SOME NEW INTEGRAL MEANS INEQUALITIES AND INCLUSION PROPERTIES FOR A CLASS OF ANALYTIC FUNCTIONS INVOLVING CERTAIN INTEGRAL OPERATORS

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Abstract. In this paper we investigate integral means inequalities for the integral operators $Q^\mu_\lambda$ and $P^\mu_\lambda$ applied to suitably normalized analytic functions. Further, we obtain some neighborhood and inclusion properties for a class of functions $G^\alpha(\phi, \psi)$ (defined below). Several corollaries exhibiting the applications of the main results are considered in the concluding section.

1. Introduction and Preliminaries

Let $A$ denote the class of functions $f(z)$ normalized by $f(0) = f'(0) - 1 = 0$, and analytic in the open unit disk $U = \{ z : z \in \mathbb{C}, |z| < 1 \}$, then $f(z)$ can be expressed as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

We denote by $M(\alpha)$, $N(\alpha)$ and $\Lambda^\alpha_\lambda$ the three subclasses of the class $A$, which are defined (for $\alpha > 1$) as follows (see [9]):

$$M(\alpha) = \left\{ f : f \in A, \Re \left( \frac{zf'(z)}{f(z)} \right) < \alpha \ (\alpha > 1; \ z \in U) \right\}, \quad (1.2)$$

$$N(\alpha) = \left\{ f : f \in A, \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) < \alpha \ (\alpha > 1; \ z \in U) \right\} \quad (1.3)$$

and

$$\Lambda^\alpha_\lambda = \left\{ f : f \in A, \Re \left( \frac{D^\lambda f(z)}{D^\lambda f(z)} \right) < \alpha \ (\alpha > 1; \lambda > -1; \ z \in U) \right\}, \quad (1.4)$$

where the operator $D^\lambda$ involved in (1.4) is the familiar Ruscheweyh operator [10]. The classes $M(\alpha)$ and $N(\alpha)$ were studied recently by Owa and Nishiwaki [9].

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In fact, for $1 < \alpha \leq 4/3$, these classes were investigated earlier by Uralegaddi et al. [15], and the class $\Lambda_\alpha(\lambda)$ was recently studied by Raina and Bansal [9].

It follows from (1.2) and (1.3) that
\[
 f(z) \in N(\alpha) \iff zf'(z) \in M(\alpha).
\]
(1.5)
If $f, h \in \mathcal{A}$, where $f(z)$ is given by (1.1), and $h(z)$ is defined by
\[
h(z) = z + \sum_{n=2}^{\infty} c_n z^n,
\]
(1.6)
then their Hadamard product (or convolution) $f \ast h$ is defined (as usual) by
\[
(f \ast h)(z) = z + \sum_{n=2}^{\infty} a_n c_n z^n = (h \ast f)(z).
\]
(1.7)

In order to prove our main results, we need the following definitions and lemmas.

**Definition 1** (Raina and Bansal [9, p. 3686]). Let the functions $\phi(z)$ and $\psi(z)$ be given by
\[
\phi(z) = z + \sum_{n=2}^{\infty} \lambda_n z^n,
\]
(1.8)
and
\[
\psi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n,
\]
(1.9)
where $\lambda_n \geq \mu_n > 0$ $(\forall n \in \mathbb{N}\setminus\{1\})$. Then, we say that $f \in \mathcal{A}$ is in $S_\alpha(\phi, \psi)$ if
\[
\Re \left\{ \frac{(f \ast \phi)(z)}{(f \ast \psi)(z)} \right\} < \alpha \ (\alpha > 1; \ z \in \mathbb{U}),
\]
(1.10)
provided that $(f \ast \psi)(z) \neq 0$.

Several new and known subclasses can be obtained from the class $S_\alpha(\phi, \psi)$ by suitably choosing the functions $\phi(z)$ and $\psi(z)$. We mention below some of these subclasses of $S_\alpha(\phi, \psi)$ consisting of functions $f(z) \in \mathcal{A}$.

For example, using (1.8) to (1.10), it evidently follows that
\[
S_\alpha \left( \frac{z}{(1-z)^{\lambda+2}}, \frac{z}{(1-z)^{\lambda+1}} \right) = \Lambda_\lambda(\lambda)
\]
(1.11)
where $\lambda_n = \frac{\Gamma(n + \lambda + 1)}{\Gamma(n) \Gamma(\lambda + 2)}; \mu_n = \frac{\Gamma(n + \lambda)}{\Gamma(n) \Gamma(\lambda + 1)}$. 

\[
(1.8)
\]
and
\[
(1.9)
\]
SOME NEW INTEGRAL MEANS INEQUALITIES AND INCLUSION PROPERTIES

\[ S_\alpha \left( \frac{z}{(1-z)^2}, \frac{z}{(1-z)} \right) = \mathcal{M}(\alpha) \text{ (where } \lambda_n = n; \mu_n = 1) \quad (1.12) \]

and

\[ S_\alpha \left( \frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2} \right) = \mathcal{N}(\alpha) \text{ (where } \lambda_n = n^2; \mu_n = n) . \quad (1.13) \]

**Definition 2** (Jung-Kim-Srivastava [3]). Let \( f(z) \in \mathcal{A} \) be defined by (1.1), then

\[ \begin{align*}
Q_\lambda^\mu f(z) &= \left( \frac{\lambda + \mu}{\lambda} \right) \frac{\mu}{2\lambda} \int_0^z \left( 1 - \frac{t}{z} \right)^{\mu-1} f(t) dt \\
&= z + \frac{\Gamma(\lambda + \mu + 1)}{\Gamma(\lambda + 1)} \sum_{n=2}^{\infty} \frac{\Gamma(n + \lambda)}{\Gamma(n + \lambda + \mu)} a_n z^n. \quad (1.14)
\end{align*} \]

(\( \lambda > -1; \mu > 0; f \in \mathcal{A} \))

For \( \mu = 1 \), (1.14) reduces to the generalized Libera operator [7] given by

\[ Q_\lambda^1 f(z) = B_\lambda f(z) = z + \sum_{n=2}^{\infty} \left( \frac{\lambda + 1}{\lambda + n} \right) a_n z^n. \quad (1.15) \]

**Definition 3** (Komatu [4]). Let \( f(z) \in \mathcal{A} \) be defined by (1.1), then

\[ \begin{align*}
P_\lambda^\mu f(z) &= \frac{(\lambda + 1)^\mu}{z^{\lambda+1} \Gamma(\mu)} \int_0^z \left( \log \frac{z}{t} \right)^{\mu-1} f(t) dt \\
&= z + \sum_{n=2}^{\infty} \left( \frac{\lambda + 1}{\lambda + n} \right)^\mu a_n z^n. \quad (1.16)
\end{align*} \]

(\( \lambda > -1; \mu > 0; f \in \mathcal{A} \))

The operators (1.14) and (1.16) contain the familiar Jung-Kim-Srivastava and Komatu operator (see the details in [3], [4]).

**Lemma 1** (Raina & Bansal [9, Theorem 2.1, p. 3687]). If \( f(z) \in \mathcal{A} \) and satisfies

\[ \sum_{n=2}^{\infty} L(\lambda_n, \mu_n, k, \alpha) |a_n| \leq 2(\alpha - 1), \quad (1.17) \]

where

\[ L(\lambda_n, \mu_n, k, \alpha) = \{ (\lambda_n - k\mu_n) + |\lambda_n + (k-2\alpha)\mu_n| \}, \quad (1.18) \]

for some \( k (0 \leq k \leq 1) \) and some \( \alpha \ (\alpha > 1) \), then \( f(z) \in \mathcal{S}_\alpha(\phi, \psi) \).

**Lemma 2.** Let \( L(\lambda_n, \mu_n, k, \alpha) \) be defined by (1.18), then \( \{ L(\lambda_n, \mu_n, k, \alpha) \}_{n=2}^{\infty} \) is a nonvanishing, positive and nondecreasing sequence provided that the sequences \( \{ \mu_n \} \) and \( \{ \lambda_n \} \) are nondecreasing, and

\[ \lambda_n > \mu_n > 0 \ ; n \in \mathbb{N} \setminus \{1\} ; \alpha > 1 ; 0 \leq k \leq 1. \quad (1.19) \]
Proof. See details in [9, p. 3692].

**Lemma 3** (Littlewood [5]). If \( f(z) \) and \( h(z) \) are analytic in \( U \) with \( f(z) \prec h(z) \), then for \( p > 0 \) and \( z = re^{i\theta}(0 < r < 1) \):

\[
\int_0^{2\pi} |f(z)|^p d\theta \leq \int_0^{2\pi} |h(z)|^p d\theta. \tag{1.20}
\]

Corresponding to the neighborhood definition given by Frasin and Darus [2], let \( f \in \mathcal{A} \) be of the form (1.1) and \( s \geq 0 \), then a \((q - s)\) neighborhood of the function \( f \) is defined by

\[
M^q_s(f) = \left\{ h \in \mathcal{A} : h(z) = z + \sum_{n=2}^{\infty} c_n z^n, \sum_{n=2}^{\infty} n^{q+1} |a_n - c_n| \leq s \right\}. \tag{1.21}
\]

For \( e(z) = z \), we observe that

\[
M^q_s(e) = \left\{ h \in \mathcal{A} : h(z) = z + \sum_{n=2}^{\infty} c_n z^n, \sum_{n=2}^{\infty} n^{q+1} |c_n| \leq s \right\}, \tag{1.22}
\]

where \( q \in \mathbb{N} \cup \{0\} \). We note that \( M^0_s(f) = N_s(f) \) and \( M^1_s(f) = M_s(f) \), where \( N_s(f) \) denotes the \( s \)-neighborhood of \( f \) introduced by Ruscheweyh [11], and \( M_s(f) \) is the neighborhood defined by Silverman [12].

In view of Lemma 1, we further define the following subclasses of the class \( \mathcal{S}_\alpha(\phi, \psi) \).

**Definition 4.** Let \( G_\alpha(\phi, \psi) \) denote the class of functions \( f \in \mathcal{S}_\alpha(\phi, \psi) \) (defined by (1.10)) whose coefficients satisfy the coefficient inequality (1.17).

Corresponding to the subclasses \( \Lambda_\alpha(\lambda), \mathcal{M}(\alpha) \) and \( \mathcal{N}(\alpha) \) defined by (1.11) to (1.13), we also have the following set of subclasses of the class \( G_\alpha(\phi, \psi) \) (see [9, p. 3691]):

\[
G_\alpha \left( \frac{z}{(1-z)^{\lambda+2}}, \frac{z}{(1-z)^{\lambda+1}} \right) \equiv \Lambda^*_\alpha(\lambda) \tag{1.23}
\]

\[
G_\alpha \left( \frac{z}{(1-z)^2}, \frac{z}{1-z} \right) \equiv \mathcal{M}^*(\alpha) \tag{1.24}
\]

and

\[
G_\alpha \left( \frac{z + z^2}{(1-z)^3}, \frac{z}{(1-z)^2} \right) \equiv \mathcal{N}^*(\alpha). \tag{1.25}
\]

Obviously, we have the relationships

\[
\Lambda^*_\alpha(\lambda) \subset \Lambda_\alpha(\lambda); \mathcal{M}^*(\alpha) \subset \mathcal{M}(\alpha); \mathcal{N}^*(\alpha) \subset \mathcal{N}(\alpha).
\]
Among many others, the classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ were studied recently by Choi [1], Srivastava and Attiya [13] and Owa and Nishiwaki [6]. In this paper we investigate the integral means inequalities for the integral operators $Q_{\lambda}^{\mu}$ and $P_{\lambda}^{\mu}$ involving suitably normalized analytic functions. We also derive some neighborhood and inclusion relationships for the class $G_{\alpha}(\phi, \psi)$ (defined above). Several corollaries depicting some interesting consequences of the main results are also mentioned.

2. THE MAIN RESULTS

In this section we give integral means inequalities in Theorems 1 and 2, a neighborhood property for the class $G_{\alpha}(\phi, \psi)$ in Theorem 3, and some inclusion properties for class $G_{\alpha}(\phi, \psi)$ in Theorem 4, involving the integral operators $Q_{\lambda}^{\mu}$ and $P_{\lambda}^{\mu}$ (defined above by (1.14) and (1.16), respectively).

**Theorem 1.** Let $f(z) \in \mathcal{A}$ and $g(z)$ be defined by

$$g(z) = z + b_{j}z^{j} \quad (b_{j} \neq 0; j \geq 2) \quad (2.1)$$

and suppose that

$$\sum_{n=2}^{\infty} L(\lambda_{n}, \mu_{n}, k, \alpha)|a_{n}| \leq \frac{|b_{j}| \Gamma(\delta + \eta + 1) \Gamma(\delta + j)L(\lambda_{2}, \mu_{2}, k, \alpha)(\lambda + \mu + 1)}{\Gamma(\delta + 1) \Gamma(\delta + \eta)(\lambda + 1)}, \quad (2.2)$$

where $L(\lambda_{n}, \mu_{n}, k, \alpha)$ is given by (1.18). If $\langle \lambda_{n} \rangle$, $\langle \mu_{n} \rangle$ and $\langle \lambda_{n}/\mu_{n} \rangle$ are non-decreasing sequences and $\lambda_{n} > \mu_{n} > 0 \ (n \in \mathbb{N}\{1\})$, $0 \leq k \leq 1$, then for $\lambda > -1$, $\delta > -1$, $\mu > 0$, $\eta > 0$, $p > 0$ and $z = re^{i\theta} (0 < r < 1)$:

$$\int_{0}^{2\pi} |Q_{\lambda}^{\mu}f(z)|^{p}d\theta \leq \int_{0}^{2\pi} |Q_{\lambda}^{\mu}g(z)|^{p}d\theta. \quad (2.3)$$

**Proof.** Let $f(z)$ be given by (1.1). In view of (1.14), we obtain

$$Q_{\lambda}^{\mu}f(z) = z \left[ 1 + \frac{\Gamma(\lambda + \mu + 1)}{\Gamma(\lambda + 1)} \sum_{n=2}^{\infty} \frac{\Gamma(n + \lambda)}{\Gamma(n + \lambda + \mu)}a_{n}z^{n-1} \right]$$

and

$$Q_{\lambda}^{\mu}g(z) = z \left[ 1 + \frac{\Gamma(\lambda + \mu + 1)}{\Gamma(\lambda + 1)} \frac{\Gamma(j + \delta)}{\Gamma(j + \delta + \eta)}b_{j}z^{j-1} \right].$$

To prove (2.3), it is sufficient to show by means of Lemma 3 that

$$1 + \frac{\Gamma(\lambda + \mu + 1)}{\Gamma(\lambda + 1)} \sum_{n=2}^{\infty} \frac{\Gamma(n + \lambda)}{\Gamma(n + \lambda + \mu)}a_{n}z^{n-1} \leq 1 + \frac{\Gamma(\lambda + \mu + 1)}{\Gamma(\lambda + 1)} \frac{\Gamma(j + \delta + \eta)}{\Gamma(j + \delta + \eta)}b_{j}z^{j-1}. \quad (2.4)$$

By setting

$$1 + \frac{\Gamma(\lambda + \mu + 1)}{\Gamma(\lambda + 1)} \sum_{n=2}^{\infty} \frac{\Gamma(n + \lambda)}{\Gamma(n + \lambda + \mu)}a_{n}z^{n-1} = 1 + \frac{\Gamma(\lambda + \mu + 1)}{\Gamma(\lambda + 1)} \frac{\Gamma(j + \delta + \eta)}{\Gamma(j + \delta + \eta)}b_{j}[w(z)]^{j-1}$$

and

$$\int_{0}^{2\pi} \left[ 1 + \frac{\Gamma(\lambda + \mu + 1)}{\Gamma(\lambda + 1)} \sum_{n=2}^{\infty} \frac{\Gamma(n + \lambda)}{\Gamma(n + \lambda + \mu)}a_{n}z^{n-1} \right]^{p}d\theta \leq \int_{0}^{2\pi} \left[ 1 + \frac{\Gamma(\lambda + \mu + 1)}{\Gamma(\lambda + 1)} \frac{\Gamma(j + \delta + \eta)}{\Gamma(j + \delta + \eta)}b_{j}[w(z)]^{j-1} \right]^{p}d\theta.$$
we find that
\[ [w(z)]^{j-1} = \frac{\Gamma(\lambda + \mu + 1)\Gamma(\delta + 1)\Gamma(j + \delta + \eta)}{b_j \Gamma(\delta + \eta + 1)\Gamma(\lambda + 1)\Gamma(j + \delta)} \sum_{n=2}^{\infty} L(\lambda_n, \mu_n, k, \alpha)\theta(n)a_n z^{n-1}, \] 
(2.5)
where
\[ \theta(n) = \frac{\Gamma(n + \lambda)}{L(\lambda_n, \mu_n, k, \alpha)\Gamma(n + \lambda + \mu)} \] 
(2.6)

\( (\lambda_n > \mu_n > 0 \ (\forall n \in \mathbb{N}\{1\}), \ 0 \leq k \leq 1, \ \alpha > 1, \ \lambda > -1, \ \mu > 0). \)

If \( \langle \mu_n \rangle \) and \( \langle \lambda_n / \mu_n \rangle \) are nondecreasing sequences, then by virtue of Lemma 2 we observe that \( \frac{1}{L(\lambda_n, \mu_n, k, \alpha)} \) is a positive nonincreasing sequence. Also, \( \Gamma(n + \lambda + \mu) \) is a nonincreasing positive sequence. Thus, \( \theta(n) \ (n \in \mathbb{N}\{1\}) \) is also a nonincreasing sequence of \( n \) (being the product of two positive nonincreasing sequences).

It readily follows that
\[ 0 < \theta(n) \leq \theta(2) = \frac{\Gamma(\lambda + 2)}{L(\lambda_2, \mu_2, k, \alpha)\Gamma(\lambda + \mu + 2)}, \]
and from (2.5), we infer that \( w(0) = 0 \), and therefore, we are lead to
\[ |w(z)|^{j-1} \leq \frac{\Gamma(\lambda + \mu + 1)\Gamma(\delta + 1)\Gamma(j + \delta + \eta)}{b_j \Gamma(\delta + \eta + 1)\Gamma(\lambda + 1)\Gamma(j + \delta)} \sum_{n=2}^{\infty} L(\lambda_n, \mu_n, k, \alpha)\theta(n) |a_n| |z|^{n-1} \]
\[ \leq |z| \frac{\Gamma(\lambda + \mu + 1)\Gamma(\delta + 1)\Gamma(j + \delta + \eta)}{b_j \Gamma(\delta + \eta + 1)\Gamma(\lambda + 1)\Gamma(j + \delta)} \theta(2) \sum_{n=2}^{\infty} L(\lambda_n, \mu_n, k, \alpha) |a_n| \]
\[ \leq |z| < 1, \]
on making use of the hypothesis (2.2) of Theorem 1. Evidently, the last inequality above establishes the subordination (2.4), which consequently proves our Theorem 1.

**Theorem 2.** Let \( f(z) \in \mathcal{A} \) and \( g(z) \) be defined by (2.1), and suppose that
\[ \sum_{n=2}^{\infty} L(\lambda_n, \mu_n, k, \alpha) |a_n| \leq \left( \frac{\delta + 1}{\delta + j} \right)^{\eta} \left( \frac{\lambda + 2}{\lambda + 1} \right)^{\mu} L(\lambda_2, \mu_2, k, \alpha) |b_j|, \]
(2.7)
where \( L(\lambda_n, \mu_n, k, \alpha) \) is given by (1.18). If \( \langle \lambda_n \rangle, \ \langle \mu_n \rangle \) and \( \langle \lambda_n / \mu_n \rangle \) are non-decreasing sequences, \( \lambda_n > \mu_n > 0 \ (\forall n \in \mathbb{N}\{1\}), \ 0 \leq k \leq 1, \) then for \( \lambda > -1, \ \delta > -1, \ \mu > 0, \ \eta > 0, \ p > 0 \) and \( z = re^{i\theta} \ (0 < r < 1) \):
\[ \int_0^{2\pi} |P_\lambda^\mu f(z)|^p d\theta \leq \int_0^{2\pi} |P_\delta^\eta g(z)|^p d\theta. \]
(2.8)
Proof. Let \( f(z) \) be given by (1.1). Using (1.16), we obtain
\[
P_\lambda^\mu f(z) = z \left[ 1 + \sum_{n=2}^{\infty} \left( \frac{\lambda + 1}{\lambda + n} \right)^\mu a_n z^{n-1} \right]
\]
and
\[
P_\eta^\delta g(z) = z \left[ 1 + \left( \frac{\delta + 1}{\delta + j} \right)^\eta b_j z^{j-1} \right].
\]
To establish (2.8), it is sufficient to show that (in view of Lemma 3)
\[
1 + \sum_{n=2}^{\infty} \left( \frac{\lambda + 1}{\lambda + n} \right)^\mu a_n z^{n-1} \preceq 1 + \left( \frac{\delta + 1}{\delta + j} \right)^\eta b_j z^{j-1}.
\]
(2.9)
Putting
\[
1 + \sum_{n=2}^{\infty} \left( \frac{\lambda + 1}{\lambda + n} \right)^\mu a_n z^{n-1} = 1 + \left( \frac{\delta + 1}{\delta + j} \right)^\eta b_j [w(z)]^{j-1},
\]
we obtain
\[
[w(z)]^{j-1} = \left( \frac{\delta + j}{\delta + 1} \right)^\eta \frac{1}{b_j} \sum_{n=2}^{\infty} L(\lambda_n, \mu_n, k, \alpha) \sigma(n) a_n z^{n-1},
\]
where
\[
\sigma(n) = \left( \frac{\lambda + 1}{\lambda + n} \right)^\mu \frac{1}{L(\lambda_n, \mu_n, k, \alpha)}.
\]
(2.11)
If \( \lambda_n > \mu_n > 0 \) (\( \forall n \in \mathbb{N}\setminus\{1\} \)), \( 0 \leq k \leq 1 \), \( \alpha > 1 \), \( \lambda > -1 \), \( \mu > 0 \)
and \( \lambda_n/\mu_n \) are nondecreasing sequences then by the application of Lemma 2, we observe that \( \frac{1}{L(\lambda_n, \mu_n, k, \alpha)} \) is a nonincreasing sequence.

Also, \( \left( \frac{\lambda + 1}{\lambda + n} \right)^\mu \) is a nonincreasing sequence. Thus, \( \sigma(n) \) is a nonincreasing sequence (being the product of two positive nonincreasing sequences).

Now \( \sigma(n) \) being a nonincreasing sequence of \( n \) implies that
\[
0 < \sigma(n) \leq \sigma(2) = \left( \frac{\lambda + 1}{\lambda + 2} \right)^\mu \frac{1}{L(\lambda_2, \mu_2, k, \alpha)},
\]
and from (2.10), we note that \( w(0) = 0 \), and hence, we obtain
\[
[w(z)]^{j-1} \leq \left( \frac{\delta + j}{\delta + 1} \right)^\eta \frac{1}{b_j} \sum_{n=2}^{\infty} L(\lambda_n, \mu_n, k, \alpha) \sigma(n) |a_n| |z|^{n-1}
\]
\[
\leq \frac{|z|}{b_j} \left( \frac{\delta + j}{\delta + 1} \right)^\eta \sigma(2) \sum_{n=2}^{\infty} L(\lambda_n, \mu_n, k, \alpha) |a_n|
\]
\[
\leq |z| < 1,
\]
by virtue of (2.7) of Theorem 2. The last inequality above establishes the subordination (2.9), which completes the proof of Theorem 2.
\( \square \)
Theorem 3. If \( \left\{ \frac{L(\lambda_n, \mu_n, k, \alpha)}{n^{q+1}} \right\}_{n=2}^{\infty} \) is a nondecreasing sequence, then \( G_\alpha(\phi, \psi) \subset M^q_s(e) \), where
\[
s = \frac{2^{q+2}(\alpha - 1)}{L(\lambda_2, \mu_2, k, \alpha)}
\tag{2.12}
\]
and \( L(\lambda_n, \mu_n, k, \alpha) \) is given by (1.18).

Proof. It follows from (1.17) that if \( f(z) \in G_\alpha(\phi, \psi) \), then
\[
\frac{L(\lambda_2, \mu_2, k, \alpha)}{2^{q+1}} \sum_{n=2}^{\infty} n^{q+1} |a_n| \leq \sum_{n=2}^{\infty} L(\lambda_n, \mu_n, k, \alpha) |a_n| \leq 2(\alpha - 1)
\]
which at once gives
\[
\sum_{n=2}^{\infty} n^{q+1} |a_n| \leq \frac{2^{q+2}(\alpha - 1)}{L(\lambda_2, \mu_2, k, \alpha)}
\]
and the result follows on using (1.22).

Theorem 4. Let \( f(z) \in G_\alpha(\phi, \psi) \), then \( Q^\mu f(z) \in G_\alpha(\phi, \psi) \) and \( P^\mu f(z) \in G_\alpha(\phi, \psi) \) (\( \lambda > -1, \mu > 0 \)).

Proof. Let \( f(z) \in G_\alpha(\phi, \psi) \), then \( f(z) \) satisfies the coefficient inequality (1.17), and
\[
Q^\mu f(z) = z + \frac{\Gamma(\lambda + \mu + 1)}{\Gamma(\lambda + 1)} \sum_{n=2}^{\infty} \frac{\Gamma(\lambda + n)}{\Gamma(\lambda + \mu + n)} a_n z^n.
\]
To show that \( Q^\mu f(z) \in G_\alpha(\phi, \psi) \), we need simply to show that
\[
\sum_{n=2}^{\infty} L(\lambda_n, \mu_n, k, \alpha) \frac{\Gamma(\lambda + \mu + 1)\Gamma(\lambda + n)}{\Gamma(\lambda + 1)\Gamma(\lambda + \mu + n)} |a_n| \leq 2(\alpha - 1),
\]
which is true in view of coefficient inequality (1.17), because evidently
\[
\frac{\Gamma(\lambda + n)}{\Gamma(\lambda + \mu + n)} \leq \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + \mu + 1)} \quad (\forall n = 2, 3, ...).
\]
The proof of second part, viz. that \( f(z) \in G_\alpha(\phi, \psi) \) implies that \( P^\mu f(z) \in G_\alpha(\phi, \psi) \) is similar to the first part, and is hence omitted.

3. APPLICATIONS OF MAIN RESULTS

In this section we consider some applications of our main results (Theorems 1 to 3).
Let us set
\[
\eta = \mu, \; \delta = \lambda, \; b_j = \frac{2(\beta - 1)}{L(\lambda_j, \mu_j, k, \alpha)} (j \geq 2).
\tag{3.1}
in Theorem 1, and suppose that \( f(z) \in G_\alpha(\phi, \psi) \) (which is given by Definition 4), then the inequality (2.3) holds if the following coefficient inequality holds true:

\[
\sum_{n=2}^{\infty} L(\lambda_n, \mu_n, k, \alpha) |a_n| \leq \frac{2(\beta - 1)\Gamma(\lambda + \mu + 2)\Gamma(\lambda + j)L(\lambda_j, \mu_j, \alpha)\Gamma(\lambda + j)}{L(\lambda_j, \mu_j, \alpha)\Gamma(\lambda + 2)\Gamma(\lambda + \mu + j)}. \tag{3.2}
\]

To show that (3.2) is true, let us choose \( \beta \) such that

\[
\beta \geq 1 + \frac{L(\lambda_j, \mu_j, k, \alpha)\Gamma(\lambda + 2)\Gamma(\lambda + \mu + j)}{L(\lambda_2, \mu_2, k, \alpha)\Gamma(\lambda + j)\Gamma(\lambda + \mu + 2)}(\alpha - 1)
\]

then (3.2) reduces to

\[
\sum_{n=2}^{\infty} L(\lambda_n, \mu_n, k, \alpha) |a_n| \leq 2(\alpha - 1)
\]

which is true in view of (1.17) of Lemma 1.

In view of the above parametric substitutions (3.1), Theorem 1 finally reduces to the following result.

**Corollary 1.** Let \( f(z) \in G_\alpha(\phi, \psi) \) and \( g(z) \) be given by

\[
g(z) = z + \frac{2(\beta - 1)}{L(\lambda_n, \mu_n, k, \alpha)} z^n \quad (n \geq 2) \tag{3.3}
\]

satisfying the conditions given by (1.19), then for \( z = re^{i\theta} \) \((0 < r < 1)\):

\[
\int_0^{2\pi} |Q_\lambda^k f(z)|^p d\theta \leq \int_0^{2\pi} |Q_\lambda^k g(z)|^p d\theta \tag{3.4}
\]

\((\lambda > -1, \mu > 0, p > 0)\)

provided that there exists \( \beta \) such that

\[
\beta \geq 1 + \frac{L(\lambda_n, \mu_n, k, \alpha)\Gamma(\lambda + 2)\Gamma(\lambda + \mu + n)}{L(\lambda_2, \mu_2, k, \alpha)\Gamma(\lambda + n)\Gamma(\lambda + \mu + 2)}(\alpha - 1) \quad (n \geq 2) \tag{3.5}
\]

where \( L(\lambda_n, \mu_n, k, \alpha) \) is given by (1.18).

Next, let us choose \( n = 2 \) in Corollary 1, then from (3.5) we get \( \beta \geq \alpha \). Consequently, Corollary 1 gives

**Corollary 2.** Let \( f(z) \in G_\alpha(\phi, \psi) \) and \( g(z) \) be given by

\[
g(z) = z + \frac{2(\beta - 1)}{L(\lambda_2, \mu_2, k, \alpha)} z^2 \quad (\beta \geq \alpha) \tag{3.6}
\]
satisfying the conditions corresponding to those given by (1.19), then for \( z = re^{i\theta} \) \((0 < r < 1)\):

\[
\int_0^{2\pi} |Q^\mu f(z)|^p d\theta \leq \int_0^{2\pi} |Q^\mu g(z)|^p d\theta,
\]

\((\lambda > -1, \mu > 0, p > 0)\)

where \( L(\lambda_2, \mu_2, k, \alpha) \) is given by (1.18).

Making similar substitutions as given by (3.1) in Theorem 2, we shall arrive at the following result:

**Corollary 3.** Let \( f(z) \in G_\alpha(\phi, \psi) \) and \( g(z) \) be given by

\[
g(z) = z + \frac{2(\beta - 1)}{L(\lambda_n, \mu_n, k, \alpha)} z^n \quad (n \geq 2)
\]

satisfying the conditions given by (1.19), then for \( z = re^{i\theta} \) \((0 < r < 1)\):

\[
\int_0^{2\pi} |P^\mu f(z)|^p d\theta \leq \int_0^{2\pi} |P^\mu g(z)|^p d\theta
\]

\((\lambda > -1, \mu > 0, p > 0)\)

provided that there exists \( \beta \) such that

\[
\beta \geq 1 + \left( \frac{\lambda + n}{\lambda + 2} \right)^\mu \frac{L(\lambda_n, \mu_n, k, \alpha)}{L(\lambda_2, \mu_2, k, \alpha)} (\alpha - 1) \quad (n \geq 2)
\]

where \( L(\lambda_n, \mu_n, k, \alpha) \) is given by (1.18).

For \( n = 2 \), Corollary 3 reduces to

**Corollary 4.** Let \( f(z) \in G_\alpha(\phi, \psi) \) and \( g(z) \) be given by

\[
g(z) = z + \frac{2(\beta - 1)}{L(\lambda_2, \mu_2, k, \alpha)} z^2 \quad (\beta \geq \alpha)
\]

satisfying the conditions corresponding to those given by (1.19), then for \( z = re^{i\theta} \) \((0 < r < 1)\):

\[
\int_0^{2\pi} |P^\mu f(z)|^p d\theta \leq \int_0^{2\pi} |P^\mu g(z)|^p d\theta
\]

\((\lambda > -1, \mu > 0, p > 0)\)

where \( L(\lambda_2, \mu_2, k, \alpha) \) is given by (1.18).

If we set the arbitrary functions \( \phi \) and \( \psi \) in Corollary 2 in accordance with (1.24) and choose \( \mu = 1 \), then in view of (1.15) we obtain the following result involving generalized Libera operator [7].
Corollary 5. Let \( f(z) \in \mathcal{M}^*(\alpha) \) and \( g(z) \) be given by
\[
g(z) = z + \frac{2(\beta - 1)}{\rho(k, \alpha)} z^2 \quad (\beta \geq \alpha) \tag{3.13}
\]
where
\[
\rho(k, \alpha) = \{(2 - k) + |2 + k - 2\alpha|\}. \tag{3.14}
\]
satisfying the conditions that \( 0 \leq k \leq 1, \alpha > 1 \), then for \( z = re^{i\theta} \) \((0 < r < 1)\):
\[
\int_0^{2\pi} |B_{\lambda} f(z)|^p d\theta \leq \int_0^{2\pi} |B_{\lambda} g(z)|^p d\theta. \tag{3.15}
\]
\((\lambda > -1, p > 0)\)

where the operator \( B_{\lambda} \) is defined by (1.15).

Making use of the relation (1.24) to reduce the class \( G_{\alpha}(\phi, \psi) \) to \( \mathcal{M}^*(\alpha) \) in Theorem 3, we obtain

Corollary 6. If \( \left\{ \frac{\Omega(n,k,\alpha)}{n^{q+1}} \right\}_{n=2}^{\infty} \) is a nondecreasing sequence, then \( \mathcal{M}^*(\alpha) \subset M^q(e) \), where
\[
s = \frac{2q+2(\alpha - 1)}{\Omega(2, k, \alpha)} \tag{3.16}
\]
and
\[
\Omega(n, k, \alpha) = \{(n - k) + |n + k - 2\alpha|\}. \tag{3.17}
\]
provided that \( 0 \leq k \leq 1, \alpha > 1 \) and \( q \in \mathbb{N} \cup \{0\} \).

Similarly, if we use the relation (1.25) to reduce the class \( G_{\alpha}(\phi, \psi) \) to \( N^*(\alpha) \) in Theorem 3, we get the following result.

Corollary 7. If \( \left\{ \frac{\Delta(n,k,\alpha)}{n^{q+1}} \right\}_{n=2}^{\infty} \) is a nondecreasing sequence, then \( N^*(\alpha) \subset M^q(e) \), where
\[
s = \frac{2q+2(\alpha - 1)}{\Delta(2, k, \alpha)} \tag{3.18}
\]
and
\[
\Delta(n, k, \alpha) = n \{(n - k) + |n + k - 2\alpha|\}. \tag{3.19}
\]
provided that \( 0 \leq k \leq 1, \alpha > 1 \) and \( q \in \mathbb{N} \cup \{0\} \).
References


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