \xrightarrow{\tau_1 q} x \implies \exists H \xrightarrow{q} x$ and $m \in N$ such that $G \geq V^m_q(H)$. Thus $\mathcal{F} \geq V^n_{\tau_1 q}(V^m_q(H)) \geq V^n_{\tau_1 q}(V^m_{\tau_1 q}(H)) = V^{n+m}_{\tau_1 q}(H)$. Thus $\mathcal{F} \xrightarrow{\tau_2 q} x$. □

**Proposition 4.6.** $\pi(\tau_1 q) = \pi_q$ and $\pi(\tau_2 q) = \pi_q(\tau_1 q)$.

**Proof.** The first equality follows from the Proposition 4.4. The second equality follows from $\pi(\tau_2 q) = \pi(\tilde{\tau}_1 q) = \pi_q(\tau_1 q)$. □

**Proposition 4.7.** If $\alpha$ is a limit ordinal, $\mathcal{V}_q^\alpha(x) = \cap_{\beta<\alpha} \mathcal{V}_q^\beta(x)$. 
\[
\text{Proof. } A \in \mathcal{V}_q^\alpha(x) \iff x \in I_q^\alpha(A) = \cap_{\beta < \alpha} I_q^\beta(A) \iff x \in I_q^\beta(A), \forall \beta < \alpha \iff A \in \mathcal{V}_q^\beta(x), \forall \beta < \alpha \iff A \in \cap_{\beta < \alpha} \mathcal{V}_q^\beta(x). \quad \blacksquare
\]

**Proposition 4.8.** \( \mathcal{V}_{\tau_n q}(x) = \mathcal{V}_q^{\omega^n}(x) \) and \( \mathcal{V}_{\tau\omega q}(x) = \mathcal{V}_q^{\omega^\omega}(x) \) for all \( x \in X \).

**Proof.** As we showed in Proposition 4.6, \( \pi(\tau_2 q) = \pi^\omega(\tau_1 q) \). Thus for any \( x \in X \), \( \mathcal{V}_{\tau_2 q}(x) = \mathcal{V}^\omega_{\tau_1 q}(x) \). Also, by Proposition 4.4, \( \mathcal{V}_{\tau_1 q}(x) = \mathcal{V}^\omega_q(x) \). By Corollary 2.5, \( \mathcal{V}^2_{\tau_1 q}(x) = \mathcal{V}_{\tau_1 q}(\mathcal{V}^\omega_{\tau_1 q}(x)) = \mathcal{V}_q(\mathcal{V}^\omega_q(x)) = \mathcal{V}^{\omega^2}_q(x) \). Similarly, \( \mathcal{V}^n_{\tau_1 q}(x) = \mathcal{V}^{\omega^n}_q(x) \). Thus \( \mathcal{V}^\omega_{\tau_1 q}(x) = \cap_{n<\omega} \mathcal{V}^{\omega^n}_q(x) = \mathcal{V}^{\omega^2}_q(x) \).

Expanding the reasoning of Proposition 4.6, we have \( \mathcal{V}_{\tau_3 q}(x) = \mathcal{V}^\omega_{\tau_2 q}(x) \), for all \( x \in X \), since \( \pi(\tau_3 q) = \pi^\omega(\tau_2 q) \). \( \mathcal{V}^2_{\tau_2 q}(x) = \mathcal{V}_{\tau_2 q}(\mathcal{V}^\omega_{\tau_2 q}(x)) = \mathcal{V}^{\omega^2}_q(x) \). Similarly, \( \mathcal{V}^n_{\tau_2 q}(x) = \mathcal{V}^{\omega^n}_q(x) \), so \( \mathcal{V}_{\tau_3 q}(x) = \mathcal{V}^{\omega^3}_q(x) \). Likewise, we obtain \( \mathcal{V}^n_{\tau_{\omega q}}(x) = \mathcal{V}^{\omega^n}_q(x) \). This implies that \( \mathcal{V}_{\tau\omega q}(x) = \mathcal{V}^\omega_q(x) \). \( \quad \blacksquare \)

For \( q \in C(X) \) and any ordinal \( \alpha \), let \( \tau_0 q \) and \( \sigma_0 q \) be defined inductively by \( \tau_0 q = \sigma_0 q \) and:

\[
\mathcal{F} \xrightarrow{\tau_0 q} x \iff \exists \mathcal{G} \xrightarrow{\tau_1 q} x, \text{ } n \in N \text{ and } \beta < \alpha \text{ such that } \mathcal{F} \geq V^n_{\sigma_\beta q}(\mathcal{G}),
\]

\[
\mathcal{F} \xrightarrow{\sigma_0 q} x \iff \exists \mathcal{G} \xrightarrow{\sigma_\beta q} x, \text{ } n \in N \text{ and } \beta < \alpha \text{ such that } \mathcal{F} \geq V^n_{\sigma_\beta q}(\mathcal{G}),
\]

Note that \( \tau_1 q = \sigma_1 q \) is the lower \( q \)-topological modification of \( q \). If \( \alpha + 1 \) is any non-limit ordinal, \( \sigma_{\alpha+1} q = \tau_1(\sigma_\alpha q) \); in other words, \( \sigma_{\alpha+1} q \) is the lower \( \sigma_\alpha q \)-topological modification of \( \sigma_\alpha q \). If \( \alpha \) is a limit ordinal, \( \sigma_\alpha q = \inf \{ \sigma_\beta q : \beta < \alpha \} \). Our first goal is to prove \( \sigma_\alpha q = \tau_\alpha q \) for every ordinal \( \alpha \).

**Proposition 4.9.** For any ordinal \( \alpha \), \( \tau_\alpha q \geq \sigma_\alpha q \).
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Proof. Assume that \( \tau_\beta q \geq \sigma_\alpha q \) for every ordinal \( \beta < \alpha \). Then
\[ F \xrightarrow{\tau_\alpha q} x \implies \exists G \xrightarrow{q} x \text{ and } \beta < \alpha \text{ such that } F \geq V^n_{\tau_\beta q}(G) \geq V^n_{\sigma_\alpha q}(G). \]

Also, since \( G \xrightarrow{q} x, G \xrightarrow{\sigma_\alpha q} x \), thus \( F \xrightarrow{\sigma_\alpha q} x \). \( \square \)

**Proposition 4.10.** For any ordinal \( \alpha \), \( \tau_\alpha q = \sigma_\alpha q \).

Proof. The result is known for \( \alpha = 1 \). Assume the equality holds for \( \beta < \alpha \). By Proposition 4.9, it remains to show that \( F \xrightarrow{\sigma_\alpha q} x \implies F \xrightarrow{\tau_\alpha q} x \).

Case 1. \( \exists \alpha' \) such that \( \alpha = \alpha' + 1 \). Let \( F \xrightarrow{\sigma_{\alpha'} q} x \). Then there exists \( F \xrightarrow{\sigma_{\alpha'} q} x \) and \( n \in \mathbb{N} \) such that \( F \geq V^n_{\sigma_{\alpha'} q}(G) = V^n_{\tau_{\alpha'} q}(G) \). Also, by induction hypothesis, \( G \xrightarrow{q} x \), so there exists \( \mathcal{H} \xrightarrow{q} x, \beta < \alpha' \) and \( m \in \mathbb{N} \) such that \( G \geq V^m_{\tau_\beta q}(H) \). Thus, \( F \geq V^n_{\tau_{\alpha'} q}(G) \geq V^m_{\tau_{\alpha'} q}(V^n_{\tau_{\alpha'} q}(H)) \geq V^{n+m}_{\tau_{\alpha'} q}(H) \), and hence \( F \xrightarrow{\tau_\alpha q} x \).

Case 2. \( \alpha \) is a limit ordinal. Then by induction hypothesis, \( \tau_\beta q = \sigma_\beta q \) for \( \beta < \alpha \), so \( \sigma_\alpha q = \inf\{\sigma_\beta q : \beta < \alpha\} = \inf\{\tau_\beta q : \beta < \alpha\} = \tau_\alpha q \). \( \square \)

**Proposition 4.11.** For any ordinal \( \alpha \), \( \tau_1(\tau_\alpha q) = \tau_{\alpha+1} q \). Thus \( V_{\tau_{\alpha+1} q}(x) = V^\omega_{\tau_\alpha q}(x) \) for all \( x \in X \).

Proof. The first assertion follows by Proposition 4.10 and the note preceding Proposition 4.9. The second follows Proposition 4.6, since \( \pi(\tau_1 p) = \pi^\omega p \) holds for any convergence structure \( p \), letting \( p = \tau_\alpha q \). \( \square \)

**Proposition 4.12.** For any ordinal \( \alpha \) and \( x \in X \), \( V_{\tau_\alpha q}(x) = V_{\tau_q}^\alpha(x) \).

Proof. We will use induction on \( \alpha \). For \( \alpha = 1 \), the result follows by Proposition 4.11. Assume the equality holds for every \( \beta < \alpha \).

Case 1. Assume that there exists \( \alpha' \) such that \( \alpha = \alpha' + 1 \). then by Proposition 4.11, \( V_{\tau_{\alpha'} q}(x) = V_{\tau_{\alpha'} q}^\omega(x) \), where by induction hypothesis,
\( \forall_{\alpha', q}(x) = \forall_q^{\omega^{\alpha'}}(x) \). Thus \( \forall_{\alpha', q}^2(x) = \forall_{\alpha', q}^1(\forall_{\alpha', q}(x)) = \forall_q^{\omega^{\alpha'}2}(x) \), and similarly \( \forall_{\alpha', q}^n(x) = \forall_q^{\omega^{\alpha'}n}(x) \). Thus \( \forall_{\alpha', q}(x) = \forall_{\alpha', q}^{\omega}(x) = \bigcap_{n<\omega} \forall_{\alpha', q}^n(x) = \bigwedge_{n<\omega} \forall_q^{\omega^{\alpha'}n}(x) = \forall_q^{\omega^{\alpha'}\omega}(x) = \forall_q^{\omega^{\alpha'+1}}(x) = \forall_q^{\omega^{\alpha'}}(x) \).

Case 2. Assume that \( \alpha \) is a limit ordinal. By induction hypothesis, \( \forall_{\alpha', q}(x) = \forall_q^{\omega^{\alpha'}}(x) \) for \( \beta < \alpha \). Thus \( \forall_{\alpha', q}(x) = \bigcap_{\beta<\alpha} \forall_q^{\omega^{\beta}}(x) = \forall_q^{\omega^{\alpha'}}(x) \).

Consequently, our last result is the following Theorems.

**Theorem 4.13.** For every ordinal \( \alpha \) and \( \beta \geq 1 \) and every \( x \in X \),

1. \( \forall_{\alpha', q}(x) = \forall_q^{\omega^{\alpha'}}(x) \).
2. \( \pi^\beta(\tau_{\alpha}) = \pi^{\omega^{\alpha'}}q \).

**Proof.** (1) We will use induction on \( \beta \). For \( \beta = 1 \), the result follows by Proposition 4.12. Assume the equality holds for every \( \gamma < \beta \).

Case 1. \( \exists \beta' \) such that \( \beta = \beta' + 1 \). Then by Corollary 2.5, \( \forall_{\alpha', q}^\beta(x) = \forall_{\alpha', q}^{\beta'+1}(x) = \forall_{\alpha', q}^\beta(\forall_{\alpha', q}(x)) = \forall_{\alpha', q}^{\omega^{\alpha'}\beta'}(x) = \forall_q^{\omega^{\alpha'}\beta}(x) \).

Case 2. \( \beta \) is a limit ordinal. By induction hypothesis, \( \forall_{\alpha', q}^{\gamma}(x) = \forall_q^{\omega^{\alpha'}\gamma}(x) \) for \( \gamma < \beta \). Thus \( \forall_{\alpha', q}(x) = \bigcap_{\gamma<\beta} \forall_q^{\gamma\alpha'}(x) = \bigcap_{\gamma<\beta} \forall_q^{\omega^{\alpha'}\gamma}(x) = \forall_q^{\omega^{\alpha'}\beta}(x) \).

(2) By (1), it is clear.

Finally, we define the lengths of decomposition series and topological series of \( q \in C(X) \), \( l_D q \), and \( l_T q \), respectively by:

\[
\begin{align*}
l_D q &= \inf\{ \lambda : \lambda \text{ is an ordinal such that } \pi^{\lambda}q = \pi^{\lambda+1}q \}, \\
l_T q &= \inf\{ \lambda : \lambda \text{ is an ordinal such that } \tau_{\lambda}q = \tau_{\lambda+1}q \}.
\end{align*}
\]

We know that \( l_D q = \inf\{ \lambda : \lambda \text{ is an ordinal s.t. } I_{\lambda}^\alpha(A) = I_{\lambda+1}^\alpha(A), \forall A \subseteq X \} = \inf\{ \lambda : \lambda \text{ is an ordinal such that } \pi^{\lambda}q = q \} \).

**Proposition 4.14.** For \( q \in C(X) \) and an ordinal \( \alpha \),

1. If \( l_T q \leq \alpha \), then \( \tau_{\alpha}q = q \);
2. If \( l_T q \leq \alpha \), then \( l_D q \leq \omega^\alpha \).
Proof. (1) Let $\lambda = l_T q$. Then $\tau_{\lambda} q = \tau_{\lambda + 1} q = \tau q$. Since $\lambda \leq \alpha$, $\tau_{\lambda} q \geq \tau_{\alpha} q \geq \tau q$. Thus $\tau_{\alpha} q = \tau q$.

(2) Since $l_T q \leq \alpha$, $\tau_{\alpha} q = \tau q$. Thus $\pi(\tau_{\alpha} q) = \pi(\tau q)$, so $\pi^{\omega_{\alpha}} q = \tau q$. Finally, $l_D q \leq \omega^\alpha$. \hfill \Box

References


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