GENERALIZED HÖLDER ESTIMATES FOR THE $\bar{\partial}$-EQUATION ON CONVEX DOMAINS IN $\mathbb{C}^2$

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Abstract. In this paper, we introduce the generalized Hölder space with a majorant function and prove the Hölder regularity for solutions of the Cauchy-Riemann equation in the generalized Hölder spaces on a bounded convex domain in $\mathbb{C}^2$.

1. Introduction and regular majorant

Let $D$ be a bounded domain in $\mathbb{C}^n$. The Hölder space of order $\alpha$, $\Lambda_\alpha(D)$ ($0 < \alpha < 1$), is defined by the set of all functions $g$ on $D$ such that there exists a constant $C = C_g > 0$ satisfying

$$|g(z) - g(\zeta)| \lesssim C|z - \zeta|^{\alpha}, \quad z, \zeta \in D.$$  

We first introduce some generalized Hölder space with a majorant function. A continuous increasing function $\omega$ on $[0, \infty)$, satisfying that $\omega(0) = 0$, $\omega(t)/t$ is non-increasing, and in addition, there is a constant $C = C(\omega)$ such that

$$\int_0^\delta \frac{\omega(t)}{t} \, dt + \delta \int_\delta^\infty \frac{\omega(t)}{t^2} \, dt \leq C\omega(\delta), \quad \text{for any } 0 < \delta < 1, \quad (1)$$

is called a regular majorant. Given a regular majorant, the generalized Hölder space, $\Lambda_\omega(D)$, is defined by the family of all functions $g$ on $D$ such that

$$|g(z) - g(\zeta)| \leq C\omega(|z - \zeta|), \quad z, \zeta \in D. \quad (2)$$

The norm $||g||_\omega$ of $g \in \Lambda_\omega(D)$ is given by $C_g + ||g||_\infty$, where $C_g \geq 0$ is the smallest constant satisfying (2) and $||g||_\infty$ is the $L^\infty$ norm in $D$. Note that with this norm $\Lambda_\omega(D)$ is a Banach space and $\Lambda_\omega(D) \subset L^\infty(D)$. The generalized Hölder spaces have been studied by many authors (see [2], [3], [4], [5], [6], [7], [8], and references in their papers).
Theorem 1.1. Let \( D \) be a bounded convex domain in \( \mathbb{C}^2 \) with \( C^2 \) boundary \( bD \). Let \( \omega \) be a regular majorant. There is a bounded linear operator \( S : C_{0,1}(\bar{D}) \to C(D) \) such that \( \bar{\partial}(Sf) = f \) on \( D \) and
\[
\|Sf\|_{\Lambda_{\omega}(D)} \leq C\|f\|_{\Lambda_{\omega}(D)} \quad \text{for all } f \text{ with } \bar{\partial}f = 0.
\]

As convention we use the notation \( A \lesssim B \) or \( A \gtrsim B \) if there are constants \( c_1, c_2 \), independent of the quantities under consideration, satisfying \( A \leq c_1 B \) and \( A \geq c_2 B \), respectively.

Remark 1. Before proving Theorem 1.1, we give some examples of regular majorants.

(i) Typical example of the regular majorant is a function \( \omega(t) = t^\alpha \) (0 < \( \alpha < 1 \)).

(ii) Non-trivial example is the type of the function, \( \omega(t) = t^\alpha |\log t|^\beta \), where \( 0 < \alpha < 1 \) and \( -\infty < \beta < \infty \).

(iii) Let \( m(t) = 1/|\log t|^\beta \), \( \beta > 0 \). Then \( m(t) \) is continuous and increasing near 0, but it is not a regular majorant.

2. Henkin’s solution operator of the \( \bar{\partial} \)-equation

Let \( D \) be a bounded and convex domain in \( \mathbb{C}^2 \) with \( C^2 \) boundary \( bD \). We choose a \( C^2 \) defining function \( \rho \) for \( D \), so that in a neighborhood \( U \) of \( bD \)
\[
\rho(z) = \begin{cases} -\text{dist}(z, bD) & \text{for } z \in U \cap \bar{D} \\ +\text{dist}(z, bD) & \text{for } z \in U \setminus D. \end{cases}
\]

We define
\[
\phi(\zeta, z) = \sum_{j=1}^{2} \frac{\partial \rho}{\partial \zeta_j}(\zeta_j - z_j).
\]

The following estimate for a sufficiently small neighborhood \( U \) of \( bD \)-is a well-known consequence of the convexity of \( D \) :
\[
|\phi(\zeta, z)| \gtrsim |\text{Im } \phi(\zeta, z)| + \rho(\zeta) - \rho(z) \quad \text{for } \zeta, z \in U \tag{3}
\]
and \( d_\zeta \phi(\zeta, z)|_{z=\zeta} = \bar{\partial} \rho(\zeta) \).

Lemma 2.1. ([9]) Let \( (\zeta_0, z_0) \in \partial D \times \partial D \) such that \( \phi(\zeta_0, z_0) = 0 \). Then there exist neighborhoods \( V \) of \( \zeta_0 \) and \( W \) of \( z_0 \) such that for each \( z \in W \), there exists a \( C^1 \) local coordinate system \( \zeta \mapsto t^{(z)}(\zeta) = (t_1, t_2, t_3, t_4) \) on \( V \) with the following properties:
\[
t_1(\zeta) = \rho(\zeta), \quad t_2(\zeta) = \text{Im } \phi(\zeta, z), \quad t_3(z) = t_4(z) = 0;
\]
\[
|t_1^{(z)}(\zeta) - t_1^{(\zeta')}(\zeta')| \sim |\zeta - \zeta'|
\]
for all \( \zeta, \zeta' \in V \) with the constants in (3) independent of \( z \in W \).
We have the Henkin’s solution operator \( Sf = \mathbb{H}f + Kf \) of the \( \bar{\partial} \)-equation, where
\[
\mathbb{H}f(z) = c \int_{\zeta \in bD} f(\zeta) \wedge \frac{\partial \rho}{\partial \zeta_1}(\bar{\zeta} - \bar{z}) \frac{\partial \rho}{\partial \zeta_2}(\bar{\zeta} - \bar{z}) d\zeta_1 \wedge d\zeta_2,
\]
and
\[
Kf(z) = \int_{\zeta \in bD} f(\zeta) \wedge K(\zeta, z),
\]
where \( K(\zeta, z) \) is the Bochner-Martinelli kernel [10].

It is well-known that the Bochner-Martinelli integral, \( Kf \) has a good regularity such that \( Kf \) is a bounded operator from bounded forms to the forms whose coefficients are in \( \Lambda^\alpha(D) \) for any \( 0 < \alpha < 1 \) [10]. However, it is not at all clear that this kind of generalized Hölder regularity for the Bochner-Martinelli integral still holds.

**Proposition 2.2.** Let \( \omega \) be a regular majorant. Then \( K : L^\infty_{0,1}(D) \to \Lambda^\omega(D) \) is a bounded operator, where \( L^\infty_{0,1}(D) \) is the space of bounded forms of type \((0, 1)\).

**Proof.** It suffices to show that for arbitrary \( z, z' \in D \),
\[
|Kf(z) - Kf(z')| \lesssim \omega(|z - z'|)|f|_\infty.
\]
(5)
The proof of the regularity of the Bochner-Martinelli integral in the classical Hölder space implies that
\[
|Kf(z) - Kf(z')| \lesssim |z - z'|(1 + \log |z - z'|)|f|_\infty
\]
(for the details, see the chapter 4 of Range’s book [?]). Since \( \omega(t)/t \) is non-increasing, it follows that \( t \lesssim \omega(t) \) for \( t \) with \( 0 < t < R \), where \( R = \sup_{z, \zeta \in D} |z - \zeta| \). Thus we have \( |Kf(z) - Kf(z')| \lesssim \omega(|z - z'|)|f|_\infty \).

\[\square\]

3. Proof of Theorem 1.1

To prove that a function \( g \) belongs to the generalized Hölder space \( \Lambda^\omega(D) \), we need a variant of the Hardy-Littlewood Lemma.

**Lemma 3.1.** ([1]) Let \( D \subset \mathbb{R}^n \) be a bounded domain with the \( C^1 \)-boundary defining function \( \rho \). If \( g \) is a \( C^1(D) \)-function and \( \omega \) is a regular majorant such that for some constant \( c_g \) depending on \( g \),
\[
|dg(x)| \leq c_g \frac{\omega(|\rho(x)|)}{\rho(x)}, \quad x \in D,
\]
then we have
\[
|g(x) - g(y)| \leq c \cdot c_g \omega(|x - y|).
\]

**Lemma 3.2.** For sufficiently small \( \epsilon > 0 \), \( \omega(t)/t^{1-\epsilon} \) is decreasing.
Proof. Put
\[ F(x) = \int_x^\infty \frac{\omega(t)}{t^2} dt. \]
Since \( \omega \) is a regular majorant, by the second term of the left hand side of (??), it follows that
\[ F(x) = \int_x^\infty \frac{\omega(t)}{t^2} dt \lesssim \frac{\omega(x)}{x}. \]
Since \( \omega \) is increasing, we have
\[ F(x) = \int_x^\infty \frac{\omega(t)}{t^2} dt \geq \omega(x) \int_x^\infty \frac{1}{t^2} dt = \frac{\omega(x)}{x}. \]
Thus it follows that
\[ F(x) \sim \frac{\omega(x)}{x} = -xF'(x). \]
We will show that, for sufficiently small \( \epsilon > 0 \), \( x^\epsilon F(x) \) is decreasing. We choose sufficiently small \( \epsilon > 0 \) such that
\[ F(x) \leq \frac{1}{\epsilon} \frac{\omega(x)}{x} = \frac{1}{\epsilon}xF'(x). \]
Thus it follows that
\[ (x^\epsilon F(x))' = \epsilon x^{\epsilon-1} F(x) + x^\epsilon F'(x) \leq 0 \]
and so that we get the result. \( \square \)

By Lemma 3.1, for the proof of Theorem 1.1, it is enough to prove the following result.

**Proposition 3.3.** There exists a constant \( C_\omega > 0 \) such that
\[ |d_z \bar{\mathbb{H}} f(z)| \leq C_\omega \|f\|_{\Lambda_\omega(D)} \frac{\omega(|\rho(z)|)}{|\rho(z)|} \quad \text{for} \quad z \in D. \]  \( (6) \)

**Proof.** By differentiating under the integral sign in (4), one obtains \( d\bar{\mathbb{H}} f(z) = G_1 f(z) + G_2 f(z) \), where
\[ G_1 f(z) = \int_{bD} f \wedge \frac{A_1(\zeta, z)}{(\phi(\zeta, z))^j |\zeta - z|^2}, \]
\[ G_2 f(z) = \int_{bD} f \wedge \frac{A_2(\zeta, z)}{\phi(\zeta, z)^j |\zeta - z|^2}, \]
where \( A_j(\zeta, z) \) are smooth double forms, of degree 1 in \( z \) and type (2,1) in \( \zeta \), which satisfy \( A_j(\zeta, z) \| \lesssim |\zeta - z|^j, j = 1, 2 \). The compactness of \( bD \), and a
partition of unity, the estimation of $G_1 f(z)$ is reduced to proving the estimate

$$|I(z)| \leq C_\omega \|f\|_{A_\omega(D)} \frac{\omega(|\rho(z)|)}{|\rho(z)|} \quad \text{for } z \in W \cap D,$$

where

$$I(z) = \int_{bD \cap V} f(\zeta) \frac{\chi(\zeta) A_1(\zeta, z)}{(\phi(\zeta, z))^2 |\zeta - z|^2} d\sigma(\zeta),$$

where $V, W$ are neighborhoods as given in Lemma 2.1, and $\chi$ has compact support in $V$. Note that the new coordinate system in Lemma 2.1 satisfies $t'(\zeta) = (0, t')$ for $\zeta \in V \cap bD$, where $t' = (t_2, t_3, t_4)$. We choose $\zeta' \in V \cap bD$ satisfying $t'(\zeta') = (0, 0, t_3, t_4)$. Then we have $I(z) \leq I_1(z) + I_2(z)$, where

$$I_1(z) = \left| \int_{bD \cap V} \frac{(f(\zeta) - f(\zeta'))\chi(\zeta) A_1(\zeta, z)}{(\phi(\zeta, z))^2 |\zeta - z|^2} d\sigma(\zeta) \right|,$$

$$I_2(z) = \left| \int_{bD \cap V} \frac{f(\zeta')\chi(\zeta) A_1(\zeta, z)}{(\phi(\zeta, z))^2 |\zeta - z|^2} d\sigma(\zeta) \right|.$$
By the second term of the left hand side of (1), one obtains
\[
\int_{|\rho(z)|}^{1} \frac{\omega(t_2)}{(t_2 + |\rho(z)|)^2} dt_2 \lesssim \int_{|\rho(z)|}^{1} \frac{\omega(t_2)}{t_2^2} dt_2 \\
\lesssim \frac{\omega(|\rho(z)|)}{|\rho(z)|}.
\]
These imply
\[
I_1(z) \lesssim ||f||_{\Lambda_\omega(D)} \frac{\omega(|\rho(z)|)}{|\rho(z)|}.
\]

For \(I_2(z)\), we need somewhat different method. An integration by parts
allows one to lower the singularity order of the Henkin kernel. This kind of
method was used in [11].

We see that
\[
\frac{1}{\phi^2} = - \left( \frac{\partial \phi}{\partial t_2} \right)^{-1} \frac{\partial}{\partial t_2} \left( \frac{1}{\phi} \right).
\]

Therefore, by the integration by parts, we have
\[
I_2(z) \lesssim \int_{|t'| \leq 1} \left| f(0,0,t'') \frac{1}{\phi} \left( \frac{\partial}{\partial t_2} \right)^2 \chi(t') A_1(t',z) \right| dt' \\
= \int_{|t'| \leq 1} \left| f(0,0,t'') \frac{1}{\phi} \left( \frac{\partial}{\partial t_2} \right)^{-1} \chi(t') A_1(t',z) \right| dt' \\
= \int_{|t'| \leq 1} \left| f(0,0,t'') \frac{1}{\phi} \chi(t') A_1(t',z) \right| dt' \quad (8)
\]
where
\[
B(t',z) = - \frac{\partial^2 \phi}{\partial t_2^2} \frac{\chi(t') A_1(t',z)}{|t'|^2} + \frac{\partial \phi}{\partial t_2} \frac{\partial}{\partial t_2} \left( \frac{\chi(t') A_1(t',z)}{|t'|^2} \right).
\]

In the second equality of (8), we have use the fact that \(f(0,0,t'')\) does not
depend on \(t_2\). Since \(t_2 = \text{Im } \phi\), we have \(|\partial \phi/\partial t_2| \geq 1\). Therefore we have
\[
I_2(z) \lesssim ||f||_\infty \int_{|t'| \leq 1} \frac{dt'}{|\phi||t'|^2}.
\]

In [1], we proved that
\[
\int_{|t'| \leq 1} \frac{dt'}{|\phi||t'|^2} \lesssim ||f||_\infty |\rho(z)|^{-\epsilon}.
\]
Thus we have
\[
I_2(z) \lesssim ||f||_\infty |\rho(z)|^{-\epsilon}.
\]

By Lemma 3.2, for sufficiently small \(\epsilon > 0\) we get
\[
|\rho(z)|^{-\epsilon} = \frac{|\rho(z)|^{1-\epsilon}}{|\rho(z)|} \lesssim \frac{\omega(|\rho(z)|)}{|\rho(z)|}.
\]
Thus it follows that
\[ I_2(z) \lesssim \|f\|_{\infty} \frac{\omega(|\rho(z)|)}{|\rho(z)|} \]
and so that
\[ |G_1f(z)| \lesssim \|f\|_{\infty} \frac{\omega(|\rho(z)|)}{|\rho(z)|}. \]

The estimate for \( G_2f(z) \) is exactly like the last part of the estimate for \( I_2 \).

Thus we complete the proof of Theorem 1.1 by using Proposition 3 and Lemma 3.1. □

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