A NOTE ON THE AUSTIN’S GROUPOIDS

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Abstract. On a groupoid satisfying the Austin’s identity, every
n-ary linear term is essentially n-ary. That is, if a term has no
variables appearing more than once, then the term depends on
every variable it involves.

1. Introduction

A groupoid is a pair \((G, \cdot)\) of a set \(G\) and a binary operation \(\cdot\)
defined on \(G\). A term or a word in a set \(X = \{x_1, x_2, \cdots\}\) of symbols
is an expression built up from \(X\) using the groupoid operation. We
use the notation \(x^2\) for the term \(xx\). Thus \(x^2x, xx^2\) and \(x^2x^2\) represent
\((xx)x, x(xx)\) and \((xx)(xx)\), respectively.

A term is called \(n\)-ary if it involves \(n\) distinct variables in its expres-
sion, and linear if each variable appears at most once in the expression.
On a groupoid \((G, \cdot)\), an \(n\)-ary term \(f(x_1, x_2, \cdots, x_n)\) defines a map-
ing of \(G^n\) into \(G\) by substitution. A mapping defined by a term in
this way is called a term function. An \(n\)-ary term is called essentially
\(n\)-ary over a groupoid \((G, \cdot)\) if, as a term function, it depends on each
\(x_i\) for \(i = 1, 2, \cdots, n\). That is,

\[ f(a_1, \cdots, a_{i-1}, b, a_{i+1}, \cdots, a_n) \neq f(a_1, \cdots, a_{i-1}, c, a_{i+1}, \cdots, a_n) \]

for some elements \(a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n, b, c\) in \(G\).
By $p_n(G, \cdot)$, we denote the number of all essentially $n$-ary terms over $(G, \cdot)$ for all $n \geq 0$. We say that a groupoid $(G, \cdot)$ is term infinite if $p_n(G, \cdot)$ is infinite for all $n \geq 2$. Of course, term infinite algebras are infinite but not conversely.

A groupoid $(G, \cdot)$ is called nontrivial if $G$ has more than one element, and proper if the basic operation $xy$ is essentially binary. In another word, a groupoid which is neither a left-zero semigroup nor right-semigroup is proper.

The groupoid identity

\[(A) \quad ((y^2y)x)(y^2(y^2y)z) = x\]

is called the Austin’s identity, and a groupoid $(G, \cdot)$ satisfying this identity is called an Austin’s groupoid. Since its appearance in [1], this identity appeared in many papers ([3], [4], [5], [6], [8]), because the identity initiated the research on identities which have no nontrivial finite models.

A nontrivial Austin’s groupoid has the following interesting properties.

**Theorem 1.** ([1], [2]) *Every nontrivial Austin’s groupoid is infinite.*

**Theorem 2.** ([6]) *Every nontrivial Austin’s groupoid is term-infinite.*

In [7], in comparison with the Austin’s identity, it was shown that the identity $((y^2y)x)(y^2z) = x$ is the shortest groupoid identity which has no nontrivial finite models.

In this paper, we show the following theorem.

**Theorem 3.** *On a non-trivial Austin’s groupoid, every n-ary linear term is essentially n-ary for all $n \geq 1$.*
2. Some properties of Austin’s groupoids

An element \( a \) of a groupoid is called \textit{idempotent} if \( a^2 = a \).

For every \( n \geq 1 \), define two special terms \( f_n \) and \( g_n \) by
\[
\begin{align*}
  f_n(x_1, x_2, \ldots, x_n) &= (\cdots((x_1x_2)\cdots)x_{n-1})x_n \\
  g_n(x_1, x_2, \ldots, x_n) &= x_1(x_2(\cdots(x_{n-2}(x_{n-1}x_n))\cdots)).
\end{align*}
\]

With a groupoid \((G, \cdot)\) and an element \( a \) of \( G \), we define a mapping \( T_a : G \rightarrow G \) by \( T_a(x) = (a^2a)x \) for all \( x \) in \( G \).

**Lemma 2.1.** If \((G, \cdot)\) is a nontrivial Austin’s groupoid, then we have the following.

i. For each \( a \) in \( G \), the mapping \( T_a \) is injective.

ii. \((G, \cdot)\) is proper.

iii. \((G, \cdot)\) has no idempotent element.

iv. The terms \( x \), \( x^2 \) and \( x^2x \) are essentially unary and pairwise distinct.

v. The terms \( f_n \) and \( g_n \) are essentially \( n \)-ary for all \( n \geq 1 \).

**Proof.** (i) If \( T_a(x) = T_a(y) \) then, by the Austin’s identity,
\[
  x = ((a^2a)x)((a^2(a^2)a)z) = (T_a(x))((a^2(a^2)a)z) = (T_a(y))((a^2(a^2)a)z) = ((a^2a)y)((a^2(a^2)a)z) = y.
\]

(ii) Assume that \( xy \) does not depend on \( x \), then we have \( xy = y^2 \). Putting \( (u^2a)v \) for \( x \) and \( (u^2(u^2a))z \) for \( y \) in this identity, we get
\[
  v = ((u^2a)v)((u^2(u^2a))z) = xy = y^2 = (u^2(u^2a))z^2,
\]
which is impossible in a nontrivial groupoid. Assume now that \( xy \) does not depend on \( y \), then we have \( xy = x^2 \) and so \( T_a(b) = (a^2a)b = (a^2a)^2 \) for all \( b \) in \( G \). That is, \( T_a \) is constant, which contradicts (i). Therefore, \( xy \) is essentially binary. (iii) Suppose to the contrary that \((G, \cdot)\) has an idempotent element, say \( a \). Note that \( a^2a = a^2(a^2a) = a \) and so \( (ax)(ay) = ((a^2a)x)(a^2(a^2a))y = x \). In particular, \( a(ay) = (aa)(ay) = a \). Putting \( au \) for \( x \) in \( x = (ax)(ay) \), we obtain that \( au = (a(au))(ay) = a(ay) = a \) and hence \( x = (ax)(ay) = aa = a \), a contradiction. (iv) By (iii), \( x^2 \) is essentially unary and \( x^2 \neq x \). Assume \( x^2x = c \), a constant. Then \( x = ((y^2y)x)((y^2(y^2y))z) = (cx)((y^2c)z) \).
Putting $y = c$, we get $x = (cx)((c^2)c) = (cx)(cz)$. Putting $c$ for $x$ and $y$, we have $c = c^2c^2$. Putting $c^2$ for $x$ in $c = x^2x$ we have $c = (c^2)c^2 = cc^2$. Now putting $c^2$ for $x$ in $(cx)(cz)$, we have $c^2 = (cc^2)(cz) = c(cz)$ and so $c^2 = cc^2 = c$, a contradiction to (iii). Thus $x^2x$ is essentially unary. Now we show that $x^2x \neq x$. Assume $x^2x = x$, then $x = ((y^2y)x)((y^2y)z) = (yx)((y^2y)z) = (xy)(yz)$. Putting $x^2$ for $y$, we have $x = (x^2x)(x^2z)$ and hence $xx = (x^2x)(x^2x) = x$, a contradiction to (iii). Thus $x^2x \neq x$. Now assume $x^2x = x^2$, then  

\[ x^2 = (y^2x^2)((y^2y)^2)z. \]

Putting $x$ for $y$ and $x^2$ for $z$ in (1), we have $x^2 = (x^2x^2)((x^2x^2)x^2) = (x^2x^2)((x^2x^2)x^2) = (x^2x^2)(x^2x^2) = (x^2x^2)x^2 = (x^2x^2) = (x^2)^2$. That is, $x^2$ is an idempotent element, which is a contradiction to (iii). Thus $x^2x \neq x^2$. (v) We use induction on $n$. It is clear for $n = 1, 2$ as $(G, \cdot)$ is proper. Let $n \geq 3$ and assume that $g_k$ are essentially $k$-ary for $1 \leq k \leq n - 1$. By (A), we have

\[
g_{n-1}(x_2, \ldots, x_{n-1}) = x_2(x_3(\cdots (x_{n-1}x_n)\cdots))
\]

\[
= [(y^2y)(x_2(x_3(\cdots (x_{n-1}x_n)\cdots)))]((y^2y)^2)z)
\]

\[
= g_n(y^2y, x_2, \ldots, x_n)(y^2y)z).
\]

By induction hypothesis, $g_{n-1}$ and hence $g_n$ depends on $x_2, \ldots, x_n$. We also have

\[
g_{n-1}(x_1, x_2, \ldots, x_{n-1}) = x_1(x_2(\cdots (x_{n-2}x_{n-1})\cdots))
\]

\[
= x_1(x_2(\cdots (x_{n-2}(y^2y)_{n-1})(y^2y))z))\cdots))
\]

\[
= g_n(x_1, x_2, \ldots, x_{n-2}, (y^2y)_{n-1}, (y^2y)z).
\]

By induction hypothesis, $g_{n-1}$ and hence $g_n$ depends on $x_1$. Thus $g_n$ depends on all its variables. To prove $f_n$ is essentially $n$-ary for $n \geq 3$, we first show that $f_3(x, y, z) = (xy)z$ is essentially ternary. Assume $f_3$ does not depend on $x$. Then $(xy)z = (uy)z$. Putting $(x^2x)y$ for $x$ and $(x^2(x^2x))z$ for $y$ in this identity, we have $[(x^2x)y][(x^2x^2)z]z = [u((x^2(x^2x))^2)]z$, and so $yz = [u((x^2(x^2x))^2)]z$. This implies that $yz$ does not depend on $y$, which is a contradiction to (ii). Thus, $f_3$ depends on $x$. Since $y = ((x^2x)y)((x^2x^2)z) = f_3(x^2x, y, x^2(x^2x))z$ we
infer that $f_3$ depends on $y$. Using (A) again, we infer that $f_3$ depends on $z$ by the identity $uz = (((x^2x)u)((y^2(y^2)))v)z = f_3((x^2x)u, (y^2(y^2))v, z)$. Thus we have proved that $f_3$ is essentially ternary. Now, suppose $n > 3$ and assume that $f_k$ is essentially $k$-ary for $2 \leq k \leq n - 1$. Observe that, by the identity (A), we have 

$$f_{n-1}(x_2, x_3, \ldots, x_n) = f_n((x^2x)x_2, (x^2(x^2)x)z, x_3, \ldots, x_n)$$

Thus we have proved that $f_n$ is essentially ternary. Now, suppose $n > 3$ and assume that $f_k$ is essentially $k$-ary for $2 \leq k \leq n - 1$. Observe that, by the identity (A), we have

$$f_{n-1}(x_2, x_3, \ldots, x_n) = f_n((x^2x)x_2, (x^2(x^2)x)z, x_3, \ldots, x_n)$$

Thus, by induction hypothesis, we deduce that $f_n$ depends on all variables. That is, $f_n$ is essentially $n$ ary for all $n \geq 2$.

**Corollary 2.2.** For a nontrivial Austin’s groupoid $(G, \cdot)$, we have $p_n(G, \cdot) \geq 2$ for all $n \geq 1$.

**Lemma 2.3.** Let $(G, \cdot)$ be a nontrivial Austin’s groupoid. Then we have

i. For mappings $\phi_1, \phi_2 : G \to G$, we have $\phi_1 = \phi_2$ if and only if $\phi_1(xy) = \phi_2(xy)$ for all $x, y \in G$. Here, $xy$ can be replaced by $f_n$ or $g_n$ for any $n \geq 1$.

ii. For any $a$ in $G$, the mappings $T_a$ is not the identity mapping.

iii. The mapping $n \mapsto T_a^n$ is injective or there exists an integer $m$ such that $T_a^n(x) = x$ for all $x$ in $G$.

iv. The term $(x^2x)y$ depends on $y$ and $(x^2x)y \neq y$.

**Proof.** (i) Assume that $\phi_1(xy) = \phi_2(xy)$. Putting $(y^2y)x$ for $x$ and $(y^2(y^2))z$ for $y$, we have $\phi_1(x) = \phi_2(x)$. Further proof proceeds by induction on the arity of the terms. (ii) Assume that $T_a(x) = x$ for some $a$ and all $x \in G$. Then we have $(a^2a)x = x$. Putting $x = a^2a$ we see that $a^2a$ is idempotent, which contradicts Lemma 2.1(iii). Thus $T_a \neq Id$. (iii) Suppose the mapping $n \mapsto T_a^n$ is not injective, then $T_a^j = T_a^k$ for some $j < k$. Then $T_a^{k-j}(T_a^j(b)) = T_a^k(b) = T_a^j(b)$ for all $b$ in $G$. Since $T_a$ and hence $T_a^j$ is injective by Lemma 2.1(i), we see that $T_a^k-auto(x) = x$ for all $x$ in $G$. That is, $T_a^{k-j} = Id$. (iv) The fact that $(x^2x)y$ depends on $y$ follows from Lemma 2.1(i). If $(x^2x)y = y$, then $(x^2x)(x^2x) = x^2x$, which contradicts Lemma 2.1(iii). Thus $(x^2x)y \neq y$. □
3. Proof of Theorem

In this section, we prove Theorem 3 by induction on the arity of linear terms.

For \( n = 1, 2, 3 \), the conclusion follows by Lemma 2.1. Let \( n \geq 4 \) and assume that the assertion is true for all \( k \)-ary linear terms for \( 1 \leq k \leq n-1 \). Let \( f \) be an \( n \)-ary linear term. So, all variables in \( f \) are mutually distinct. We have two cases: (1) \( f \) contains at least two subterms of the form \( x_ix_j \) and (2) \( f \) contains only one subterm of the form \( x_ix_j \). Assume case (1) and so \( f \) contains subterms \( x_1x_2 \) and \( x_3x_4 \) after relabeling of variables if needed. Then, there are \((n-1)\)-ary linear terms \( g \) and \( h \) such that

\[
 f(x_1, x_2, \ldots, x_n) = g(x_1x_2, x_3, \ldots, x_n) = h(x_1, x_2, x_3x_4, x_5, \ldots, x_n).
\]

Then using (A) we have

\[
 f((y^2y)x_1, (y^2y)x_3, \ldots, x_n) = g(x_1, x_3, \ldots, x_n)
\]

and

\[
 f(x_1, x_2, (y^2y)x_3, (y^2y)x_5, \ldots, x_n) = h(x_1, x_2, x_3x_4 \ldots, x_n).
\]

Since \( g \) and \( h \) are essentially by induction hypothesis, these identities show that \( f \) depends on all \( x_1, x_2, \ldots, x_n \), i.e., \( f \) is essentially \( n \)-ary. Now, consider case (2). Since \( n \geq 4 \), \( f \) contains a subterm of the form \((x_1x_2)x_3 \) or \( x_3(x_1x_2) \). There is a \((n-1)\)-ary linear term \( g \) such that \( f(x_1, x_2, \ldots, x_n) = g(x_1x_2, x_3, \ldots, x_n) \), and then we have

\[
 f((y^2y)x_1, (y^2y)x_3, \ldots, x_n) = g(x_1, x_3, \ldots, x_n).
\]

By induction \( g \) depends on \( x_1, x_3, \ldots, x_n \), and hence so does \( f \). If \( f \) contains \((x_1x_2)x_3 \) as a subterm, let \( f(x_1, \ldots, x_n) = h((x_1x_2)x_3, x_4, \ldots, x_n) \) for some \((n-2)\)-ary linear term \( h \). Then \( f(y^2y, x_2, (y^2y)x_4, \ldots, x_n) = h(x_2, x_4, \ldots, x_n) \) and so \( f \) also depends on \( x_2 \). If \( f \) contains \( x_3(x_1x_2) \) as a subterm, since \( g_n \) is essentially \( n \)-ary by Lemma 2.1(v), we may assume that \( f \) is not of the form of \( g_n \). Then \( f \) is of the form

\[
 f(x_1, x_2, \ldots, x_n) = \ldots ([x_k(\ldots (x_3(x_1x_2)) \ldots)]x_{k+1}) \ldots
\]

for some \( k \geq 3 \). Putting \( y^2y \) for \( x_k \) and \((y^2y)z \) for \( x_{k+1} \), we have by (A) that

\[
 f(x_1, x_2, \ldots, x_{k-1}, y^2y, (y^2y)z, x_{k+2}, \ldots, x_n)
 = \ldots (\ldots (x_3(x_1x_2)) \ldots) \ldots,
\]
where the right-hand side is a linear term without the variables $x_k$ and $x_{k+1}$. By induction, $f$ depends on each variables appearing on the right-hand side, in particular on $x_2$ as well. Consequently, $f$ depends on every variable it involves. This completes the proof.

REFERENCES


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