$G_\delta$-CONNECTEDNESS AND $G_\delta$-DISCONNECTEDNESS
IN FUZZY BITOPOLOGICAL SPACES

E. Roja, M. K. Uma, and G. Balasubramanian

Abstract. In this paper, the concepts of pairwise fuzzy $G_\delta$-connected spaces and pairwise fuzzy $G_\delta$-extremally disconnected spaces are introduced. The concept of pairwise fuzzy $G_\delta$-basically disconnected spaces is defined. Characterizations of the above spaces are given besides giving several examples. Interrelations among the spaces introduced are discussed and some relevant counter examples are given.

1. Introduction and Preliminaries

Ever since the introduction of fuzzy set by L.A. Zadeh [8], the fuzzy concept has invaded almost all branches of Mathematics. The concept of fuzzy topological spaces was introduced in [4] by C.L. Chang. Since then many fuzzy topologists [6 & 7] have extended various notions in classical topology to fuzzy topological spaces. In 1989, Kandil [5] introduced the concept of fuzzy bitopological spaces and since then many concepts in classical topology have been extended to fuzzy bitopological spaces. The purpose of this paper is to introduce pairwise fuzzy $G_\delta$-connected spaces and pairwise fuzzy $G_\delta$-disconnected spaces. Pairwise fuzzy connected and pairwise fuzzy extremally disconnected spaces were found in [3]. Pairwise fuzzy basically disconnected space was found in [2].

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Definition 1.0.1. Let \((X, T)\) be a fuzzy topological space and \(\lambda\) be a fuzzy set in \(X\). \(\lambda\) is called a fuzzy \(G_\delta\)-set \([1]\) if \(\lambda = \bigwedge_{i=1}^{\infty} \lambda_i\) where each \(\lambda_i \in T\).

Definition 1.0.2. Let \((X, T)\) be a fuzzy topological space and \(\lambda\) be a fuzzy set in \(X\). \(\lambda\) is called a fuzzy \(F_\sigma\)-set if \(\lambda = \bigvee_{i=1}^{\infty} \lambda_i\) where each \(1 - \lambda_i \in T\).

Definition 1.0.3. Let \((X, T)\) be any fuzzy topological space. For any fuzzy set \(\lambda\) in \(X\) we define the \(\sigma\)-closure of \(\lambda\), denoted by \(\text{cl}_\sigma \lambda\), to be the intersection of all fuzzy \(F_\sigma\)-sets containing \(\lambda\). That is,

\[
\text{cl}_\sigma \lambda = \bigwedge \{ \mu : \mu \text{ is a fuzzy } F_\sigma\text{-set and } \mu \geq \lambda \}.
\]

Definition 1.0.4. Let \((X, T)\) be any fuzzy topological space. For any fuzzy set \(\lambda\) in \(X\), we define the \(\delta\)-interior of \(\lambda\), denoted by \(\text{int}_\delta \lambda\), to be the union of all fuzzy \(G_\delta\)-sets contained in \(\lambda\). That is,

\[
\text{int}_\delta \lambda = \bigvee \{ \mu : \mu \text{ is a fuzzy } G_\delta\text{-set and } \mu \leq \lambda \}.
\]

Definition 1.0.5. A fuzzy bitopological space \([5]\) is a triple \((X, T_1, T_2)\) where \(X\) is a set, \(T_1\) and \(T_2\) are any two fuzzy topologies on \(X\).

Note 1. \(G_\delta F_\sigma\) denotes the fuzzy set which is both fuzzy \(G_\delta\) and fuzzy \(F_\sigma\).

2. Main Results

2.1. Pairwise fuzzy \(G_\delta\)-connected spaces

In this section, the concept of pairwise fuzzy \(G_\delta\)-connected spaces is introduced. Besides giving examples, characterizations of pairwise fuzzy \(G_\delta\)-connected spaces are also studied.

Definition 2.1.1. A fuzzy bitopological space \((X, T_1, T_2)\) is said to be pairwise fuzzy \(G_\delta\)-connected iff \((X, T_1, T_2)\) has no proper fuzzy sets \(\lambda_1\) and \(\lambda_2\) which are \(T_1\)-fuzzy \(G_\delta\) and \(T_2\)-fuzzy \(G_\delta\) respectively such that \(\lambda_1 + \lambda_2 = 1\). A fuzzy bitopological space \((X, T_1, T_2)\) is pairwise fuzzy \(G_\delta\)-disconnected if it is not pairwise fuzzy \(G_\delta\)-connected.
Remark 2.1.1. The pairwise fuzzy $G_{\delta}$-connectedness of $(X, T_1, T_2)$ is not governed by the fuzzy $G_{\delta}$-connectedness of the spaces $(X, T_1)$ and $(X, T_2)$ as the following example shows.

Example 2.1.1. Let $X = \{a, b\}$, $T_1$ be the discrete fuzzy topology, $T_2$ be the indiscrete fuzzy topology, $T_3 = \{0, 1, \lambda\}$, where $\lambda: X \to [0, 1]$ is such that $
abla(a) = 1$, $\lambda(b) = 0$, and $T_4 = \{0, 1, \mu\}$, where $\mu: X \to [0, 1]$ is such that $\mu(a) = 0$ and $\mu(b) = 1$.

Then $(X, T_1, T_2)$ is pairwise fuzzy $G_{\delta}$-connected but $(X, T_1)$ is not fuzzy $G_{\delta}$-connected and $(X, T_2)$ is fuzzy $G_{\delta}$-connected. Also, $(X, T_3, T_4)$ is not pairwise fuzzy $G_{\delta}$-connected but $(X, T_3)$ and $(X, T_4)$ are both fuzzy $G_{\delta}$-connected.

Proposition 2.1.1. The following statements are equivalent for a fuzzy bitopological space $(X, T_1, T_2)$.

(a) $(X, T_1, T_2)$ is pairwise fuzzy $G_{\delta}$-connected.

(b) There exists no $T_1$-fuzzy $G_{\delta}$-set $\lambda_1 \neq 0$ and $T_2$-fuzzy $G_{\delta}$-set $\lambda_2 \neq 0$ such that $\lambda_1 + \lambda_2 = 1$.

(c) There exists no $T_1$-fuzzy $F_\sigma$-set $\lambda_1 \neq 1$ and $T_2$-fuzzy $F_\sigma$-set $\lambda_2 \neq 1$ such that $\lambda_1 + \lambda_2 = 1$.

(d) $(X, T_1, T_2)$ contains no fuzzy set $\lambda \neq 0, 1$ and it is both $T_1$-fuzzy $G_{\delta}$ and $T_2$-fuzzy $F_\sigma$, or both $T_2$-fuzzy $G_{\delta}$ and $T_1$-fuzzy $F_\sigma$.

Proof. (a) $\Rightarrow$ (b). Assume that (a) is true. Then (b) follows from the Definition 2.1.1.

(b) $\Rightarrow$ (c). Assume that (b) is true. Let us suppose that there exists a $T_1$-fuzzy $F_\sigma$-set $\lambda_1 \neq 1$ and a $T_2$-fuzzy $F_\sigma$-set $\lambda_2 \neq 1$ such that $\lambda_1 + \lambda_2 = 1$. Then, $1 - \lambda_1 \neq 1 - 1 \neq 0$ is a non-zero $T_1$-fuzzy $G_{\delta}$-set. Similarly, we get $1 - \lambda_2$ is a non-zero $T_2$-fuzzy $G_{\delta}$-set. Now, $(1 - \lambda_1) + (1 - \lambda_2) = 2 - (\lambda_1 + \lambda_2) = 2 - 1 = 1$. Contradiction. Hence (c).

(c) $\Rightarrow$ (d). Assume that (c) is true. Suppose that $(X, T_1, T_2)$ contains a fuzzy set $\lambda \neq 0, 1$ which is both $T_1$-fuzzy $G_{\delta}$ and $T_2$-fuzzy $F_\sigma$. Then $(1 - \lambda)$ is a proper $T_1$-fuzzy $F_\sigma$-set. Also by assumption, $\lambda$ is $T_2$-fuzzy $F_\sigma$. Now, $(1 - \lambda) + \lambda = 1$. Contradiction. Hence (d).
(d) ⇒ (a). Assume that (d) is true. Let us assume that \((X, T_1, T_2)\) is not pairwise fuzzy \(G_\delta\)-connected. Then \((X, T_1, T_2)\) has proper fuzzy sets \(\lambda_1\) and \(\lambda_2\) where \(\lambda_1\) is \(T_1\)-fuzzy \(G_\delta\)-set and \(\lambda_2\) is \(T_2\)-fuzzy \(G_\delta\)-set respectively such that \(\lambda_1 + \lambda_2 = 1\). Now, \(\lambda_1 + \lambda_2 = 1\) implies \(\lambda_1 = 1 - \lambda_2\). This implies that \(\lambda_1\) is both \(T_2\)-fuzzy \(F_\sigma\) and \(\lambda_2\) is non-zero \(T_2\)-fuzzy \(G_\delta\)-set. Clearly, \(\lambda_1 \neq 0, 1\) is in \((X, T_1, T_2)\). Contradiction. Hence (a).

### 2.2. Pairwise fuzzy super \(G_\delta\)-connected spaces

In this section, the concept of pairwise fuzzy super \(G_\delta\)-connected spaces is introduced. More examples are given to illustrate the concept introduced in this section. Characterizations of such spaces are also studied.

**Definition 2.2.1.** Let \((X, T_1, T_2)\) be any fuzzy bitopological space and let \(\lambda\) be any fuzzy set in \((X, T_1, T_2)\). Then

1. \(\lambda\) is called \((1, 2)\) fuzzy regular \(G_\delta\) if \(\text{int}_{\delta(T_1)} \text{cl}_\sigma(T_2) \lambda = \lambda\).
2. \(\lambda\) is called \((2, 1)\) fuzzy regular \(G_\delta\) if \(\text{int}_{\delta(T_2)} \text{cl}_\sigma(T_1) \lambda = \lambda\) and
3. \(\lambda\) is called pairwise fuzzy regular \(G_\delta\) if \(\lambda\) is both \((1, 2)\) fuzzy regular \(G_\delta\) and \((2, 1)\) fuzzy regular \(G_\delta\).

**Definition 2.2.2.** Let \((X, T_1, T_2)\) be any fuzzy bitopological space. Then \((X, T_1, T_2)\) is called pairwise fuzzy super \(G_\delta\)-connected if it has no proper \((\neq 0, 1)\) pairwise fuzzy regular \(G_\delta\)-set.

**Example 2.2.1.** Let \(X = \{a, b\}\), \(T_1 = \{0, 1, \lambda\}\) and \(T_2 = \{0, 1, \mu\}\), where \(\lambda: X \to [0, 1]\) is such that

\[
\lambda(a) = 1, \quad \lambda(b) = 3/4,
\]

and \(\mu: X \to [0, 1]\) is such that

\[
\mu(a) = 0 \quad \text{and} \quad \mu(b) = 1.
\]

Then the fuzzy bitopological space \((X, T_1, T_2)\) is pairwise fuzzy super \(G_\delta\)-connected and pairwise fuzzy \(G_\delta\)-connected.

**Proposition 2.2.1.** If \((X, T_1, T_2)\) is any fuzzy bitopological space, then (a) ⇒ (b) and (b) ⇒ (c), where

(a) \((X, T_1, T_2)\) is pairwise fuzzy super \(G_\delta\)-connected space.
(b) the $T_2$-$\sigma$-closure or $T_1$-$\sigma$-closure of a pairwise fuzzy regular $G_\delta$-set which is different from 0 is 1.
(c) the $T_2$-$\delta$-interior or $T_1$-$\delta$-interior of a pairwise fuzzy regular $F_\sigma$-set which is different from 1 is 0.

Proof. (a)$\Rightarrow$(b). Assume (a). Suppose there exists a pairwise fuzzy regular $G_\delta$-set $\lambda \neq 0$ such that $\text{cl}_\sigma(T_2)\lambda \neq 1$. Then

\[
\text{int}_\delta(T_1)\text{cl}_\sigma(T_2)\lambda \neq 1.
\]
But since $\lambda$ is pairwise fuzzy regular $G_\delta$-set,

\[
\text{int}_\delta(T_1)\text{cl}_\sigma(T_2)\lambda = \lambda.
\]
From (1) and (2) we get $\lambda \neq 1$. Thus we find that $(X, T_1, T_2)$ has a proper pairwise fuzzy regular $G_\delta$-set $\lambda$. Contradiction.

Similarly, we can show that $T_1$-$\sigma$-closure of a pairwise fuzzy regular $G_\delta$-set which is different from 0 is one. Hence (b).

(b)$\Rightarrow$(c). Assume (b). Suppose (c) is not true. This means that there exists a pairwise fuzzy regular $F_\sigma$-set $\lambda \neq 0$ such that $\text{int}_\delta(T_2)\lambda \neq 0$. Now, $\mu = 1 - \lambda \neq 0$ and $\mu$ is a non-zero pairwise fuzzy regular $G_\delta$-set. Then $\text{cl}_\sigma(T_2)\mu = 1 - \text{int}_\delta(T_2)(1 - \mu) = 1 - \text{int}_\delta(T_2)\lambda \neq 1$ (since $\text{int}_\delta(T_2)\lambda \neq 0$). Contradiction.

Similarly, we can prove that $T_1$-$\delta$-interior of a pairwise fuzzy regular $F_\sigma$-set which is different from 1 is 0. Hence (c).

Remark 2.2.1. The following examples give the relation between pairwise fuzzy super $G_\delta$-connectedness, pairwise fuzzy $G_\delta$-connectedness and pairwise fuzzy $G_\delta$-disconnectedness.

Example 2.2.2. Let $X = \{a, b\}$, $T_1 = \{0, 1, \lambda\}$ and $T_2 = \{0, 1, \mu\}$, where $\lambda: X \rightarrow [0, 1]$ is such that

\[
\lambda(a) = 1/4, \quad \lambda(b) = 0
\]
and $\mu: X \rightarrow [0, 1]$ is such that

\[
\mu(a) = 0, \quad \mu(b) = 1.
\]
Now, $\text{int}_\delta(T_1)\text{cl}_\sigma(T_2)\lambda = \lambda$, $\text{int}_\delta(T_2)\text{cl}_\sigma(T_1)\mu = \mu$, $\lambda$ is (1, 2) fuzzy regular and $\mu$ is (2, 1) fuzzy regular. Therefore the fuzzy bitopological space $(X, T_1, T_2)$ is not pairwise fuzzy super $G_\delta$-connected. Also, $\lambda + \mu = 1$. Now the fuzzy bitopological space $(X, T_1, T_2)$ is pairwise fuzzy $G_\delta$-connected.
Example 2.2.3. Let $X = \{a, b\}$, $T_1 = \{0, 1, \lambda\}$ and $T_2 = \{0, 1, \mu\}$, where $\lambda : X \to [0, 1]$ is such that
\[\lambda(a) = 1/4, \quad \lambda(b) = 0\]
and $\mu : X \to [0, 1]$ is such that
\[\mu(a) = 3/4, \quad \mu(b) = 1.\]
Then the fuzzy bitopological space $(X, T_1, T_2)$ is not pairwise fuzzy super $G_\delta$-connected and not pairwise fuzzy $G_\delta$-connected.

2.3. Pairwise fuzzy strongly $G_\delta$-connected spaces

In this section, the concept of pairwise fuzzy strongly $G_\delta$-connected spaces is introduced. Interesting properties and characterizations are also discussed.

Definition 2.3.1. A fuzzy bitopological space $(X, T_1, T_2)$ is said to be pairwise fuzzy strongly $G_\delta$-connected if it has no proper $T_1$-fuzzy $F_\sigma$-sets or $T_2$-fuzzy $F_\sigma$-sets $\lambda_1$ and $\lambda_2$ such that $\lambda_1 + \lambda_2 \leq 1$. If $(X, T_1, T_2)$ is not pairwise fuzzy strongly $G_\delta$-connected, then it will be called pairwise fuzzy weakly $G_\delta$-connected.

Proposition 2.3.1. A fuzzy bitopological space $(X, T_1, T_2)$ is pairwise fuzzy strongly $G_\delta$-connected iff it has no proper $T_1$-fuzzy $G_\delta$-sets or $T_2$-fuzzy $G_\delta$-sets $\lambda, \mu$ such that $\lambda + \mu \geq 1$.

Proof. $(X, T_1, T_2)$ is pairwise fuzzy weakly $G_\delta$-connected iff it has proper $T_1$-fuzzy $F_\sigma$-sets or $T_2$-fuzzy $F_\sigma$-sets $f, k$ such that $f + k \leq 1$ $\iff$ it has proper $T_1$-fuzzy $G_\delta$-sets or $T_2$-fuzzy $G_\delta$-sets $\lambda, \mu$ where $\lambda = 1 - f, \mu = 1 - k$ such that $\lambda + \mu \geq 1$. 

Remark 2.3.1. Pairwise fuzzy strongly $G_\delta$-connectedness implies pairwise fuzzy $G_\delta$-connectedness. However the converse is not true as shown in Example 2.3.1.

Example 2.3.1. Let $X = [0, 1]$. Define $T_1 = \{0, 1, \lambda\}$ and $T_2 = \{0, 1, \mu\}$ where $\lambda : X \to [0, 1]$ is such that $\lambda(x) = 2/3$, for all $x \in X$ and $\mu : X \to [0, 1]$ is such that $\mu(x) = 3/4$, for all $x \in X$. Then the fuzzy bitopological space $(X, T_1, T_2)$ is pairwise fuzzy $G_\delta$-connected but not pairwise fuzzy strongly $G_\delta$-connected.
**Proposition 2.3.2.** Let \((X, T_1, T_2)\) be any fuzzy bitopological space and \(A \subset X\) be any subset. Then the following statements are equivalent.

(a) \((A, T_1/A, T_2/A)\) is pairwise fuzzy strongly \(G_\delta\)-connected subspace of \((X, T_1, T_2)\).

(b) For any proper \(T_1\)-fuzzy \(G_\delta\)-sets or \(T_2\)-fuzzy \(G_\delta\)-sets \(\lambda_1, \lambda_2\) such that \(1_A \leq \lambda_1/A + \lambda_2/A\) implies either \(1_A = \lambda_1/A\) or \(1_A = \lambda_2/A\).

**Proof.** (b) \(\Rightarrow\) (a). Suppose \(A\) is not pairwise fuzzy strongly \(G_\delta\)-connected subset of \(X\). Then there exist proper \(T_1\)-fuzzy \(F_\sigma\)-set or \(T_2\)-fuzzy \(F_\sigma\)-sets \(f, k\) such that \(f + k \leq 1_A\). Therefore, we can find proper \(T_1\)-fuzzy \(G_\delta\)-sets or \(T_2\)-fuzzy \(G_\delta\)-sets \(\lambda_1, \lambda_2\) such that

\[
\lambda_1/A = 1_A - f, \quad \lambda_2/A = 1_A - k.
\]

Then

\[
\lambda_1/A + \lambda_2/A = 1_A - f + 1_A - k = 2 - (f + k).
\]

That is,

\[
\lambda_1/A + \lambda_2/A \geq 1_A \quad \text{(since } f + k \leq 1_A)\tag{3}
\]

Since

\[
0 < \lambda_1/A < 1_A \quad \text{(4)}
\]

and

\[
0 < \lambda_2/A < 1_A, \quad \text{(5)}
\]

we have from (3), (4) and (5) that \(1_A \neq \lambda_1/A\) and \(1_A \neq \lambda_2/A\). Contradiction. This proves (a).

(a) \(\Rightarrow\) (b). Suppose there exist proper \(T_1\)-fuzzy \(G_\delta\)-sets or \(T_2\)-fuzzy \(G_\delta\)-sets \(\lambda_1, \lambda_2\) such that \(1_A \leq \lambda_1/A + \lambda_2/A\) but both \(1_A \neq \lambda_1/A\) and \(1_A \neq \lambda_2/A\). This shows by Proposition 2.3.1, \(A\) is not pairwise fuzzy strongly \(G_\delta\)-connected. Contradiction. This proves (b). \(\square\)

**Proposition 2.3.3.** Let \((X, T_1, T_2)\) be any fuzzy bitopological space. Let \(F \subset X\) be such that \(\chi_F\) is \(T_1\)-fuzzy \(F_\sigma\) or \(T_2\)-fuzzy \(F_\sigma\). Then \((X, T_1, T_2)\) is pairwise fuzzy strongly \(G_\delta\)-connected implies \((F, T_1/F, T_2/F)\) is pairwise fuzzy strongly \(G_\delta\)-connected.
Proof. Let $F \subset X$ be such that $\chi_F$ is $T_1$-fuzzy $F_\sigma$ or $T_2$-fuzzy $F_\sigma$. We want to show that $(F, T_1/F, T_2/F)$ is pairwise fuzzy strongly $G_\delta$-connected. Suppose $(F, T_1/F, T_2/F)$ is not pairwise fuzzy strongly $G_\delta$-connected. This means there exist proper $T_1$-fuzzy $F_\sigma$-sets or $T_2$-fuzzy $F_\sigma$-sets $f, k$ such that

$$f + k \leq 1.$$ (6)

Hence, we can find proper $T_1$-fuzzy $F_\sigma$ or $T_2$-fuzzy $F_\sigma$ sets $\lambda_1, \lambda_2$ such that $f = \lambda_1/F, k = \lambda_2/F$. Now, consider $(\lambda_1 \wedge \chi_F) + (\lambda_2 \wedge \chi_F)$. Since $\chi_F$ is $T_1$-fuzzy $F_\sigma$ or $T_2$-fuzzy $F_\sigma$, $\lambda_1 \wedge \chi_F$ and $\lambda_2 \wedge \chi_F$ are $T_1$-fuzzy $F_\sigma$ or $T_2$-fuzzy $F_\sigma$. From (6) we find that

$$(\lambda_1 \wedge \chi_F) + (\lambda_2 \wedge \chi_F) \leq 1_X.$$ This shows $(X, T_1, T_2)$ is not pairwise fuzzy strongly $G_\delta$-connected, which is a contradiction. Hence the proposition.

2.4. Pairwise fuzzy $G_\delta$-extremally disconnected spaces

In this section, the concept of pairwise fuzzy $G_\delta$-extremally disconnected spaces is introduced. Characterizations and some interesting properties are also given with necessary examples.

**Definition 2.4.1.** A fuzzy bitopological space $(X, T_1, T_2)$ is said to be pairwise fuzzy $G_\delta$-extremally disconnected if $T_1\sigma$-closure of each $T_2$-fuzzy $G_\delta$-set is $T_2$-fuzzy $G_\delta$ and $T_2\sigma$-closure of each $T_1$-fuzzy $G_\delta$-set is $T_1$-fuzzy $G_\delta$.

**Example 2.4.1.** Let $X = \{a, b, c, d\}$. Define $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \mu_1, \mu_2, \mu_3, \mu_4 : X \rightarrow [0, 1]$ as follows:

- $\lambda_1(a) = \lambda_1(c) = \lambda_3(d) = 0$, $\lambda_1(b) = 1$,
- $\lambda_2(a) = \lambda_2(b) = 1$, $\lambda_2(c) = \lambda_2(d) = 0$,
- $\lambda_3(b) = \lambda_3(d) = 1$, $\lambda_3(a) = \lambda_3(c) = 0$,
- $\lambda_4(a) = \lambda_4(b) = \lambda_4(d) = 1$, $\lambda_4(c) = 0$,
- $\mu_1(a) = \mu_1(b) = \mu_1(d) = 0$, $\mu_1(c) = 1$,
- $\mu_2(a) = \mu_2(c) = 1$, $\mu_2(b) = \mu_2(d) = 0$,
- $\mu_3(a) = \mu_3(b) = 0$, $\mu_3(c) = \mu_3(d) = 1$,
- $\mu_4(a) = \mu_4(c) = \mu_4(d) = 1$, $\mu_4(b) = 0$. 
Clearly, \( T_1 = \{0, 1, \lambda_1, \lambda_2, \lambda_3, \lambda_4\} \) and \( T_2 = \{0, 1, \mu_1, \mu_2, \mu_3, \mu_4\} \) are fuzzy topologies on \( X \). Then we can easily see that \((X, T_1, T_2)\) is pairwise fuzzy \(G_\delta\)-extremally disconnected even though both \((X, T_1)\) and \((X, T_2)\) are fuzzy \(G_\delta\)-connected spaces.

**Proposition 2.4.1.** For any fuzzy bitopological space \((X, T_1, T_2)\), the following are equivalent.

(a) \((X, T_1, T_2)\) is pairwise fuzzy \(G_\delta\)-extremally disconnected.

(b) Whenever \( \lambda \) is a \( T_1 \)-fuzzy \( F_\sigma\)-set, \( \text{Int}_{\delta(T_2)} \lambda \) is a \( T_1 \)-fuzzy \( F_\sigma\)-set. Similarly, whenever \( \mu \) is a \( T_2 \)-fuzzy \( F_\sigma\)-set, \( \text{Int}_{\delta(T_1)} \mu \) is a \( T_2 \)-fuzzy \( F_\sigma\)-set.

(c) Whenever \( \lambda \) is a \( T_1 \)-fuzzy \( G_\delta\)-set, we have
\[
\text{cl}_{\sigma(T_1)}(1 - \text{cl}_{\sigma(T_2)} \lambda) = 1 - \text{cl}_{\sigma(T_2)} \lambda.
\]
Similarly, whenever \( \lambda \) is a \( T_2 \)-fuzzy \( G_\delta\)-set, we have
\[
\text{cl}_{\sigma(T_2)}(1 - \text{cl}_{\sigma(T_1)} \lambda) = 1 - \text{cl}_{\sigma(T_1)} \lambda.
\]

(d) For every pair of \( T_1 \)-fuzzy \( G_\delta\)-set \( \lambda \) and \( T_2 \)-fuzzy \( G_\delta\)-set \( \mu \) in \((X, T_1, T_2)\) with \( \text{cl}_{\sigma(T_2)} \lambda + \mu = 1 \), we have \( \text{cl}_{\sigma(T_2)} \lambda + \text{cl}_{\sigma(T_1)} \mu = 1 \). Similarly, for every pair of \( T_2 \)-fuzzy \( G_\delta\)-set \( \lambda \) and \( T_1 \)-fuzzy \( G_\delta\)-set \( \mu \) in \((X, T_1, T_2)\) with \( \text{cl}_{\sigma(T_1)} \lambda + \mu = 1 \), we have \( \text{cl}_{\sigma(T_2)} \lambda + \text{cl}_{\sigma(T_1)} \mu = 1 \).

**Proof.** (a) \(\Rightarrow\) (b). Suppose (a) is true. Let \( \lambda \) be any \( T_1 \)-fuzzy \( F_\sigma\)-set. Then \( 1 - \lambda \) is a \( T_1 \)-fuzzy \( G_\delta\)-set. Then from (a), \( \text{cl}_{\sigma(T_2)}(1 - \lambda) \) is a \( T_1 \)-fuzzy \( G_\delta\)-set. Clearly, \( 1 - \text{cl}_{\sigma(T_2)}(1 - \lambda) \) is \( T_1 \)-fuzzy \( F_\sigma\)-set. But
\[
1 - \text{cl}_{\sigma(T_2)}(1 - \lambda) = \text{Int}_{\delta(T_2)} \lambda
\]
and so \( \text{Int}_{\delta(T_2)} \lambda \) is \( T_1 \)-fuzzy \( F_\sigma\)-set. Similar statement holds for \( T_2 \)-fuzzy \( F_\sigma\)-sets. Thus (b) is proved.

(b) \(\Rightarrow\) (c). Assume that (b) is true. Suppose \( \lambda \) is a \( T_1 \)-fuzzy \( G_\delta\)-set. Then \( 1 - \lambda \) is a \( T_1 \)-fuzzy \( F_\sigma\)-set. Now, \( \text{cl}_{\sigma(T_2)} \lambda \) is \( T_1 \)-fuzzy \( G_\delta\) and therefore \( 1 - \text{cl}_{\sigma(T_2)} \lambda \) is \( T_1 \)-fuzzy \( F_\sigma\). Therefore,
\[
\text{cl}_{\sigma(T_1)}(1 - \text{cl}_{\sigma(T_2)} \lambda) = 1 - \text{cl}_{\sigma(T_2)} \lambda.
\]
Similarly, we can show that \( \text{cl}_{\sigma(T_2)}(1 - \text{cl}_{\sigma(T_1)} \lambda) = 1 - \text{cl}_{\sigma(T_1)} \lambda \) when \( \lambda \) is a \( T_2 \)-fuzzy \( G_\delta\)-set. Hence (c).
(c) \implies (d). Assume for every \( T_1\)-fuzzy \( G_\delta\)-set \( \lambda \), we have
\[
\text{cl}_{\sigma(T_1)}(1 - \text{cl}_{\sigma(T_2)} \lambda) = 1 - \text{cl}_{\sigma(T_2)} \lambda,
\]
and for every \( T_2\)-fuzzy \( G_\delta\)-set \( \lambda \),
\[
\text{cl}_{\sigma(T_2)}(1 - \text{cl}_{\sigma(T_1)} \lambda) = 1 - \text{cl}_{\sigma(T_1)} \lambda.
\]
Suppose that \( \lambda \) is \( T_1\)-fuzzy \( G_\delta \) and \( \mu \) is \( T_2\)-fuzzy \( G_\delta\)-set such that
\[
(7) \quad \text{cl}_{\sigma(T_2)} \lambda + \mu = 1.
\]
Then,
\[
\text{cl}_{\sigma(T_2)} \lambda + \text{cl}_{\sigma(T_1)}(1 - \text{cl}_{\sigma(T_2)} \lambda) = 1 = \text{cl}_{\sigma(T_2)} \lambda + \mu
\]
implies \( \mu = \text{cl}_{\sigma(T_1)}(1 - \text{cl}_{\sigma(T_2)} \lambda) \) and so \( 1 - \text{cl}_{\sigma(T_2)} \lambda = \text{cl}_{\sigma(T_1)}(1 - \text{cl}_{\sigma(T_2)} \lambda) \). That is, \( 1 - \text{cl}_{\sigma(T_2)} \lambda \) is \( T_1\)-fuzzy \( F_\sigma \) and hence \( \text{cl}_{\sigma(T_1)} \mu = 1 - \text{cl}_{\sigma(T_2)} \lambda \). That is, \( \text{cl}_{\sigma(T_1)} \mu + \text{cl}_{\sigma(T_2)} \lambda = 1 \). Similarly, we can prove the other part. Hence (d).

(d) \implies (a). Assume that (d) is true. Let \( \lambda \) be any \( T_1\)-fuzzy \( G_\delta\)-set. Put \( \text{cl}_{\sigma(T_2)} \lambda + \mu = 1 \). That is, \( \mu = 1 - \text{cl}_{\sigma(T_2)} \lambda \). By (d), \( \text{cl}_{\sigma(T_2)} \lambda + \text{cl}_{\sigma(T_1)} \mu = 1 \) and hence \( \text{cl}_{\sigma(T_2)} \lambda \) is \( T_1\)-fuzzy \( G_\delta \) in \( (X, T_1, T_2) \). Similarly, we can show that \( T_1\)-\( \sigma \)-closure of a \( T_2\)-fuzzy \( G_\delta\)-set is \( T_2\)-fuzzy \( G_\delta \). Therefore, \( (X, T_1, T_2) \) is pairwise fuzzy \( G_\delta\)-extremally disconnected.

\[ \square \]

**Proposition 2.4.2.** Let \( (X, T_1, T_2) \) be a pairwise fuzzy \( G_\delta\)-extremally disconnected space. If \( A \subset X \) is such that \( \chi_A \) is \( T_1\)-fuzzy \( G_\delta \) and \( T_2\)-fuzzy \( G_\delta \), then the fuzzy subspace \( (A, T_1/A, T_2/A) \) is pairwise fuzzy \( G_\delta\)-extremally disconnected.

**Proof.** Let \( A \subset X \) be such that \( \chi_A \) is \( T_1\)-fuzzy \( G_\delta \) and \( T_2 \) fuzzy \( G_\delta \). Let \( \lambda_1 \) be \( T_1\)-\( A\)-fuzzy \( G_\delta \) and let \( \lambda_2 \) be \( T_2\)-\( A\)-fuzzy \( G_\delta \) in \( A \) such that \( \text{cl}_{\sigma(T_2/A)} \lambda_1 + \lambda_2 = 1 \). Then, there exist \( T_1\)-fuzzy \( G_\delta\)-set \( \mu_1 \) and \( T_2\)-fuzzy \( G_\delta\)-set \( \mu_2 \) in \( (X, T_1, T_2) \) such that \( \mu_1/A = \lambda_1 \) and \( \mu_2/A = \lambda_2 \). That is, \( \mu_1 \land \chi_A = \lambda_1 \) and \( \mu_2 \land \chi_A = \lambda_2 \). Since \( \chi_A \) is \( T_1\)-fuzzy \( G_\delta \) and \( T_2\)-fuzzy \( G_\delta \), \( \lambda_1 \land \chi_A \) is \( T_1\)-fuzzy \( G_\delta \) and \( \lambda_2 \land \chi_A \) is \( T_2\)-fuzzy \( G_\delta \). That is, \( \lambda_1 \) is \( T_1\)-fuzzy \( G_\delta \) and \( \lambda_2 \) is \( T_2\)-fuzzy \( G_\delta \) in \( (X, T_1, T_2) \). Since \( (X, T_1, T_2) \) is pairwise fuzzy \( G_\delta\)-extremally disconnected, \( \text{cl}_{\sigma(T_2)} \lambda_1 + \text{cl}_{\sigma(T_1)} \lambda_2 = 1 \) in \( (X, T_1, T_2) \) and therefore in \( (A, T_1/A, T_2/A) \). Thus, \( (A, T_1/A, T_2/A) \) is pairwise fuzzy \( G_\delta\)-extremally disconnected.

\[ \square \]
2.5. Pairwise fuzzy $G_\delta$-basically disconnected spaces

In this section, the concept of pairwise fuzzy $G_\delta$-basically disconnected spaces is introduced. Characterizations and properties are discussed with examples.

**Definition 2.5.1.** A fuzzy bitopological space $(X, T_1, T_2)$ is said to be **pairwise fuzzy $G_\delta$-basically disconnected** if the $T_1$-$\sigma$-closure of each $T_2$-fuzzy $G_\delta$, $T_2$-fuzzy $F_\sigma$-set is $T_2$-fuzzy $G_\delta$ and $T_1$-fuzzy $F_\sigma$-set is $T_1$-fuzzy $G_\delta$.

**Example 2.5.1.** Let $X = \{a, b, c, d\}$, $T_1 = \{0, 1, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ and $T_2 = \{0, 1, \mu_1, \mu_2, \mu_3, \mu_4\}$, where $\lambda_i : X \to [0, 1], i = 1, 2, 3, 4$ and $\mu_j : X \to [0, 1], j = 1, 2, 3, 4$ are defined as follows:

- $\lambda_1(a) = \lambda_1(c) = \lambda_1(d) = 0$, $\lambda_1(b) = 1$,
- $\lambda_2(a) = \lambda_2(b) = 1$, $\lambda_2(c) = \lambda_2(d) = 0$,
- $\lambda_3(b) = \lambda_3(d) = 1$, $\lambda_3(a) = \lambda_3(c) = 0$,
- $\lambda_4(a) = \lambda_4(b) = \lambda_4(d) = 1$, $\lambda_4(c) = 0$,
- $\mu_1(a) = \mu_1(b) = \mu_1(d) = 0$, $\mu_1(c) = 1$,
- $\mu_2(a) = \mu_2(b) = \mu_2(c) = 1$, $\mu_2(d) = 0$,
- $\mu_3(a) = \mu_3(b) = 0$, $\mu_3(c) = \mu_3(d) = 1$,
- $\mu_4(a) = \mu_4(c) = \mu_4(d) = 1$, $\mu_4(b) = 0$.

Clearly, $(X, T_1)$ and $(X, T_2)$ are fuzzy topological spaces. Also, we can easily see that they are both fuzzy $G_\delta$-connected spaces (since both $(X, T_1)$ and $(X, T_2)$ have no proper fuzzy $G_\delta F_\sigma$-sets).

Also, in fuzzy topological space $(X, T_1)$, there is no such $T_1$-fuzzy $G_\delta$, $T_1$-fuzzy $F_\sigma$-set and also there is no such $T_2$-fuzzy $G_\delta$, $T_2$-fuzzy $F_\sigma$-set in $(X, T_2)$. Therefore, the fuzzy bitopological space $(X, T_1, T_2)$ is pairwise fuzzy $G_\delta$-basically disconnected even though both $(X, T_1)$ and $(X, T_2)$ are fuzzy $G_\delta$-connected.

**Example 2.5.2.** Let $X = \{a, b, c\}$. Suppose

- $T_1 = \{0, 1, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\}$
and $T_2 = \{0, 1\}$, where $\lambda_i : X \to [0, 1], i = 1$ to 6 are defined as follows:

\[
\begin{align*}
\lambda_1(a) &= \lambda_1(b) = 1, \quad \lambda_1(c) = 0, \\
\lambda_2(b) &= \lambda_2(c) = 1, \quad \lambda_2(a) = 0, \\
\lambda_3(a) &= \lambda_3(c) = 1, \quad \lambda_3(b) = 0, \\
\lambda_4(a) &= 1, \quad \lambda_4(b) = \lambda_4(c) = 0, \\
\lambda_5(a) &= \lambda_5(c) = 0, \quad \lambda_5(b) = 1, \\
\lambda_6(a) &= \lambda_6(b) = 0, \quad \lambda_6(c) = 1.
\end{align*}
\]

Clearly, $(X, T_1)$ is a fuzzy topological space and $(X, T_2)$ is the indiscrete fuzzy topological space. Clearly, $(X, T_1)$ is a fuzzy $G_\delta$-disconnected space and $(X, T_2)$ is a fuzzy $G_\delta$-connected space.

We claim the fuzzy bitopological space $(X, T_1, T_2)$ is a pairwise fuzzy $G_\delta$-basically disconnected space.

Let $\lambda$ be any non-zero $T_1$-fuzzy $G_\delta$, $T_1$-fuzzy $F_\sigma$-set. Then $\text{cl}_{\sigma(T_2)} \lambda = 1$ which is clearly $T_1$-fuzzy $G_\delta$. Similarly, we can see that $\text{cl}_{\sigma(T_1)} \mu = 1$ whenever $\mu$ is a non-zero $T_2$-fuzzy $G_\delta$, fuzzy $F_\sigma$-set and clearly $\text{cl}_{\sigma(T_1)} \mu$ is $T_2$-fuzzy $G_\delta$.

Therefore, the fuzzy bitopological space $(X, T_1, T_2)$ is pairwise fuzzy $G_\delta$-basically disconnected space.

Remark 2.5.1. Every pairwise fuzzy $G_\delta$-extremally disconnected space is pairwise fuzzy $G_\delta$-basically disconnected, but the converse is not true as shown in Example 2.5.3.

Example 2.5.3. Let $X = \{a, b\}$. Suppose $T_1 = \{0, 1, \lambda_1, \lambda_2\}$ and $T_2 = \{0, 1, \mu_1, \mu_2\}$, where $\lambda_i : X \to [0, 1], i = 1, 2$ and $\mu_j : X \to [0, 1], j = 1, 2$ are defined as follows:

\[
\begin{align*}
\lambda_1(a) &= 1/2, \quad \lambda_1(b) = 1, \\
\lambda_2(a) &= 0, \quad \lambda_2(b) = 1/3, \\
\mu_1(a) &= 1/2, \quad \mu_1(b) = 1/3, \\
\mu_2(a) &= 1/4, \quad \mu_2(b) = 1/3.
\end{align*}
\]

Then clearly, $(X, T_1)$ and $(X, T_2)$ are fuzzy topological spaces. Also, we can easily say in the fuzzy topological space $(X, T_1)$ there is no such $T_1$-fuzzy $G_\delta$, $T_1$-fuzzy $F_\sigma$-set and also there is no such $T_2$-fuzzy
For any fuzzy bitopological space \((X, T_2, T_2)\). Therefore, the fuzzy bitopological space \((X, T_1, T_2)\) is pairwise fuzzy \(G_\delta\)-basically disconnected but not pairwise fuzzy \(G_\delta\)-extremally disconnected.

**Proposition 2.5.1.** For any fuzzy bitopological space \((X, T_1, T_2)\), the following are equivalent.

(a) \((X, T_1, T_2)\) is pairwise fuzzy \(G_\delta\)-basically disconnected.

(b) Whenever \(\lambda\) is a \(T_1\)-fuzzy \(G_\delta\) and \(T_1\)-fuzzy \(F_\sigma\)-set, \(\text{int}_{\delta(T_1)} \text{cl}_{\sigma(T_2)}\) \(\lambda\) is \(T_2\)-fuzzy \(F_\sigma\). Similar statement holds when \(\lambda\) is a \(T_2\)-fuzzy \(G_\delta\) and \(T_2\)-fuzzy \(F_\sigma\)-set.

(c) Whenever \(\lambda\) is a \(T_1\)-fuzzy \(G_\delta\) and \(T_1\)-fuzzy \(F_\sigma\)-set, we have \(\text{cl}_{\sigma(T_1)}\) \(\lambda\) \(\leq\) \(1 - \text{cl}_{\sigma(T_2)}(1 - \text{cl}_{\sigma(T_2)} \lambda)\). Similar statement holds when \(\lambda\) is a \(T_2\)-fuzzy \(G_\delta\) and \(T_2\)-fuzzy \(F_\sigma\)-set.

(d) Whenever \(\lambda\) is a \(T_1\)-fuzzy \(G_\delta\)-set and \(\mu\) is a \(T_2\)-fuzzy \(G_\delta\)-set such that \(\lambda + \mu \leq 1\) and \(\lambda\) being a \(T_1\)-fuzzy \(F_\sigma\)-set or \(\mu\) being a \(T_2\)-fuzzy \(F_\sigma\)-set, we have \(\text{cl}_{\sigma(T_1)} \lambda + \text{cl}_{\sigma(T_1)} \mu \leq 1\).

**Proof.** (a) \(\Rightarrow\) (b). Let \(\lambda\) be any \(T_1\)-fuzzy \(G_\delta\) and \(T_1\)-fuzzy \(F_\sigma\)-set. Now,

\[
\text{int}_{\delta(T_1)} \text{cl}_{\sigma(T_2)}\lambda = 1 - \text{cl}_{\sigma(T_1)}(1 - \text{cl}_{\sigma(T_2)} \lambda).
\]

By (a), \(\text{cl}_{\sigma(T_2)}\lambda\) is \(T_1\)-fuzzy \(G_\delta\) and therefore from (8) it follows that \(\text{int}_{\delta(T_1)}\) \(\text{cl}_{\sigma(T_2)}\lambda\) is \(T_2\)-fuzzy \(F_\sigma\). Similar argument holds when \(\lambda\) is a \(T_2\)-fuzzy \(G_\delta\) and \(T_2\)-fuzzy \(F_\sigma\)-set.

(b) \(\Rightarrow\) (c). Let \(\lambda\) be any \(T_1\)-fuzzy \(G_\delta\) and \(T_1\)-fuzzy \(F_\sigma\)-set and suppose that \(\text{cl}_{\sigma(T_2)}\lambda \not\leq 1 - \text{cl}_{\sigma(T_1)}(1 - \text{cl}_{\sigma(T_2)} \lambda)\). Then there exists an \(x \in X\) such that \(\text{cl}_{\sigma(T_2)}\lambda(x) \not\leq (1 - \text{cl}_{\sigma(T_1)}(1 - \text{cl}_{\sigma(T_2)} \lambda))(x)\). Now by (b), \(\text{int}_{\delta(T_1)}\) \(\text{cl}_{\sigma(T_2)}\lambda\) is \(T_2\)-fuzzy \(F_\sigma\). Also, \(\text{cl}_{\sigma(T_1)}(1 - \text{cl}_{\sigma(T_2)} \lambda) = 1 - \text{int}_{\delta(T_1)}\) \(\text{cl}_{\sigma(T_2)}\lambda\). Hence it follows that

\[
\text{cl}_{\sigma(T_2)}\lambda(x) \not\leq (1 - \text{int}_{\delta(T_1)}\text{cl}_{\sigma(T_2)}\lambda)(x) \not\leq \text{int}_{\delta(T_1)}\text{cl}_{\sigma(T_2)}\lambda(x)
\]

which is not possible; for by (b), \(\text{int}_{\delta(T_1)}\) \(\text{cl}_{\sigma(T_2)}\lambda\) is \(T_2\)-fuzzy \(F_\sigma\) containing \(\lambda\). Hence, \(\text{cl}_{\sigma(T_2)}\lambda \leq 1 - \text{cl}_{\sigma(T_1)}(1 - \text{cl}_{\sigma(T_2)} \lambda)\). Similar proof holds when \(\lambda\) is a \(T_2\)-fuzzy \(G_\delta\) and \(T_2\)-fuzzy \(F_\sigma\)-set.

(c) \(\Rightarrow\) (d). Let \(\lambda\) be any \(T_1\)-fuzzy \(G_\delta\), \(T_1\)-fuzzy \(F_\sigma\)-set and \(\mu\) be any \(T_2\)-fuzzy \(G_\delta\)-set such that \(\lambda + \mu \leq 1\). We know that \(\mu \leq 1 - \text{cl}_{\sigma(T_2)}\lambda\) and \(\lambda \leq 1 - \text{cl}_{\sigma(T_1)}\mu\). But by hypothesis, \(\text{cl}_{\sigma(T_1)}\lambda \leq 1 - \text{cl}_{\sigma(T_1)}(1 - \text{cl}_{\sigma(T_2)} \lambda)\)
and therefore \( \mu \leq 1 - \text{cl}_{\sigma(T_2)} \lambda \). Since \( \text{cl}_{\sigma(T_1)} \mu \) is the smallest \( T_1 \)-fuzzy \( F_\sigma \)-set containing \( \mu \), we have

\[
\text{cl}_{\sigma(T_1)} \mu \leq \text{cl}_{\sigma(T_1)}(1 - \text{cl}_{\sigma(T_2)} \lambda).
\]

Also, since \( \text{cl}_{\sigma(T_2)} \lambda + \text{cl}_{\sigma(T_1)}(1 - \text{cl}_{\sigma(T_2)} \lambda) \leq 1 \), it follows from (9) that \( \text{cl}_{\sigma(T_2)} \lambda + \text{cl}_{\sigma(T_1)} \mu \leq 1 \).

(d) \( \Rightarrow \) (a). Let \( \lambda \) be any \( T_1 \)-fuzzy \( G_\delta \), \( T_1 \)-fuzzy \( F_\sigma \)-set. We shall show that \( \text{cl}_{\sigma(T_2)} \lambda \) is \( T_1 \)-fuzzy \( G_\delta \). Let \( \mu = 1 - \text{cl}_{\sigma(T_2)} \lambda \). Clearly, \( \mu \) is \( T_2 \)-fuzzy \( G_\delta \) and \( \mu + \lambda \leq 1 \). Hence by (d), we have \( \text{cl}_{\sigma(T_2)} \lambda + \text{cl}_{\sigma(T_1)} \mu \leq 1 \) and therefore by construction of \( \mu \), we have \( 1 - \text{cl}_{\sigma(T_1)} \mu = \text{cl}_{\sigma(T_2)} \lambda \). This shows \( \text{cl}_{\sigma(T_2)} \lambda \) is \( T_1 \)-fuzzy \( G_\delta \). Similarly, we can show for any \( T_2 \)-fuzzy \( G_\delta \) and \( T_2 \)-fuzzy \( F_\sigma \)-set \( \lambda \), \( \text{cl}_{\sigma(T_1)} \lambda \) is \( T_2 \)-fuzzy \( G_\delta \).

**Proposition 2.5.2.** Let \( (X, T_1, T_2) \) be a pairwise fuzzy \( G_\delta \)-basically disconnected space and let \( (Y, T_1/Y, T_2/Y) \) be any pairwise fuzzy subspace of \( (X, T_1, T_2) \). Then \( (Y, T_1/Y, T_2/Y) \) is pairwise fuzzy \( G_\delta \)-basically disconnected.

**Proof.** Let \( \lambda_1 \) and \( \lambda_2 \) be \( T_1/Y \)-fuzzy \( G_\delta \)-set and \( T_2/Y \)-fuzzy \( G_\delta \)-set in \( (Y, T_1/Y, T_2/Y) \) respectively such that \( \lambda_1 + \lambda_2 \leq 1 \) and suppose that \( \lambda_1 \) is \( T_1/Y \)-fuzzy \( F_\sigma \)-set. Define \( \lambda_1^1 : X \rightarrow [0, 1] \) and \( \lambda_2^2 : X \rightarrow [0, 1] \) on \( X \) as follows:

\[
\lambda_1^1(x) = \begin{cases} 
\lambda_1(x), & \text{if } x \in X, \\
0, & \text{otherwise},
\end{cases}
\]

and

\[
\lambda_2^2 = \begin{cases} 
\lambda_2(x), & \text{if } x \in X, \\
0, & \text{otherwise}.
\end{cases}
\]

We know that \( \lambda_1^1 \) and \( \lambda_2^2 \) are \( T_1 \)-fuzzy \( G_\delta \)-set and \( T_2 \)-fuzzy \( G_\delta \)-set respectively such that \( \lambda_1^1 + \lambda_2^2 \leq 1 \) and that \( \lambda_1^1 \) is \( T_1 \)-fuzzy \( F_\sigma \)-set. Since \( (X, T_1, T_2) \) is pairwise fuzzy \( G_\delta \)-basically disconnected, it follows that \( \text{cl}_{\sigma(T_2)}(\lambda_1^1) + \text{cl}_{\sigma(T_1)}(\lambda_2^2) \leq 1 \) and this in turn implies

\[
\text{cl}_{\sigma(T_2/Y)}(\lambda_1^1) + \text{cl}_{\sigma(T_1/Y)}(\lambda_2^2) \leq 1.
\]

We arrive at the same conclusion when we assume \( \lambda_2 \) is \( T_2/Y \)-fuzzy \( F_\sigma \)-set. Hence the proposition holds. \( \square \)
Proposition 2.5.3. The fuzzy bitopological sum of a family of disjoint pairwise fuzzy $G_\delta$-basically disconnected spaces is pairwise fuzzy $G_\delta$-basically disconnected.

Proof. Let $\{(X_\alpha, T_\alpha, T_\alpha^*) : \alpha \in \Delta\}$ be a family of disjoint pairwise fuzzy $G_\delta$-basically disconnected spaces. Let $(X, \oplus_{\alpha \in \Delta} T_\alpha, \oplus_{\alpha \in \Delta} T_\alpha^*)$ be the fuzzy bitopological sum of these spaces. Let $\lambda_1$ and $\lambda_2$ be $\oplus_{\alpha \in \Delta} T_\alpha$-fuzzy $G_\delta$ and $\oplus_{\alpha \in \Delta} T_\alpha^*$-fuzzy $G_\delta$-sets in $X$ respectively such that $\lambda_1 + \lambda_2 \leq 1$. Also, we shall assume that $\lambda_1$ is $\oplus_{\alpha \in \Delta} T_\alpha$-fuzzy $F_\sigma$-set. Now, from the assumptions, it is clear that $\lambda_1/X_\alpha$ and $\lambda_2/X_\alpha$ are $T_\alpha$-fuzzy $G_\delta$-set and $T_\alpha^*$-fuzzy $G_\delta$-set in $X_\alpha$ respectively for each $\alpha \in \Delta$. Also, $\lambda_1/X_\alpha + \lambda_2/X_\alpha \leq 1$ and $\lambda_1/X_\alpha$ is $T_\alpha$-fuzzy $F_\sigma$-set in $X_\alpha$. Since $(X_\alpha, T_\alpha, T_\alpha^*)$ is pairwise fuzzy $G_\delta$-basically disconnected, we have

$$\text{cl}_{\sigma(T_\alpha^*)}(\lambda_1/X_\alpha) + \text{cl}_{\sigma(T_\alpha)}(\lambda_2/X_\alpha) \leq 1, \quad \alpha \in \Delta.$$ 

Hence,

$$\text{cl}_{\sigma(\oplus_{\alpha \in \Delta} T_\alpha^*)}(\lambda_1) + \text{cl}_{\sigma(\oplus_{\alpha \in \Delta} T_\alpha)}(\lambda_2) \leq 1.$$

This proves that the fuzzy bitopological sum is a pairwise fuzzy $G_\delta$-basically disconnected space. \qed

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E. Roja and M. K. Uma
Department of Mathematics
Sri Sarada College for Women
Salem 636 016
Tamil Nadu, India.

G. Balasubramanian
Department of Mathematics
Periyar University, Salem 636 011
Tamil Nadu, India.