A NOTE ON FUNCTIONAL LIMIT THEOREM FOR THE INCREMENTS OF FBM IN SUP-NORM

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Abstract. In this paper, using large deviation results for Gaussian processes, we establish some functional limit theorems for increments of a fractional Brownian motion in the usual sup-norm via estimating large deviation probabilities for increments of a fractional Brownian motion.

1. Introduction and Results

The analogue of Strassen law of functional limit theorems is known for many Gaussian processes which have suitable scaling properties. Mueller [10] and Chen and Csörgő [4] studied functional modulus of continuity of a Wiener process. Révész [12] obtained functional large increment theorem which generalized Strassen’s functional LIL of a Wiener process. Wang [13] obtained functional limit results for increments of a fractional Brownian motion (FBM). Goodman and Kuelbs [5], Kuelbs, Li and Talagrand [8] and Monrad and Rootzén [9] investigated functional LIL for general Gaussian process. Let \( \{X(t), t \geq 0\} \) be FBM of order \( 2\alpha \) with \( 0 < \alpha < 1 \) and \( X(0) = 0 \), then \( \{X(t), t \geq 0\} \) has a covariance function

\[
\Gamma(s, t) = E(X(s)X(t)) = \frac{1}{2} (s^{2\alpha} + t^{2\alpha} - |s-t|^{2\alpha})
\]

for \( s, t \geq 0 \), and we have the representation

\[
X(t) = \int_{\mathbb{R}} \frac{1}{q_\alpha} \{ |x-t|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2} \} dW(x),
\]

where

(a) \( q_\alpha^2 = \int_{\mathbb{R}} \{ |x-1|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2} \}^2 dx \),

(b) \( \{W(t), -\infty < t < \infty\} \) is a Wiener process.
\( \{X(t), t \geq 0\} \) has stationary increments with \( E(X(s+t) - X(s))^2 = t^{2\alpha}, \ t \geq 0 \) and is a Wiener process when \( \alpha = 1/2 \).

Let \( C_0[0,1] \) be the space of continuous functions on \([0,1]\) with value zero at the origin and \( \|\cdot\|_{\infty} \) be the usual sup-norm on \( C_0[0,1] \). The limit set associated with functional LIL for \( \{X(t), t \geq 0\} \) is \( K_{\alpha} \), the subset of functions in \( C_0[0,1] \) with the form

\[
f(t) = \int_{\mathbb{R}} \frac{1}{q_{\alpha}} \left( |x-t|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2} \right) g(x) \, dx, \quad 0 \leq t \leq 1.
\]

Here the function \( g(\cdot) \) range over the unit ball of \( L^2(\mathbb{R}) \), and hence \( \int_{\mathbb{R}} g^2(x) \, dx \leq 1 \) (cf. [7]). Let \( H_\alpha \subseteq C_0 \) be the reproducing kernel Hilbert space (RKHS) of the kernel \( \Gamma(s,t), 0 \leq s, t \leq 1 \). \( H_\alpha \) is the RKHS corresponding to the centered Gaussian measure \( \mu \) on separable Banach space \( C_0[0,1] \) under the sup-norm induced by \( \{X(t), 0 \leq t \leq 1\} \), then the set \( K_{\alpha} \) is the unit ball of \( H_\alpha \). If \( f \in H_\alpha \), then \( \|f\|_\alpha \) denotes the \( H_\alpha \)-norm of \( f \), and \( |f(t) - f(s)| \leq |t-s|^{\alpha} \|f\|_\alpha \) for all \( s, t \in [0,1] \). Throughout this paper put \( K = K_{\alpha} \) and \( \log x = \log_0(x \vee 1) \).

For any \( \varepsilon > 0 \), let \( P_\varepsilon \) be probability measure on \( C_0[0,1] \) corresponding to \( W = \{ \sqrt{2}W(t), 0 \leq t < \infty \} \). Define the mapping \( I : C_0[0,1] \to [0, \infty) \) by

\[
I(f) = \begin{cases} 
\frac{1}{2} \int_0^1 (f'(x))^2 \, dx, & \text{if } f \text{ is an absolutely continuous function} \\
\infty, & \text{otherwise},
\end{cases}
\]

and the subset \( S \) of \( C_0[0,1] \) by \( S = \{ f \in C_0[0,1] : I(f) \leq 1/2 \} \), then it is easy to show that \( I(f) \) is lower semi-continuous and \( \{ f \in C_0[0,1] : I(f) \leq a \} \) is compact for any fixed \( a > 0 \).

Let \( a_T (0 < a_T \leq T) \) be a continuous function such that

(i) \( a_T \) is nondecreasing,

(ii) \( T/a_T \) is nondecreasing,

(iii) \( \lim_{T \to \infty} \frac{T}{a_T \log \log T} = \infty \).

For \( t \in [0, T - a_T] \), define

\[
Y_{t,T}(x) = \frac{W(t + xa_T) - W(t)}{\sqrt{a_T \gamma_T}}, \quad 0 \leq x \leq 1,
\]

\[
Z_{t,T}(x) = \frac{X(t + xa_T) - X(t)}{a_T^2 \gamma_T}, \quad 0 \leq x \leq 1,
\]

where

\[
\gamma_T = \left( \frac{2 \log \frac{T}{a_T \log \log T}}{a_T \log \log T} \right)^{1/2}.
\]

The following theorem is in [3].
Theorem A. Let \( \{ W(t), 0 \leq t < \infty \} \) be a standard Wiener process. Assume that \( 0 < a_T \leq T \) satisfies conditions (i), (ii) and (iii). Then
\[
\liminf_{T \to \infty} \sup_{0 \leq t \leq T - a_T} \inf_{f \in S} \| Y_{t,T}(\cdot) - f(\cdot) \|_{\infty} = 0 \quad \text{a.s.}
\]
and for any \( f \in S \)
\[
\lim_{T \to \infty} \inf_{0 \leq t \leq T - a_T} \| Y_{t,T}(\cdot) - f(\cdot) \|_{\infty} = 0 \quad \text{a.s.}
\]

Using large deviation estimates for Wiener processes in Hölder norm (see [1]), Wei [14] generalized to \( k \)-dimensional Wiener processes. Wang [13] investigated functional limit results for small and large increments of FBM in the sup-norm, but he only obtained the results related to Csörgő and Révész’s type limit theorems. In this paper, we will consider the functional limit theorem with the normalizing factor of Csáki and Révész’s type. Using large deviation results for Gaussian processes, we establish some functional limit theorems for increments of FBM in the sup-norm. Our main result is as follows:

Theorem 1.1. Let \( \{ X(t), t \geq 0 \} \) be FBM of order \( 2\alpha \) with \( 0 < \alpha < 1 \) and \( X(0) = 0 \). Assume that \( 0 < a_T \leq T \) satisfies conditions (i), (ii) and (iii). Then we have
\[
\liminf_{T \to \infty} \sup_{0 \leq t \leq T - a_T} \inf_{f \in K} \| Z_{t,T}(\cdot) - f(\cdot) \|_{\infty} = 0 \quad \text{a.s.}
\]
and for any \( f \in K \)
\[
\lim_{T \to \infty} \inf_{0 \leq t \leq T - a_T} \| Z_{t,T}(\cdot) - f(\cdot) \|_{\infty} = 0 \quad \text{a.s.}
\]

2. Some lemmas

Let \( B \) denote a real separable Banach space with norm \( \| \cdot \| \) and topological dual \( B^* \), and let \( \mu \) be a centered Gaussian measure on \( B \). Let \( H_\mu \) be the Hilbert space which generates the Gaussian measure \( \mu \) on \( B \) under the norm \( \| \cdot \| \). Define continuous operators
\[
\Pi_d(x) = \sum_{k=1}^{d} \alpha_k(x)\Delta_{\alpha_k} \quad \text{and} \quad Q_d(x) = x - \Pi_d(x) \quad (d \geq 1)
\]
taking \( B \) into \( B \). Here \( \{ \alpha_k; k \geq 1 \} \) is a sequence in \( B^* \) orthonormal in \( L^2(\mu) \), and \( \{ \Delta_{\alpha_k}; k \geq 1 \} \) is a complete orthonormal system in \( H_\mu \) defined by the Bochner integral \( \Delta_{\alpha_k} = \int_B x\alpha_k(x)\mu(dx) \). If \( \xi \) is a \( B \)-valued random vector with a mean zero Gaussian measure \( \mu \), then it is well known [6] that \( \lim_{d \to \infty} \| Q_d(\xi) \| = 0 \) with probability one, and
\[
E\| Q_d(\xi) \| \downarrow 0 \quad \text{as} \quad d \uparrow \infty.
\]

The following Lemma 2.1 is from [13] (cf. [5]).
Lemma 2.1. Let \( \{\xi(t), t \geq 0\} \) be a centered Gaussian process with values in \( B \). Let \( P \) be the probability measure generated by \( \xi(\cdot) \). Let \( Q_d(d \geq 1) \) be the linear operators of (2.1), \( U_a = \{f \in H_\mu : \|f\|_\mu \leq a\} \), where \( a > 0 \), and \( H_\mu \) the RKHS with \( \mu = \mathcal{L}(\xi) \). Let \( \lambda \geq 1 \) and \( d_\lambda \) be an integer such that

\[
d_\lambda \geq \inf\{m \geq 1 : E\|Q_m(\xi)\|/m \leq 2r \log \lambda/\lambda\},
\]

and \( \varepsilon_\lambda = \gamma d_\lambda \log \lambda/\lambda^2 \) with some constant \( \gamma > 3\tau \), where \( \tau = \sup_{x \in U_a} \|x\| \).

Then for every \( \varepsilon \geq \varepsilon_\lambda \) (where \( \varepsilon \) may depend on \( \lambda \)) we have

\[
P\left( \inf_{f \in U_a} \|\frac{\xi}{\lambda} - f\| \geq \varepsilon \right)
\leq \frac{1}{\sqrt{2\pi(d_\lambda + 1)}} \exp\left( -\frac{(a\lambda(1 + \varepsilon))^2}{2} + \frac{d_\lambda - 1}{2} \log \frac{(a\lambda(1 + \varepsilon))^2 \varepsilon}{d_\lambda - 1} \right)
\]

for any \( \lambda \geq \lambda_0 \) with some \( \lambda_0 > 0 \).

In order to prove Lemma 2.3, we need the following lemmas. The detailed proofs of the following Lemma 2.2 are similar in [11].

Lemma 2.2. For any \( \varepsilon > 0 \) there exists a positive constant \( C = C(\varepsilon) \) such that

\[
P\left( \sup_{0 \leq t \leq T, |t-s| \leq h} \frac{|X(t) - X(s)|}{(t-s)\alpha} \geq x \right) \leq C \frac{T}{h} \exp\left( -\frac{x^2}{2 + \varepsilon} \right)
\]

for any \( x \geq x_0 \) with some \( x_0 > 0 \).

The following Lemma 2.3 is a generalization of Lemma 2.1 in the case of \( B = C_0[0,1] \) with the usual sup-norm \( \| \cdot \|_\infty \) and \( \xi \) is FBM of order \( 2\alpha \) with \( 0 < \alpha < 1 \) and \( X(0) = 0 \). For the proofs we refer the reader to [13].

Lemma 2.3. Let \( \{X(t), t \geq 0\} \) be FBM of order 2\( \alpha \) with 0 < \( \alpha < 1 \) and \( X(0) = 0 \), and \( K \) be defined as in Section 1. Then for any \( \varepsilon > 0 \) there exists a positive number \( \lambda_0 = \lambda_0(\varepsilon) \) such that

\[
P\left( \sup_{0 \leq t \leq T-h} \inf_{f \in K} \left\| \frac{X(t+h) - X(t)}{\lambda h^\alpha} - f(\cdot) \right\|_\infty \geq \varepsilon \right)
\leq C \frac{T}{h} \exp\left( -\frac{(\lambda(1 + \varepsilon/3))^2}{2 + \varepsilon} \right)
\]

for any \( \lambda > \lambda_0 \) and every 0 < \( h \leq T \).

Remark 2.4. From Lemma 2.3 we have

\[
(2.2) \quad P\left( \sup_{0 \leq t \leq 1} \inf_{f \in K} \left\| \frac{X(t+h) - X(t)}{\lambda h^\alpha} - f(\cdot) \right\|_\infty \geq \varepsilon \right)
\leq C \exp\left( -\frac{(\lambda(1 + \varepsilon/3))^2}{2 + \varepsilon} \right).
\]

The following inequalities are well known (see [2], [9]).
Lemma 2.5. Let \( V \) be a convex, symmetric, measurable subset of \( B \). Then for all \( f \in H \)
\[
\mu(V) \geq \mu(f + V) \geq \mu(V) \exp \left\{ -\frac{1}{2} \| f \|_2^2 \right\}.
\]

In order to prove our result, we need the following lemma. The detailed proof can be found in [7].

Lemma 2.6. Let \( 0 < \alpha < 1 \) and fix \( 0 < \eta < \alpha \). Let \( s_k = k^{-\eta} \) and \( d_k = k^{k+1-r} \) for \( k \geq 1 \). Set for \( 0 \leq t \leq 1 \)
\[
H^{(k)}(s_k, t) = \int_{|x| \in (s_k d_{k-1}, s_k d_k]} \frac{1}{q_\alpha} \left\{ |x - s_k f|\left(\frac{2\alpha-1}{2}\right) - |x|\left(\frac{2\alpha-1}{2}\right) \right\} dW(x),
\]
where \( \{W(x), -\infty < x < \infty\} \) is a standard Wiener process. Let \( 0 < \beta < r \). Then for \( \delta = \min \{2\beta(\alpha - \eta), r - \beta, (1 - r)(2 - 2\alpha), (2\alpha - 2\eta)r\} \) there exists a positive constant \( C \) depending on \( \alpha \) such that uniformly in \( t, h, k \)
\[
\sigma^2_k(t, h) = \mathbb{E} \left( H^{(k)}(s_k, t + h) - H^{(k)}(s_k, t) \right)^2 \leq C h^2 s_k^{2\alpha - \delta}.
\]

3. Proof of Theorem 1.1.

Proof of (1.1). If \( \limsup_{T \to \infty} \frac{\log \left( T/a_T \right)}{\log \log \log T} = \infty \), then for any \( M > 0 \) there exists a sequence of positive numbers \( \{T_n\} \) such that
\[
T_n \geq (\log \log T_n)^M
\]
for large \( n \). For any \( \varepsilon > 0 \) we have by Lemma 2.3 and (3.1)
\[
P \left( \inf_{0 \leq t \leq T_n - a_T} \sup_{f \in K} \|Z_t - f\|_\infty \geq \varepsilon \right)
\leq C \frac{T_n}{a_T} \exp \left( - \frac{2(1 + \varepsilon/3)^2}{2 + \varepsilon} \log \frac{T_n}{a_T} \log \log T_n \right)
\leq C (\log \log T_n)^{-1} \longrightarrow 0 \text{ as } n \to \infty.
\]
Hence, by the arbitrariness of \( \varepsilon \), we get (1.1).

Let \( T_n = e^n \) with \( n \geq 1 \). If \( \limsup_{T \to \infty} \frac{\log \left( T/a_T \right)}{\log \log \log T} < \infty \), then for any \( \varepsilon > 0 \) there exists constant \( r_0 > 1 \) with \( r_0 \varepsilon > 4 \) such that
\[
(\log \log T_n)^{r_0 - \varepsilon} \leq \frac{T_n}{a_T} \leq (\log \log T_n)^{r_0 + \varepsilon}
\]
for large \( n \). It suffices to show that
\[
\limsup_{n \to \infty} \sup_{0 \leq t \leq T_n - a_T} \inf_{f \in K} \|Z_t - f\|_\infty = 0 \quad \text{a.s.}
\]
Let $d_k$ be as in Lemma 2.5, and set for $0 \leq t \leq 1$

$$X^{(k)}(t) = \int_{|x| \in (d_{k-1}, d_k]} \frac{1}{q_\alpha} \{ |x - t|^{(2\alpha - 1)/2} - |x|^{(2\alpha - 1)/2} \} dW(x),$$

where $W(\cdot)$ is a Wiener process, then $\tilde{X}^{(k)}(t) = X(t) - X^{(k)}(t)$ and $\{X^{(k)}(t)\}$, $k = 1, 2, \cdots$ are independent. Note that by self-similarity of FBM

$$\sup_{0 \leq t \leq T_n - aT_n} \inf_{f \in F} \|Z_{t,T_n}(\cdot) - f(\cdot)\|_\infty \leq \sup_{0 \leq t \leq T_n - aT_n} \inf_{f \in F} \left\| \frac{X^{(n)}(t + aT_n) - X^{(n)}(t)}{a^2 T_n} \right\|_\infty + \sup_{0 \leq t \leq T_n - aT_n} \left\| \frac{\tilde{X}^{(n)}(t + aT_n) - \tilde{X}^{(n)}(t)}{a^2 T_n} \right\|_\infty =: L_1 + L_2.$$

For any positive number $t$ and integer $k > 0$, let $t_0 = 0$ and $t_k = aT_n \left[ t2^k/aT_n \right]/2^k$, then we define

$$\tilde{X}^{(n)}(t) := \sum_{k=1}^\infty (\tilde{X}^{(n)}(t_k) - \tilde{X}^{(n)}(t_{k-1})).$$

For $0 \leq t \leq T_n - aT_n$, we write

$$\sup_{0 \leq x \leq 1} |\tilde{X}^{(n)}(t_k/aT_n) - \tilde{X}^{(n)}(t_{k-1}/aT_n + x)|$$

$$\leq 2 \sup_{0 \leq x \leq 1} \left| \tilde{X}^{(n)}(t_k/aT_n) + \tilde{X}^{(n)}(t_{k-1}/aT_n) \right| \sum_{k=1}^\infty \left| \tilde{X}^{(n)}(t_k/aT_n + x) - \tilde{X}^{(n)}(t_{k-1}/aT_n + x) \right|.$$

We now apply the similar arguments as in Lemma 2.5, with $k$ replaced to $n$, to obtain that

$$E(H^{(n)}(s_n, t + h) - H^{(n)}(s_n, t))^2 \leq Ch^{2\eta} s_n^{2\alpha - \delta},$$

where $\delta$, $\eta$ and $\alpha$ are in Lemma 2.5. There exists a positive number $k_0 = k_0(\alpha, \varepsilon)$ such that for $k \geq k_0$

$$\frac{\varepsilon^{2\alpha k + 1}}{\sqrt{C(k + 1)^2}} \geq \sqrt{(1 + \varepsilon/2)(k + 1)}.$$
Letting $x_k = (k + 1)^{-2}$, $k \geq 1$, we have

$$P\{L_2 \geq \varepsilon\}$$

$$\leq P\left\{ \sup_{0 \leq t \leq T_n - a T_n} \sup_{0 \leq s \leq 1} \sum_{k=1}^{\infty} \left| \bar{X}^{(n)}(t_k/a T_n + x) - \bar{X}^{(n)}(t_{k-1}/a T_n + x) \right| \geq 2 \varepsilon \gamma T_n \sum_{k=1}^{\infty} x_k \right\}$$

$$\leq \sum_{k=1}^{\infty} 2^k \frac{T_n - a T_n}{a T_n} \sup_{0 \leq t \leq T_n - a T_n} P\left\{ \sup_{0 \leq s \leq 1} \left| \bar{X}^{(n)}(t_k/a T_n + x) - \bar{X}^{(n)}(t_{k-1}/a T_n + x) \right| \geq 2 \varepsilon \gamma T_n x_k \right\}$$

$$\leq C \sum_{k=k_0}^{\infty} 2^k \frac{T_n - a T_n}{a T_n} \sup_{0 \leq t \leq T_n - a T_n} P\left\{ \sup_{0 \leq s \leq 1} \left| H^{(n)}(s, t_k/a T_n + x) - H^{(n)}(s, t_{k-1}/a T_n + x) \right| \geq \frac{\sqrt{C(2^{-h})} s^\alpha n^{-\delta}}{\sqrt{(2^{-h})} s^\alpha n^{-\delta}} \right\}$$

(3.6)

$$\leq C \sum_{k=k_0}^{\infty} 2^k \frac{T_n - a T_n}{a T_n} \exp \left( - (k + 1)n^\delta \log \left( \frac{T_n}{a T_n \log \log T_n} \right) \right)$$

$$\leq C \frac{T_n - a T_n}{a T_n} \exp \left( - n^\delta \log \left( \frac{T_n}{a T_n \log \log T_n} \right) \right) \sum_{k=k_0}^{\infty} (2/e)^k$$

$$\leq C (\log \log T_n)^{(r_0 + \varepsilon) - n^\delta (r_0 - \varepsilon - 1)}$$

$$\leq Cn^{-(n^\delta (r_0 - \varepsilon - 1) - (r_0 + \varepsilon))}$$

for large $n$, which implies

$$\sum_{n=1}^{\infty} P\{L_2 \geq \varepsilon\} < \infty.$$  

By the Borel-Cantelli lemma and the arbitrariness of $\varepsilon$, we obtain

$$\limsup_{n \to \infty} L_2 = 0 \quad \text{a.s.}$$  

(3.7)
Consider $L_4$. We have for any $\varepsilon > 0$

$$L_4 := P\{L_4 \leq 3\varepsilon\}$$

$$\geq P\left\{ \sup_{0 \leq t \leq T_n - aT_n} \inf_{f \in K} \left\| X(t + aT_n) - X(t) \right\|_\infty \leq 2\varepsilon \right\}$$

(3.8)

$$\geq P\left\{ \sup_{0 \leq t \leq T_n - aT_n} \inf_{f \in K} \left\| \frac{X^{(n)}(t + aT_n)}{aT_n} - \frac{X^{(n)}(t)}{aT_n} \right\|_\infty \geq 2\varepsilon \right\}$$

$$=: L_4 - L_5.$$

It is clear from (3.6) that

(3.9)

$$\sum_{n=1}^{\infty} L_5 < \infty.$$

We need only consider $L_4$. By the properties of FBM we have

$$L_4 \geq P\left\{ \max_{0 \leq j \leq T_n/aT_n} \sup_{0 \leq t \leq 2T_n} \inf_{f \in K} \left\| \frac{X(t + \cdot) - X(t)}{\gamma T_n} - f(\cdot) \right\|_\infty \leq 2\varepsilon \right\}$$

(3.10)

$$\geq P\left\{ \max_{0 \leq j \leq T_n/aT_n} \sup_{0 \leq t \leq 2T_n} \inf_{f \in K} \left\| \frac{X^{(j)}(t + \cdot)}{\gamma T_n} - X^{(j)}(t) \right\|_\infty \leq \varepsilon \right\}$$

$$- P\left\{ \max_{0 \leq j \leq T_n/aT_n} \sup_{0 \leq t \leq 2T_n} \inf_{f \in K} \left\| \frac{\hat{X}^{(j)}(t + \cdot) - \hat{X}^{(j)}(t)}{\gamma T_n} \right\|_\infty \geq \varepsilon \right\}$$

$$=: L_4 - L_7.$$
for $n$ large enough. Taking $j_0$ to be large enough such that $\varepsilon^2(r_0 - \varepsilon - 1)^{\beta}/(2C) - (r_0 + \varepsilon) > 1$, we have

\begin{equation}
\sum_{n=1}^{\infty} L_7 < \infty.
\end{equation}

Using the fact that $-\log(1 - x) \leq m_\nu x$ for $x \in (0, 1 - \nu)$ and for some $m_\nu > 0$, we have by (2.2) and independence of $X^{(j)}$

\begin{align*}
L_6 &= \left(1 - P\left\{ \sup_{0 \leq t \leq 1} \inf_{f \in K} \|(X^{(j)}(t + \cdot) - X^{(j)}(t)) - f(\cdot)\gamma_{T_n}\|_\infty \geq \varepsilon_{T_n}\right\}\right)^{\frac{T_n}{T_n} + 1} \\
&\geq \left(1 - C \exp\left\{ - \frac{(1 + \varepsilon/3)^2}{2 + \varepsilon} \log \frac{T_n}{aT_n \log \log T_n} \right\}\right)^{\frac{T_n}{T_n} + 1} \\
&\geq \exp\left( - m_\nu C T_n \left( \frac{aT_n \log \log T_n}{T_n} \right)^{1+\varepsilon/3} \right) \\
&\geq \exp\left( - C_\nu \left( \log \log T_n \right)^{-\varepsilon(r_0+\varepsilon)/3+1+\varepsilon/3} \right) \\
&\geq \exp\left( - C_\nu n^{1-\varepsilon(r_0+\varepsilon-1)/3} \right)
\end{align*}

for sufficiently large $n$, by hence

\begin{equation}
\sum_{n=1}^{\infty} L_6 = \infty.
\end{equation}

In combination with (3.12) and (3.13), it follows that

\begin{equation}
\sum_{n=1}^{\infty} L_4 = \infty.
\end{equation}

Combining (3.8) with (3.9) and (3.14), we have

\[ \sum_{n=1}^{\infty} L_3 = \infty. \]

Since the events of $L_3$ are independent, by the Borel Cantelli lemma and the arbitrariness of $\varepsilon$, we have

\begin{equation}
\limsup_{n \to \infty} L_1 = 0 \quad \text{a.s}
\end{equation}

From (3.7) and (3.15) we obtain (3.3), and (1.1) is proved.
Proof of (1.2). Let \( f \in K \), put \( T_n = \theta^n \) with \( \theta > 1 \). We have that for \( T_n \leq T \leq T_{n+1} \)

\[
\inf_{0 \leq t \leq T - a_T} \| Z_{t,T} (\cdot) - f (\cdot) \|_\infty \\
\leq \sup_{T_n \leq t \leq T_{n+1}} \inf_{0 \leq t \leq T_n - a_T} \| Z_{t,T_{n+1}} (\cdot a_T/a_{T_{n+1}}) - f (\cdot a_T/a_{T_{n+1}}) \|_\infty \\
+ \sup_{T_n \leq t \leq T_{n+1}} \inf_{0 \leq t \leq T_{n+1} - a_{T_{n+1}}} \| Z_{t,T} (\cdot) - Z_{t,T_{n+1}} (\cdot a_T/a_{T_{n+1}}) \|_\infty \\
+ \sup_{T_n \leq t \leq T_{n+1}} \| f (\cdot a_T/a_{T_{n+1}}) - f (\cdot) \|_\infty
\]

(3.16)

\[
\leq \inf_{0 \leq t \leq T_n - a_T} \| Z_{t,T_{n+1}} (\cdot) - f (\cdot) \|_\infty \\
+ \left| \frac{a_{T_{n+1}}}{a_T} \gamma_{T_{n+1}} - 1 \right| \sup_{T_n \leq t \leq T_{n+1}} \sup_{0 \leq t \leq T_{n+1} - a_{T_{n+1}}} \| Z_{t,T_{n+1}} (\cdot a_T/a_{T_{n+1}}) \|_\infty \\
+ \sup_{T_n \leq t \leq T_{n+1}} \| f (\cdot a_T/a_{T_{n+1}}) - f (\cdot) \|_\infty
\]

\[
= L_8 + L_9 + L_{10}.
\]

Consider \( I_8 \). Let \( \rho_n = \left[ (T_n - a_{T_n})/a_{T_{n+1}} \right] \), then it is sufficient to show that

(3.17)

\[
\limsup_{n \to \infty} \min_{0 \leq j \leq \rho_n} \left\| \frac{X(j + \cdot) - X(j)}{\gamma_{T_{n+1}}} - f (\cdot) \right\|_\infty = 0 \quad \text{a.s.}
\]

Let \( d_n, X^{(n)} \) and \( \tilde{X}^{(n)} \) be as in (3.4). By standard Borel-Cantelli arguments, (3.17) follows if we prove that

(3.18)

\[
\sum_{n=1}^{\infty} P \left\{ \min_{0 \leq j \leq \rho_n} \left\| \frac{X^{(n)}(j + \cdot) - X^{(n)}(j)}{\gamma_{T_{n+1}}} - f (\cdot) \right\|_\infty \leq \varepsilon \right\} = \infty
\]

and

(3.19)

\[
\sum_{n=1}^{\infty} P \left\{ \max_{0 \leq j \leq \rho_n} \left\| \frac{\tilde{X}^{(n)}(j + \cdot) - \tilde{X}^{(n)}(j)}{\gamma_{T_{n+1}}} \right\|_\infty \geq \varepsilon \right\} < \infty,
\]

since the events in (3.18) are independent.
Similar to (3.6) and (3.11), it is easily seen (3.19). Hence it remains to show (3.18). For any $\varepsilon > 0$ we have

$$P\left\{ \min_{0 \leq j \leq \rho_n} \left\| \frac{X^{(n)}(j+\cdot) - X^{(n)}(j)}{\gamma_{n+1}} - f(\cdot) \right\|_{\infty} \leq 4\varepsilon \right\}$$

$$\geq P\left\{ \min_{0 \leq j \leq \rho_n} \left\| \frac{X(j+\cdot) - X(j)}{\gamma_{n+1}} - f(\cdot) \right\|_{\infty} \leq 2\varepsilon \right\}$$

$$- P\left\{ \max_{0 \leq j \leq \rho_n} \left\| \frac{X^{(n)}(j+\cdot) - X^{(n)}(j)}{\gamma_{n+1}} \right\|_{\infty} \geq 2\varepsilon \right\}$$

$$\geq P\left\{ \min_{0 \leq j \leq \rho_n} \left\| \frac{X(j+\cdot) - X(j)}{\gamma_{n+1}} - f(\cdot) \right\|_{\infty} \leq \varepsilon \right\}$$

$$- P\left\{ \max_{0 \leq j \leq \rho_n} \left\| \frac{X(j+\cdot) - X(j)}{\gamma_{n+1}} \right\|_{\infty} \geq \varepsilon \right\}$$

$$- P\left\{ \max_{0 \leq j \leq \rho_n} \left\| \frac{X^{(n)}(j+\cdot) - X^{(n)}(j)}{\gamma_{n+1}} \right\|_{\infty} \geq 2\varepsilon \right\}$$

$$=: L_{11} - L_{12} - L_{13}.$$

It is clear from (3.19) that

$$\sum_{n=1}^{\infty} (L_{12} + L_{13}) < \infty.$$

We now turn to the case $L_{11}$. Put $f^{(c)} = (1-\varepsilon/2)f$ $(0 < \varepsilon < 1)$ for $f \in K$, then $f^{(c)} \in K$ and $\|f - f^{(c)}\|_{\infty} < \varepsilon/2$. We have by Lemma 2.4 and independence of $X^{(j)}$

$$P\left\{ \min_{0 \leq j \leq \rho_n} \left\| \frac{X^{(j)}(j+\cdot) - X^{(j)}(j)}{\gamma_{n+1}} - f(\cdot) \right\|_{\infty} \leq \varepsilon \right\}$$

$$= \sum_{j=0}^{\rho_n} P\left\{ \|X^{(j)}(j+\cdot) - X^{(j)}(j)\|_{\infty} \leq \varepsilon/2\gamma_{n+1} \right\}$$

$$\geq \sum_{j=0}^{\rho_n} \exp \left\{ - \frac{1}{2} \|f^{(c)}\|^2 \gamma_{n+1} \right\} P\left\{ \|X^{(j)}(j+\cdot) - X^{(j)}(j)\|_{\infty} \leq \varepsilon/2\gamma_{n+1} \right\}$$

$$\geq C \sum_{j=0}^{\rho_n} \exp \left\{ - \frac{(1+\varepsilon/2)^2}{2} \|f^{(c)}\|^2 \gamma_{n+1} \right\}$$

$$\geq C \frac{T_{n+1}}{\alpha T_{n+1}} \exp \left\{ - (1+\varepsilon/2)^2 \frac{T_{n+1}}{\alpha T_{n+1}} \log \log T_{n+1} \right\}$$

$$\geq C (\log \log T_{n+1})^{(1+\varepsilon/2)^2} = C (\log ((n+1) \log \theta))^{(1+\varepsilon/2)^2}$$
for large enough \( n \), which implies

\[(3.22) \quad \sum_{n=1}^\infty L_{11} = \infty.\]

Combining (3.20) with (3.21) and (3.22), we obtain (3.18).

Since \( \|f\|_\infty \leq 1 \) for all \( f \in K \), we have by (3.17)

\[(3.23) \quad L_9 \leq 2(\theta^{q+1} - 1) \quad \text{a.s.}\]

for large \( n \). Next, for \( T_n \leq T \leq T_{n+1} \) and \( 0 \leq x \leq 1 \), we have

\[|f(xa_T/a_{T_{n+1}}) - f(x)| \leq |a_T/a_{T_{n+1}} - 1|^\alpha \leq |\theta - 1|^\alpha,\]

thus we get

\[(3.24) \quad L_{10} \leq |\theta - 1|^\alpha.\]

Combining (3.16) with (3.17), (3.23) and (3.24), and letting \( \theta \downarrow 1 \), we obtain (1.2), and the proof is complete. \( \square \)

Since \( K \) is a compact subset of \( C_0[0,1] \), Theorem 1.1 implies:

**Corollary 1.** With probability one, \( \{Z_{t,T}(x) : 0 \leq x \leq 1, 0 \leq t \leq T - a_T, T \geq 3\} \) (as \( T \to \infty \)) is relatively compact in \( C_0[0,1] \), and the set of its limit point is \( K \).

**Corollary 2.** We have

\[\liminf_{T \to \infty} \sup_{0 \leq t \leq T - a_T} \sup_{0 \leq s \leq a_T} \frac{|X(t + s) - X(t)|}{a_T^{q+1}} = \liminf_{T \to \infty} \sup_{0 \leq t \leq T - a_T} \frac{|X(t + a_T) - X(t)|}{a_T^{q+1}} = 1 \quad \text{a.s.}\]

References


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