A STUDY ON WEAK BI-IDEALS OF NEAR-RINGS

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Abstract. From the notion of bi-ideals in near-rings, various generalizations of regularity conditions have been studied. In this paper, we generalize further the notion of bi-ideals and introduce the notion of weak bi-ideals in near-rings and obtain some characterizations using this concept in left self distributive near-rings.

1. Introduction

In this paper, by a near-ring we mean a right near-ring. For basic definitions and notations, we may refer to Pilz [3]. Tamizh Chelvam and Ganesan [4] introduced the notion of bi-ideals in near-rings. Further Tamizh Chelvam [5] introduced the concept of b-regular near-rings and obtained equivalent conditions for regularity in terms of bi-ideals. In this paper the notion weak bi-ideals has been introduced and studied to the extent possible.

Let N be a right near-ring. For two subsets A and B of N, AB = \{ab | a ∈ A, b ∈ B\} and A*B = \{a_1(a_2 + b) - a_1a_2 | a_1, a_2 ∈ A and b ∈ B\}. A subgroup B of (N, +) is said to be a bi-ideal of N if BNB \cap (BN)*B ⊆ B [4]. In the case of a zero-symmetric near-ring, a subgroup B of (N, +) is a bi-ideal if BNB ⊆ B. A subgroup Q of (N, +) is called a quasi-ideal of N if QN \cap NQ \cap N*Q ⊆ Q [4]. If N is zero-symmetric, a subgroup Q of (N, +) is a quasi-ideal of N if QN \cap NQ ⊆ Q.

A near-ring N is said to be left (right)-unital if a ∈ Na(a ∈ aN) for all a ∈ N. A near-ring N is said to be unital if it is both left as well as right unital. An element a ∈ N is said to be regular if a = aba for some b ∈ N. A near-ring N is said to be regular if every element in N is regular. It may be noted that a regular near-ring is a unital near-ring, but not the converse. An element a ∈ N is said to be strongly regular if a = ba^2, for some b ∈ N. A near-ring N is called strongly regular if every element in N is strongly regular. N is said to satisfy IFP (Insertion of Factors Property) if ab = 0 implies axb = 0 for all x ∈ N.

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A near-ring is called left bi-potent if $Na = Na^2$ for $a \in N$. A subgroup $M$ of $(N, +)$ is said to be a left (right) $N$-subgroup if $NM \subseteq M(MN \subseteq M)$. A near-ring $N$ is said to be two sided if every left $N$-subgroup is a right $N$-subgroup and vice versa. A near-ring $N$ is called $b$-regular near-ring if $a \in (a)_N(N(a))$ for every $a \in N$ where $(a)_N$ is the right (left) $N$-subgroup generated by $a \in N$.[6]. Note that every regular near-ring is $b$-regular.

A near-ring $N$ is said to be left self distributive if $abc = abac$ for all $a, b, c \in N$. Let $E$ be the set of all idempotents of $N$ and $L$ the set of all nilpotent elements of $N$.

2. A Study on Weak Bi-ideals

In this section, we introduce weak bi-ideals and obtain some of the properties of this concept.

**Definition 2.1.** A subgroup $B$ of $(N, +)$ is said to be a weak bi-ideal if $B^3 \subseteq B$.

**Example 2.2.** Every bi-ideal is a weak bi-ideal, but the converse is not true. For, consider the near-ring $N$ constructed on the Klein’s 4-group according to the scheme $(0, 0, 2, 1)$ (p. 408, Pilz [3]). In this near-ring, one can check that $\{0, b\}$ and $\{0, c\}$ are weak bi-ideals. Note that $\{0, b\}N\{0, b\} = \{0, c, b\}$ and hence $\{0, b\}$ is not a bi-ideal of $N$.

**Proposition 2.3.** If $B$ is a weak bi-ideal of a near-ring $N$ and $S$ is a sub near-ring of $N$, then $B \cap S$ is a weak bi-ideal of $N$.

**Proof.** Let $C = B \cap S$. Now $C^3 = (B \cap S)((B \cap S)(B \cap S)) \subseteq (B \cap S)(BB \cap SS) \subseteq (B \cap S)BB \cap (B \cap S)SS \subseteq BBB \cap SSS = B^3 \cap SS \subseteq B \cap S = C$, i.e., $C^3 \subseteq C$. Therefore $C$ is a weak bi-ideal of $N$. □

**Proposition 2.4.** Let $B$ be a weak bi-ideal of $N$. Then $Bb$ and $b'B$ are the weak bi-ideals of $N$ where $b, b' \in B$ and $b'$ is a distributive element.

**Proof.** Clearly $Bb$ is a subgroup of $(N, +)$. Also $(Bb)^3 = BbBbBb \subseteq BbB \subseteq Bb$. Since $b'$ is distributive, $b'B$ is a subgroup of $(N, +)$ and $(b'B)^3 = b'Bb'Bb'B \subseteq b'BBB \subseteq b'B^3 \subseteq b'B$. Hence $Bb$ and $b'B$ are weak bi-ideals of $N$. □

**Corollary 2.5.** Let $B$ be a weak bi-ideal of $N$. For $b, c \in B$, if $b$ is distributive, then $bBc$ is a weak bi-ideal of $N$.

**Proposition 2.6.** Let $N$ be a left self-distributive left-unital near-ring. Then $B^3 = B$ for every weak bi-ideal $B$ of $N$ if and only if $N$ is strongly regular.

**Proof.** Let $B$ be a weak bi-ideal of $N$. If $N$ is strongly regular, then $N$ has no non-zero nilpotent elements. This further implies that $N$ has IFP. Let $x \in N$ and $x = ax^2$ for $a \in N$. Now $(xax - x)x = 0$ and so $x(xax - x) = 0$ as $N$ has IFP. Hence $(xax - x)^2 = 0$ and so $xax - x = 0$. i.e., $x$ is regular.
and \( N \) is regular. Let \( b \in B \). Since \( N \) is regular, \( b = bab \) for some \( a \in N \).

By our assumption that \( N \) is left self-distributive, we have \( bab = babb \). Thus \( b = bab = babb = bbb = b^3 \subseteq B^3 \), i.e., \( B \subseteq B^3 \). Hence \( B = B^3 \) for every weak bi-ideal \( B \) of \( N \). Conversely let \( a \in N \). Since \( Na \) is a weak bi-ideal of \( N \) and \( N \) is a left-unital near-ring, we get \( a \in Na = (Na)^3 = NaNaNa \subseteq NaNa \), i.e., \( a = n_1an_2a \). Since \( N \) is left self-distributive, \( a = n_1an_2a \), i.e., \( N \) is strongly regular.

**Proposition 2.7.** Let \( N \) be a left self-distributive left unital near-ring. Then \( B = NB^2 \) for every strong bi-ideal \( B \) of \( N \) if and only if \( N \) is strongly regular.

**Proof.** Assume that \( B = NB^2 \) for every strong bi-ideal \( B \) of \( N \). Since \( Na \) is a strong bi-ideal of \( N \) and \( N \) is a left unital near-ring, we have \( a \in Na = N(Na)^2 = NNaNa \subseteq NaNa \), i.e., \( a = n_1an_2a \). Since \( N \) is a left self-distributive near-ring, \( a = n_1an_2a = n_1an_2a \in Na^2 \), i.e., \( N \) is strongly regular. Conversely, let \( B \) be a strong bi-ideal of \( N \). Since \( N \) is strongly regular, for \( b \in B \), \( b = nb^2 \in NB^2 \), i.e., \( B \subseteq NB^2 \). Hence \( NB^2 = B \) for every strong bi-ideal \( B \) of \( N \).

**Theorem 2.8.** Let \( N \) be a left self-distributive left unital near-ring. Then \( B^3 = B \) for every weak bi-ideal \( B \) of \( N \) if and only if \( NB^2 = B \) for every strong bi-ideal \( B \) of \( N \).

**Proof.** Follows from the Propositions 2.6 and 2.7.

**Proposition 2.9.** Let \( N \) be a left self-distributive left-unital near-ring. Then \( B = BNB \) for every bi-ideal \( B \) of \( N \) if and only if \( N \) is regular.

**Proof.** Let \( B \) be a bi-ideal of \( N \). If \( N \) is regular, then \( B = BNB \) for every bi-ideal \( B \) of \( N \). Conversely, let \( B = BNB \) for every bi-ideal \( B \) of \( N \). Since \( Na \) is a bi-ideal of \( N \) and \( N \) is a left-unital near-ring, we have \( a \in Na = NaNaNa \subseteq NaNa \), i.e., \( a = n_1an_2a \). Since \( N \) is a left self-distributive near-ring, \( a = n_1an_2a = n_1an_2a \in Na^2 \), i.e., \( N \) is strongly regular and as in the proof of Proposition 2.6, \( N \) is regular.

**Proposition 2.10.** Let \( N \) be a left self-distributive left-unital near-ring. Then \( B = B^3 \) for every weak bi-ideal \( B \) of \( N \) if and only if \( A \cap C = AC \) for any two left \( N \)-subgroups \( A \) and \( C \) of \( N \).

**Proof.** Assume that \( B = B^3 \) for every weak bi-ideal \( B \) of \( N \). By the Proposition 2.6, \( N \) is strongly regular. Therefore \( N \) is regular. Let \( A \) and \( C \) be any two left \( N \)-subgroups of \( N \). Let \( x \in A \cap C \). Since \( N \) is regular, \( x = xzx \) for some \( a \in N \). Therefore \( (xa)x \in ANC \subseteq AC \) which implies that \( A \cap C = AC \).

On the other hand, let \( x \in AC \). Since \( N \) is strongly regular, \( L = 0 \) and so \( en = ne \) for all \( e \in E \). Then \( x = yz \in AC \) with \( y \in A \) and \( z \in C \). Now \( x = yz = (yb)z \). Since \( by \) is an idempotent element \( (by)z = (by)z(by) \). Thus \( x = yz = (yb)z \in NA \subseteq A \). Thus \( x \in A \cap C \). From the two inclusions proved above, we get that \( AC = A \cap C \).
Conversely let $a \in N$. Since $Na$ is a left $N$-subgroup of $N$, from the assumption we get that $Na = Na \cap Na = NaNa$. But $Na = Na \cap N = NaN$ implies that $Na = NaNa$. Therefore $Na = Na^2$. Since $N$ is a left-unital near-ring, $a \in Na = Na^2$, i.e., $N$ is strongly regular. By the Proposition 2.6, $B = B^3$ for every weak bi-ideal $B$ of $N$. □

**Theorem 2.11.** Let $N$ be a left self-distributive left unital near-ring. Then the following conditions are equivalent.

(i) $B = B^3$ for every weak bi-ideal $B$ of $N$.

(ii) $N$ is regular and $NxNy = NyNx$ for all $x, y \in N$.

(iii) $NxNy = Nxy$ for all $x, y \in N$.

(iv) $N$ is left bi-potent.

(v) $N$ is Boolean.

Proof. (i) $\Rightarrow$ (ii) Assume that $B = B^3$ for every weak bi-ideal $B$ of $N$. By the Proposition 2.6, $N$ is strongly regular and so $N$ is regular. Again by the Proposition 2.10, $A \cap B = AB$ for two left $N$-sub groups $A$ and $B$ of $N$. Let $x, y \in N$. Since $Nx$ and $Ny$ are left $N$-sub groups of $N$, from the above fact we get that $NxNy = Nx \cap Ny = Ny \cap Nx = NyNx$.

(ii) $\Rightarrow$ (iii) Let $x, y \in N$. Let $A$ be a left $N$-subgroup of $N$. Trivially, $A^2 \subseteq A$. Since $N$ is regular, for any $a \in N, a = aba$ for some $b \in N$. Hence $a = a(ba) \in A(NA) \subseteq AA = A^2$. Thus $A = A^2$. Since $Nx \cap Ny$ is a left $N$-subgroup of $N$, $Nx \cap Ny = (Nx \cap Ny)^2 \subseteq NxNy \subseteq Ny$. Again by the assumption, $NxNy = NyNx \subseteq Nx$. Therefore $Nx \cap Ny = NxNy$. Now $Nx = Nx \cap N = NxN$ and from this we get that $Nxy = Nxy$. Therefore $Nxy = Nx \cap Ny$ for all $x, y \in N$.

(iii) $\Rightarrow$ (iv) Let $a \in N$. Then $Na = Na \cap Na = NaNa = Na^2$. i.e., $N$ is left bi-potent near-ring.

(iv) $\Rightarrow$ (v) By the assumption that $a \in Na = Na^2, N$ is strongly regular and so $N$ is regular. Let $x \in N$. Then $x = xxy = xyy = x^2$, i.e., $N$ is Boolean.

(v) $\Rightarrow$ (i) Let $B$ be a weak bi-ideal of $N$. Let $x \in B$. By the assumption, $x = x^2 = x^3 \in B^3$. Therefore $B \subseteq B^3$ and hence $B = B^3$. □

**References**


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