COINCIDENCE POINT THEOREMS FOR SINGLE AND MULTI-VALUED CONTRACTIONS

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Abstract. In this paper two coincidence point theorems in complete metric spaces for two pairs of single and multi-valued mappings have been established.

1. Introduction

Let \( (X, d) \) be a metric space and let \( f \) and \( g \) be mappings from \( X \) into itself. In [5], Sessa defined \( f \) and \( g \) to be weakly commuting if \( d(gfx, fgx) \leq d(gx, fx) \) for all \( x \in X \). It can be seen that two commuting mappings are weakly commuting, but the converse is false as shown in the example of [5]. Recently Jungck [1] extended the concept of weak commutativity in the following way.

Let \( f \) and \( g \) be mappings from a metric space \( (X, d) \) into itself. The mappings \( f \) and \( g \) are said to be compatible if

\[
\lim_{n \to \infty} d(fgx_n, gfx_n) = 0
\]

whenever \( \{x_n\} \) is a sequence in \( X \) such that

\[
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = z
\]

for some \( z \) in \( X \). It is obvious that two weakly commuting mappings are compatible, but the converse is not true, as one can see from the examples in [1].

Recently Kaneko [2] and Singh et al. [6] extended the concepts of weak commutativity and compatibility for single valued mappings to the setting of single valued and multi valued mappings, respectively. Now let \( (x, d) \) be a metric space and let \( CB(X) \) denote the family of all non-empty closed and bounded subsets of \( X \). Let \( H \) be the

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Hausdorff metric on \( CB(X) \) and it is defined as

\[
H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\}
\]

for \( A, B \in CB(X) \).

Where \( d(x, A) = \inf_{y \in A} d(X, y) \). It is well known that \( (CB(X), H) \) is a metric space. Further if \( (X, d) \) is complete, then \( (CB(X), H) \) is also complete.

The following lemma has been proved in Nadler [4].

**Lemma 1.1.** Let \( A, B \in CB(X) \) and \( k > 1 \). Then for each \( a \in A \) there exists a point \( b \in B \) such that

\[
d(a, b) \leq kH(A, B).
\]

**Definition 1.2.** Let \( X, d \) be a metric space and let \( f : X \rightarrow X \) and \( S : X \rightarrow CB(X) \) be single valued and multi valued mappings respectively. The mappings \( f \) and \( S \) are said to be weakly commuting if for all \( x \in X, fSx \in CB(X) \) and \( H(Sfx, fSx) \leq d(fx, Sx) \), where \( H \) is the Hausdorff metric defined on \( CB(X) \).

**Definition 1.3.** The mappings \( f \) and \( S \) are said to compatible if

\[
\lim_{n \to \infty} d(fy_n, Sfx_n) = 0
\]

whenever \( \{x_n\} \) and \( \{y_n\} \) are sequence in \( X \) such that

\[
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} y_n = z \quad \text{for some } z \in X,
\]

where \( y_n \in Sx_n \) for \( n = 1, 2, \ldots \).

**Remark 1.4.**

(i) Definition 1.3 is slightly different from Kaneko's [2] definition.

(ii) If \( S \) is a single valued mapping on \( X \) in Definitions 1.2 and 1.3, then Definitions 1.2 and 1.3 become the definitions of weak commutativity and compatibility for single valued mapping.

(iii) If the mappings \( f \) and \( S \) are weakly commuting, then they are compatible, but the converse is not true. In fact, suppose that \( f \) and \( S \) are weakly commuting and let \( \{X_n\} \) and \( \{y_n\} \) be two sequences in \( X \) such that \( y_n \in SX_n \) for \( n = 1, 2, \ldots \) and

\[
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} y_n = z \quad \text{for some } z \in X.
\]

From \( d(fx_n, Sx_n) \leq d(fx_n, y_n) \), it follows that

\[
\lim_{n \to \infty} d(fx_n, Sx_n) = 0.
\]
Thus, $f$ and $S$ are weakly commuting, we have

$$\lim_{n \to \infty} H(Sf y_n, f Sx_n) = 0.$$ 

On the other hand, since $d(f y_n, Sfx_n) \leq H(f Sx_n, Sfx_n)$, we have

$$\lim_{n \to \infty} d(f y_n, Sfx_n) = 0,$$

which means that $f$ and $S$ are compatible.

**Example 1.5.** Let $X = [1, \infty)$ be set with the Euclidean metric $d$ and define $f x = 2x^4 - 1$ and $Sx = [1, x^2]$ for all $x \geq 1$. Note that $f$ and $S$ are continuous and $S(X) = f(X) = X$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in $X$ defined by

$$x_n = y_n = 1 \text{ for } n = 1, 2, \ldots.$$

Then we have

$$\lim_{n \to \infty} f x_n = \lim_{n \to \infty} y_n = 1 \in X,$$

where $y_n \in Sx_n$. On the other hand, we can show that $H(f Sx_n, Sfx_n) = 2(xn^4 - 1)^2 \to 0$ if and only if $x_n \to 1$ as $n \to \infty$ and so, since $d(f y_n, Sfx_n) \leq H(f Sx_n, Sfx_n)$, we have

$$\lim_{n \to \infty} d(f y_n, Sfx_n) = 0.$$

Therefore $f$ and $S$ are compatible, but $f$ and $S$ are not weakly commuting at $x = 2$.

**2. Main Results**

In this section we prove two coincidence point theorems and some particular cases of the same as corollaries.

**Theorem 2.1.** Let $(X, d)$ be a complete metric space. Let $f, g : X \to X$ be a continuous mappings and $S, T : X \to CB(X)$ be $H$ continuous mappings. Suppose $T(X) \subseteq f(X)$, $S(X) \subseteq g(X)$, the pair $S$ and $g$ are compatible mappings and

$$H(Sfx, Tgy) \leq h \max\{d(fx, Sfx), d(gy, Tgy),
\quad d(gy, Sfx), d(fx, Tgy), d(fx, gy)\}$$

for all $x, y \in X$ and $0 < h < 1$. Then $S, f$ and $T, g$ have a unique coincidence point.
Proof. Let \( x_0 \in X \) be any arbitrary element in \( X \). Since \( S(X) \subseteq g(X) \) we have \( Sfx_0 \subseteq g(X) \). This implies that there exists and element \( x_1 \in X \) such that \( gx_1 \in Sfx_0 \). Since \( T(X) \subseteq f(X) \) we have \( Tgx_1 \subseteq f(X) \). Thus there exists \( x_2 \in X \) such that \( fx_2 \in Tgx_1 \) and

\[
d(gx_1, fx_2) \leq \frac{1}{p}H(Sfx_0, Tgx_1) \quad \text{where } P = \sqrt{2h} < 1.
\]

Similarly, there exists \( x_3 \in X \) such that \( gx_3 \in Sfx_2 \) and

\[
d(gx_3, fx_2) \leq \frac{1}{p}H(Sfx_2, Tgx_1).
\]

Now using (1), we have

\[
H(Sfx_0, Tgx_1) \leq h \max\{d(fx_0, Sfx_0), d(gx_1, Tgx_1), d(gx_1, Sfx_0),
\]

\[
d(fx_0, Tgx_1), d(fx_0, gx_1)\}
\]

and so

\[
\frac{1}{p}H(Sfx_0, Tgx_1) \leq \frac{h}{p} \max\{d(fx_0, gx_1), d(gx_1, fx_2), d(gx_1, gx_1),
\]

\[
d(fx_0, fx_2), d(fx_0, gx_1)\}
\]

\[
d(gx_1, fx_2) \leq \frac{h}{p} \max\{d(fx_0, gx_1), d(gx_1, fx_2), d(fx_0, fx_2)\}
\]

\[
\leq \frac{h}{p} \max(d(fx_0, gx_1), d(gx_1, fx_2), d(fx_0, gx_1), +d(gx_1, fx_2))
\]

\[
= \frac{h}{p} [d(fx_0, gx_1) + d(gx_1, fx_2)]
\]

\[
(P - h) d(gx_1, fx_2) \leq h d(fx_0, gx_1)
\]

This implies that

\[
d(gx_1, fx_2) \leq \frac{h}{P - h} d(fx_0, gx_1)
\]

\[
= \sqrt{r} d(fx_0, gx_1) \quad \text{where } 0 < \sqrt{r} = \frac{h}{P - h} < 1.
\]
Also
\[
d(g_3, f_2) \leq \frac{1}{P} H(Sf_2, Tg_1)
\]
\[
\leq \frac{1}{P} \max\{d(f_2, Sf_2), d(g_1, Tg_1), d(g_1, Sf_2),
\]
\[
d(f_2, Tg_1), d(f_2, g_1)\}
\]
\[
\leq \frac{1}{P} \max\{d(f_2, g_3), d(g_1, f_2), d(g_1, g_3),
\]
\[
d(f_2, f_2), d(f_2, g_1)\}\}
\]
\[
\leq \frac{1}{P} \max\{d(f_2, g_3), d(g_1, f_2), d(g_1, f_2) + d(f_2, f_3)\}
\]
and hence
\[
d(g_3, f_2) \leq \frac{1}{P} h\{d(f_2, g_3) + d(g_1, f_2)\}(P - h)d(g_3, f_2)
\]
\[
\leq h d(g_1, f_2)
\]
\[
d(g_3, f_2) \leq \frac{h}{P - h} d(g_1, f_2)
\]
\[
d(g_3, f_2) \leq \sqrt{r} d(g_1, f_2)
\]
\[
\leq \sqrt{r} \sqrt{d(f_0, g_1)} \quad \text{(using 2.2)}
\]
\[
d(g_3, f_2) \leq r d(f_0, g_1)
\]
Continuing in this way, we get a sequence \(\{x_n\}\) in \(X\) such that
\[
gx_{2n+1} \in Sf_{2n} \text{ and } f_{2n} \in Tg_{2n+1} \text{ for all } n \geq 1 \text{ and so}
\]
\[
d(g_{2n+1}, f_{2n}) \leq r^n d(f_0, g_1) \quad \text{for } n \geq 1.
\]
and
\[
d(g_{2n+1}, f_{2n+2}) \leq r^{n+1/2} d(f_0, g_1) \quad \text{for } n \geq 0.
\]
Thus \(\{g_1, f_2, g_3, f_4, \ldots, f_{2n}, g_{2n+1}\}\) is a Cauchy sequence, since \(X\) is complete there is a point \(z \in X\) such that
\[
\lim_{n \to \infty} g_{2n+1} = \lim_{n \to \infty} f_{2n} = z.
\]
Now, we will prove that \(Z\) is a coincidence point of \(f\) and \(S\). For every \(n \geq 0\), we have
\[
(3) \quad d(fg_{2n+1}, Sz) \leq d(fg_{2n+1}, Sf_{2n}) + H(Sf_{2n}, Sz)
\]
It follows from $H$-continuity of $S$ and \( fx_{2n} \to z \) as \( n \to \infty \) that
\[
(4) \quad \lim_{n \to \infty} H(Sfx_{2n}, Sz) = 0
\]
Since \( f \) and \( S \) are compatible mappings and
\[
\lim_{n \to \infty} fx_{2n} = \lim_{n \to \infty} gx_{2n+1} = z.
\]
and \( gx_{2n+1} \in Sfx_{2n} \) we have
\[
(5) \quad \lim_{n \to \infty} d(fgx_{2n+1}, Sfx_{2n}) = 0
\]
Thus from (3), (4) and (5) we get
\[
\lim_{n \to \infty} (fgx_{2n+1}, Sz) = 0
\]
and so
\[
d(fz, Sz) \leq d(fz, fgx_{2n+1}) + d(fgx_{2n+1}, Sz).
\]
Letting \( n \) tends to infinity, it follows that \( d(fz, Sz) = 0 \) this implies that \( fz \in Sz \) since \( Sz \) is closed subset of \( X \) and thus \( z \) is a coincidence point of \( f \) and \( S \). Similarly, we can prove that \( z \) is a coincidence point of \( g \) and \( T \).

To prove the uniqueness of the coincidence point, let \( z \neq y \) be another coincidence point for the pairs \( f, S \) and of \( g, T \). Then \( f(z) = g(z) = z \) and \( f(y) = g(y) = y \). Also \( f(z) \in S(z) \) and \( g(z) \in T(z), f(y) \in S(y) \) and \( g(y) \in T(y) \).

Now, we have
\[
H(Sfz, Tgy) \leq h \max\{d(fz, Sz),
\]
\[
d(gy, Tgy), d(gy, Sfz), d(fz, Tgy), d(fz, gy)\}
\]
and so
\[
H(Sz, Ty) \leq h \max\{d(z, z), d(y, y), d(y, z), d(z, y), d(z, y)\}
\]
Hence
\[
d(y, z) \leq H(z, y) \leq hd(y, z) < d(y, z),
\]
which is a contradiction.

This completes the proof of the theorem.
\[\square\]

Letting \( f = g \) as the identity mapping on \( X \), in the above Theorem 2.1, we have the following corollary, which contains the result of Bose and Mukherjee [7].
**Corollary 2.2.** Let $(X, d)$ be a complete metric space and let $S, T : X \to CB(X)$ be $H$-continuous multi-valued mappings such that

$$H(Sx, Ty) \leq h \max\{d(x, Sx), d(y, y), d(y, Sx), d(x, Ty), d(x, y)\}.$$  

for all $x, y \in X$ and $0 \leq h < 1$, then $S$ and $T$ have a unique common fixed point in $X$.

Putting $f = g$ and $S = T$ in Theorem 2.1, we have the following corollary.

**Corollary 2.3.** Let $(X, d)$ be a complete metric space. Let $f : X \to X$ be a continuous mapping and let $S : X \to CB(X)$ be an $H$-continuous mapping such that $S(X) \subseteq f(X)$ and

$$H(Sfx, Sfy) \leq h \max\{d(fx, Sfx), d(fy, Sfy), d(fx, Sfy), d(fx, fy)\}$$

for all $x, y \in X$ and $0 \leq h < 1$. Then $f$ and $S$ have a unique coincidence point.

Putting $f = g = 1$ and $T = S$ in Theorem 2.1, we have the following corollary, which includes the result of Ciric [9].

**Corollary 2.4.** Let $(X, d)$ be a complete metric space and let $X \to CB$ continuous mapping such that

$$H(Sx, Sy) \leq h \max\{d(x, Sx), d(y, Sy), d(x, Sy), d(y, Sx), d(x, y)\}$$

for all $x, y \in X$ and $0 \leq h < 1$. Then $S$ has a unique fixed point.

**Theorem 2.5.** Let $(X, d)$ be a complete metric space. Let $f, g : X \to X$ be continuous mappings and $S, T : X \to CB(X)$ be $H$-continuous mappings such that $T(X) \subseteq f(X)$ and $S(X) \subseteq g(X)$; the fair $S$ and $g$ are compatible mappings and

$$H^p(Sx, Ty) \leq \max\{ad(fx, gy)d^{p-1}(fx, Sx), ad^{p-1}(gy, Ty)d(fx, gy), ad^{p-1}(gy, Ty)\}$$

for all $x, y \in X$, integer $p \geq 2, 0 < a < 1$ and $c_1, c_2 \geq 0$, then there exists a coincidence point $z$ of $f, S$ and $g, T$. Further, if $0 < c_1 < 1$, then $z$ is unique.
Proof. Let \( x_0 \) be an arbitrary point in \( X \). Since \( Sx_0 \subseteq g(X) \), there exists a point \( x_1 \in X \) such that \( gx_1 \in Sx_0 \) and so there exists a point \( x_2 \in X \) such that \( fx_2 \in Tx_1 \).

Hence by Lemma 1.1, there is \( k = a^{-1/2p} > 1 \) with such that
\[
d(gx_1, fx_2) \leq kH(Sx_0, Tx_1).
\]
Similarly, there exists a point \( x_3 \in X \) and \( gx_3 \in Sx_2 \) such that \( d(gx_3, fx_2) \leq kH(Sx_2, Tx_1) \). Again, there exists a point \( x_4 \in X, fx_4 \in Tx_3 \) such that \( d(gx_3, fx_4) \leq kH(Sx_2, Tx_3) \). Inductively, we can obtain a sequence \( \{x_n\} \) in \( X \) such that for all \( n \geq 0, fx_{2n+2} \in Tx_{2n+1} \) and \( gx_{2n+1} \in Sx_{2n} \) and
\[
d(gx_{2n+1}, fx_{2n+2}) \leq kH(Sx_{2n}, Tx_{2n+1}).
\]

Hence
\[
d^p(gx_{2n+1}, fx_{2n+2})
\]
\[
\leq k^pH^p(Sx_{2n}, Tx_{2n+1})
\]
\[
\leq k^p \max \{ ad(fx_{2n}, gx_{2n+1}), dP^{-1}(fx_{2n}, Sx_{2n}), adP^{-1}(gx_{2n+1}, Tx_{2n+1}),
\]
\[
d(fx_{2n}, gx_{2n+1}), adP^{-1}(gx_{2n+1}, Tx_{2n+1})d(fx_{2n}, Sx_{2n}),
\]
\[
d(gx_{2n+1}, Sx_{2n})[c_1dP^{-1}(fx_{2n}, Tx_{2n+1}) + c_2dP^{-1}(gx_{2n+1}, Tx_{2n+1})]\}
\]
\[
\leq k^p \max \{ ad(fx_{2n}, gx_{2n+1}), dP^{-1}(fx_{2n}, Sx_{2n}), adP^{-1}(gx_{2n+1}, fx_{2n+2}),
\]
\[
d(fx_{2n}, gx_{2n+1}), adP^{-1}(gx_{2n+1}, fx_{2n+2})d(fx_{2n}, Sx_{2n})\} (using 2.1)
\]
\[
d(gx_{2n+1}, Sx_{2n})[c_1dP^{-1}(fx_{2n}, fx_{2n+2}) + c_2dP^{-1}(gx_{2n+1}, fx_{2n+2})]\}
\]
\[
\leq k^p a \max \{ d^p(fx_{2n}, gx_{2n+1}), dP^{-1}(gx_{2n+1}, fx_{2n+2})d(fx_{2n}, gx_{2n+1})
\]
\[
= a^2 \max \{ d^p(fx_{2n}, gx_{2n+1}), dP^{-1}(gx_{2n+1}, fx_{2n+2})d(fx_{2n}, gx_{2n+1})\}
\]
\[
d^p(gx_{2n+1}, fx_{2n+2})
\]
\[
\leq \sqrt{a} \max \{ d^p(fx_{2n}, gx_{2n+1}), dP^{-1}(gx_{2n+1}, fx_{2n+2})d(fx_{2n}, gx_{2n+1})\}
\]

If
\[
\max \{ d^p(fx_{2n}, gx_{2n+1}), dP^{-1}(gx_{2n+1}, fx_{2n+2})d(fx_{2n}, gx_{2n+1})\}
\]
\[
= dP^{-1}(gx_{2n+1}, fx_{2n+2})d(fx_{2n}, gx_{2n+1})
\]
then
\[
d^p(fx_{2n}, gx_{2n+1}) \leq dP^{-1}(gx_{2n+1}, fx_{2n+2})d(fx_{2n}, gx_{2n+1})
\]
and so

\[ d(f_{2n}, g_{2n+1}) \leq d(g_{2n+1}, f_{2n+2}) \]  

Also

\[ d^p(g_{2n+1}, f_{2n+2}) \leq \sqrt{a} d^{-1}(g_{2n+1}, f_{2n+2})d(f_{2n}, g_{2n+1}) \]  

\[ d(g_{2n+1}, f_{2n+2}) \leq \sqrt{a} d(f_{2n}, g_{2n+1}) < d(f_{2n}, g_{2n+1}) \]  

Hence from (7) and (8)

\[ \max\{d^p(f_{2n}, g_{2n+1}), d^{-1}(g_{2n+1}, f_{2n+2}), d(f_{2n}, g_{2n+1})\} = d^p(f_{2n}, g_{2n+1}) \]

Thus

\[ d^p(g_{2n+1}, f_{2n+2}) \leq \sqrt{a} d^p(g_{2n+1}, f_{2n}) \]

and hence

\[ d(g_{2n+1}, f_{2n+2}) \leq \beta d(g_{2n+1}, f_{2n}) \quad \text{for } n \geq 0. \]

where \( \beta = a^{1/2p} < 1. \)

Also

\[ d(g_{2n+1}, f_{2n+2}) \leq \beta d(g_{2n-1}, f_{2n}) \quad \text{for } n \geq 1 \]

\[ \leq \beta^n d(g_1, f_2) \to 0 \quad \text{as } n \to \infty \text{ (since } 0 < \beta < 1). \]

It follows that \( \{g_1, f_2, g_3, f_4, \ldots, g_{2n-1}, f_{2n} \ldots\} \) is a Cauchy sequence in \( X. \)

Since \((X, d)\) is a complete metric space, there is a point \( z \) in \( X \) such that

\[ \lim_{n \to \infty} g_{2n+1} = \lim_{n \to \infty} f_{2n} = z. \]

Now we will prove that \( z \) is a coincidence point of \( f \) and \( S. \) For every \( n \geq 1, \) we have

\[ d(f_{2n+1}, S_z) \leq d(f_{2n+1}, S_{f_{2n}}) + H(s_{f_{2n}}, S_z) \]

It follows from \( H \)-continuity of \( S \) that

\[ \lim_{n \to \infty} H(s_{f_{2n}}, S_z) = 0 \]

Since \( f_{2n} \to z \) as \( n \to \infty. \)

\[ \square \]
Since \( f \) and \( S \) are compatible mappings and
\[
\lim_{n \to \infty} d(fgx_{2n+1}, Sfx_{2n}) = 0
\]
Thus from the identities (9), (10) and (11) we have \( \lim_{n \to \infty} d(fgx_{2n+1}, Sz) = 0 \) and so \( d(fz, Sz) \leq (fz, fgx_{2n+1}) + d(fgx_{2n+1}, Sz) \). Letting \( n \to \infty \), it follows that \( d(fz, Sz) = 0 \). This implies that \( fz \in Sz \), since \( Sz \) is a closed subset of \( X \). Thus \( z \) is a coincidence point of \( f \) and \( S \).

Similarly, we prove that \( z \) is a coincidence point of \( g \) and \( T \).

Suppose \( z \neq y \) is an another coincidence point for the pair \( f, S \) and \( g, T \) then \( fz = gz = z \) and \( fy = gy = y \). This gives that \( f(z) \in S(z), g(z) \in T(z) \) and \( f(y) \in S(y), g(y) \in T(y) \) and so
\[
d^p(z, y) \leq H^p(Sz, Ty)
\]
\[
\leq \max\{ad(fz, gy)d^{p-1}(fz, Sz), ad^{p-1}(gy, Ty)d(fz, gy), ad^{p-1}(gy, Ty)\}
\]
\[
d(fz, Sz), d(gy, Sz)[c_1d^{p-1}(fz, Ty) + c_2d^{p-1}(gy, Ty)]\}
\]
\[
= \max\{ad(z, y), 0, 0, d(z, y), a, 0, d(y, z)\} \{c_1, d^{p-1}(z, y) + c_2x0\}
\]
\[
= c_1d(y, z)d^{p-1}(y, z)
\]
\[
< d^p(y, z) \quad \text{(since } c_1 < 1)\]
which is a contradiction.

Hence \( f, S \) and \( g, T \) have a unique coincidence point.

Allowing \( c_1 = c \) and \( c_2 = 0 \) in Theorem 2.5, we have the following corollary.

**Corollary 2.6** (Duran Turkoglu Orhan Ozer, and Brain Fisher [8]).

Let \((X, d)\) be a complete metric space. Let \( f, g : X \to X \) be continuous mappings and \( S, T : X \to CB(X) \) be \( H \)-continuous mappings such that \( T(X) \subseteq f(X) \) and \( S(X) \subseteq g(X) \), the pair \( S \) and \( g \) are compatible mappings and
\[
H^p(Sx, Ty) \leq \max\{ad(fx, gy)d^{p-1}(fx, Sy), ad(fx, gy)d^{p-1}(gy, Ty),
\]
\[
ad(fx, Sy)d^{p-1}(gy, Ty), cd^{p-1}(fx, Ty)d(gy, Sz)\}
\]
for all \( x, y \in X \), where \( p \geq 2 \) is an integer, \( 0 < a < 1 \) and \( c \leq 0 \). Then there exists a point \( z \in X \), such that \( fx \in Sz \) and \( gz \in Tz \), i.e., \( z \) is a coincidence point of \( f, S \) and of \( g, T \). Further, \( z \) is unique when \( 0 < c < 1 \).
Letting $f = g$ as identity mapping on $X$, in Theorem 2.5, we have the following corollary.

**Corollary 2.7.** Let $(X, d)$ be a complete metric space and let $S, T : X \rightarrow CB(X)$ be $H$-Continuous multi-valued mappings such that
\[
H^P(Sx, Ty) \leq \max\{ad(x, y)d^{P-1}(x, Sx), ad^{P-1}(y, Ty)d(x, y), ad^{P-1}(y, Ty)
\]
\[d(x, Sx), d(y, Sx)]c_1d^{P-1}(x, Ty) + c_2d^{P-1}(y, Ty)\}
\]
for all $x, y \in X$ where $p \geq 2$ is an integer $0 < a < 1$, and $c_1 + c_2 \geq 0$.

Then $S$ and $T$ have a common fixed-point $z$ in $X$. Also $S$ and $T$ have a unique common fixed point $z$ in $X$ when $0 < c_1 < 1$.

Putting $f = g$ and $S = T$ in Theorem 2.5, we have the following corollary.

**Corollary 2.8.** Let $(X, d)$ be a complete metric space, let $f : X \rightarrow X$ be a continuous mapping and let $S : X \rightarrow CB(X)$ be an $H$-continuous mapping such that $S(X) \subseteq f(X)$ and
\[
H^P(Sx, Sy)
\]
\[\leq \max\{ad(f_x, f_y)d^{P-1}(f_x, Sx), ad^{P-1}(f_y, Sy)d(f_x, f_y),
\]
\[ad^{P-1}(f_y, Sy)d(f_x, Sx), d(f_y, Sx)]c_1d^{P-1}(f_x, Sy) + c_2d^{P-1}(f_y, Sy)\}\[
\]
for all $x, y \in X$ where $p \geq 2$ is an integer, $0 < a < 1$ and $c_1 + c_2 \geq 0$.

Then there exists a coincidence point $z$ of $f$ and $S$.

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