ON INTERVAL VALUED FUZZY QUASI-IDEALS OF SEMIGROUPS

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Abstract. In this paper we shall introduce the notion of an i-v fuzzy interior ideal, an i-v fuzzy quasi-ideal and an i-v fuzzy bi-ideal in a semigroup. We study some properties of i-v fuzzy subsets and using their properties we characterize regular semigroups.

1. Introduction

In 1975, Zadeh ([11]) introduced a new notion of fuzzy subsets viz., interval valued fuzzy subsets (in short, i-v fuzzy subsets) where the values of the membership functions are closed intervals of numbers instead of a number. In ([3]), Biswas defined interval valued fuzzy subgroups and investigated some elementary properties. Subsequently, Jun and Kim ([7]) and Davvaz ([4]) applied a few concept of i-v fuzzy subsets in near-rings. In this paper we introduce the notion of an i-v fuzzy interior ideal, an i-v fuzzy quasi-ideal and an i-v fuzzy bi-ideal in a semigroup. We investigate some of their properties. We give examples which are i-v fuzzy interior ideal and i-v fuzzy bi-ideal but not i-v fuzzy ideal and i-v fuzzy quasi-ideal respectively. We find the equivalent conditions on which these i-v fuzzy subsets coincide. Finally we characterize regular semigroups through their i-v fuzzy subsets. We also find equivalent conditions on regular semigroups through i-v fuzzy subsets.

2. Basic definitions and preliminary results

Let $S$ be a semi group. Let $A$ and $B$ be subsets of $S$, the multiplication of $A$ and $B$ is defined as $AB = \{ ab \in S \mid a \in A \text{ and } b \in B \}$. A nonempty subset $A$ of $S$ is called a subsemigroup of $S$ if $AA \subseteq A$. A nonempty subset $A$ of $S$ is called a left (right) ideal of $S$ if $SA \subseteq A$ ($AS \subseteq A$). $A$ is called a two-sided ideal (simply ideal) of $S$ if it is both a left and a right ideal of $S$. A nonempty subset $A$ of $S$ is called an interior ideal of $S$ if $SAS \subseteq A$, and a quasi-ideal of $S$ if $AS \cap SA \subseteq A$. A subsemigroup $A$ of $S$ is called a bi-ideal of $S$ if $ASA \subseteq A$.

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A semigroup $S$ is called regular if for each element $a \in S$ there exists $x \in S$ such that $a = axa$. A function $f$ from a nonempty set $A$ to the unit interval $[0, 1]$ is called a fuzzy subset of $A$.

**Definition 2.1.** An interval number $\tau$ on $[0, 1]$ is a closed subinterval of $[0, 1]$, that is, $\tau = [a^-, a^+]$ such that $0 \leq a^- \leq a^+ \leq 1$ where $a^-$ and $a^+$ are the lower and upper end points of $\tau$ respectively.

In this notation $\Upsilon = [0, 0]$ and $\Upsilon = [1, 1]$. For any interval numbers

(i) $\tau \leq \tau'$ if and only if $a^- \leq a'^-$ and $a^+ \leq a'^+$.

(ii) $\tau = \tau'$ if and only if $a^- = a'^-$ and $a^+ = a'^+$.

**Definition 2.2.** Let $X$ be any set. A mapping $\overline{A} : X \rightarrow D[0, 1]$ is called an interval-valued fuzzy subset (briefly, i-v fuzzy subset) of $X$, where $D[0, 1]$ denotes the family of all closed subintervals of $[0, 1]$ and $\overline{A}(x) = [A^-(x), A^+(x)]$ for all $x \in X$, where $A^-$ and $A^+$ are fuzzy sets of $X$ such that $A^-(x) \leq A^+(x)$ for all $x \in X$.

Thus $\overline{A}(x)$ is an interval (a closed subset of $[0, 1]$) and not a number from the interval $[0, 1]$ as in the case of fuzzy set.

**Definition 2.3.** A mapping $\min : D[0, 1] \times D[0, 1] \rightarrow D[0, 1]$ defined by

\[
\min(A, B) = \bigcup \{ a, b : [a, b] \subseteq (A \cap B) \}
\]

is called an interval min-norm. A mapping $\max : D[0, 1] \times D[0, 1] \rightarrow D[0, 1]$ defined by

\[
\max(A, B) = \bigcup \{ a, b : [a, b] \subseteq (A \cup B) \}
\]

is called an interval max-norm.

Let $\min$ and $\max$ be the interval min-norm and max-norm on $D[0, 1]$ respectively. Then the following are true.

(i) $\min(A, \tau) = \min(A, \tau)$ and $\max(A, \tau) = \tau$ for all $\tau \in [0, 1]$.

(ii) $\min(A, \tau) = \min(A, \tau)$ and $\max(A, \tau) = \max(A, \tau)$ for all $\tau, \tau' \in [0, 1]$.

(iii) If $\tau \geq \tau' \in [0, 1]$, then $\min(A, \tau) \geq \min(A, \tau')$ and $\max(A, \tau) \geq \max(A, \tau')$ for all $\tau \in [0, 1]$.

**Definition 2.4.** Let $\overline{A}$ be an i-v fuzzy set of a set $X$ and $[t_1, t_2] \in D[0, 1]$. Then the set $\overline{U}(\overline{A} : [t_1, t_2]) = \{ x \in X | \overline{A}(x) \geq [t_1, t_2] \}$ is called the upper level set of $\overline{A}$.

Note that

\[
\overline{U}(\overline{A} : [t_1, t_2]) = \{ x \in X | [A^-(x), A^+(x)] \geq [t_1, t_2] \}
\]

\[
= \{ x \in X | A^-(x) \geq t_1 \} \cap \{ x \in X | A^+(x) \geq t_2 \}
\]

\[
= (U(A^- ; t_1)) \cap (U(A^+ ; t_2)).
\]
Definition 2.5. Let \( \overline{A}, \overline{B}, \overline{A}_i \ (i \in \Omega) \) be interval valued fuzzy subsets of \( X \).
The following are defined by

(i) \( \overline{A} \leq \overline{B} \) if and only if \( \overline{A}(x) \leq \overline{B}(x) \).
(ii) \( \overline{A} = \overline{B} \) if and only if \( \overline{A}(x) = \overline{B}(x) \).
(iii) \( (\overline{A} \cup \overline{B})(x) = \max \{\overline{A}(x) , \overline{B}(x)\} \).
(iv) \( (\overline{A} \cap \overline{B})(x) = \min \{\overline{A}(x) , \overline{B}(x)\} \).
(v) \( (\bigcap_{i \in \Omega} \overline{A}_i)(x) = \inf \{\overline{A}_i(x) | i \in \Omega\} \).
(vi) \( (\bigcup_{i \in \Omega} \overline{A}_i)(x) = \sup \{\overline{A}_i(x) | i \in \Omega\} \).

where \( \inf \{\overline{A}_i(x) | i \in \Omega\} = [\inf_{i \in \Omega} \{\overline{A}_i^{-}(x)\} , \inf_{i \in \Omega} \{\overline{A}_i^{+}(x)\}] \) is the interval valued infimum norm and \( \sup \{\overline{A}_i(x) | i \in \Omega\} = [\sup_{i \in \Omega} \{\overline{A}_i^{-}(x)\} , \sup_{i \in \Omega} \{\overline{A}_i^{+}(x)\}] \) is the interval valued supremum norm.

Definition 2.6. Let \( ' \) be a binary composition in a set \( S \). The product \( \overline{A} \odot \overline{B} \) of any two i-v fuzzy subsets \( \overline{A}, \overline{B} \) of \( S \) is defined by

\[
(\overline{A} \odot \overline{B})(x) = \begin{cases} 
\sup \{\min \{\overline{A}(a) , \overline{B}(b)\} \}, & \text{if } x \text{ is expressed as } x = a.b \\
0 & \text{otherwise.}
\end{cases}
\]

Since semigroup \( S \) is associative, the operation \( \odot \) is associative. We denote \( x'y \) instead of \( x.y \) and \( \overline{A} \overline{B} \) for \( \overline{A} \odot \overline{B} \).

Definition 2.7. Let \( I \) be a subset of a semigroup \( S \). Define a function \( \overline{\chi}_I : S \rightarrow [0, 1] \) by

\[
\overline{\chi}_I(x) = \begin{cases} 
1 & \text{if } x \in I \\
0 & \text{otherwise}
\end{cases}
\]

for all \( x \in S \). Clearly \( \overline{\chi}_I \) is an i-v fuzzy subset of \( S \). Throughout this paper \( \overline{\chi}_S \) is denoted by \( \overline{S} \) and \( \overline{S} \) will denote a semigroup unless otherwise mentioned.

Definition 2.8. An i-v fuzzy subset \( \overline{\lambda} \) of \( S \) is called an i-v fuzzy subsemigroup of \( S \) if \( \overline{\lambda}(ab) \geq \min \{\overline{\lambda}(a) , \overline{\lambda}(b)\} \), for all \( a, b \in S \).

Definition 2.9. An i-v fuzzy subset \( \overline{\lambda} \) of \( S \) is called an i-v fuzzy left (right) ideal of \( S \) if \( \overline{\lambda}(ab) \geq \overline{\lambda}(b) \) (\( \overline{\lambda}(ab) \geq \overline{\lambda}(a) \)), for all \( a, b \in S \).

An i-v fuzzy subset \( \overline{\lambda} \) of \( S \) is called an i-v fuzzy two-sided ideal (simply i-v fuzzy ideal) of \( S \) if it is both an i-v fuzzy left ideal and an i-v fuzzy right ideal of \( S \).

Every i-v fuzzy right(left, two-sided) ideal of \( S \) is an i-v fuzzy subsemigroup of \( S \). However the converse is not true in general as shown in the following example.

Example 2.10. Let \( S = \{0,1,2,3\} \) be a semigroup with the multiplication table given below:
Define $\lambda : S \to D[0, 1]$ by $\lambda(0) = [0.8, 0.9], \lambda(1) = [0.6, 0.7], \lambda(2) = [0.1, 0.2]$ and $\lambda(3) = [0.3, 0.4]$. Then $\lambda$ is an i-v fuzzy subsemigroup of $S$. $\lambda$ is not an i-v fuzzy left (right, two-sided) ideal of $S$. For, $(\lambda \lambda)(x) \leq \lambda(x)$ for all $x \in S$.

$$
(\lambda S)(2) = \sup_{2=a} \{\min_i \{\lambda(a), S(b)\}\}
$$

$$= \min_i \{\lambda(1), S(2)\} \text{ as } 2=1.2
$$

$$= [0.6, 0.7] \nless \lambda(2) = [0.1, 0.2] \text{ and}
$$

$$\lambda(3) = \min_i \{\lambda(1), \lambda(3)\} \text{ as } 3=1.3
$$

$$= [0.6, 0.7] \nless [0.3, 0.4].$$

Thus $\lambda$ is neither an i-v fuzzy right ideal nor an i-v fuzzy left ideal of $S$. That is $\lambda$ is not an i-v fuzzy ideal of $S$.

**Lemma 2.11.** Let $\lambda, \mu$ and $\nu$ be i-v fuzzy subsets of $S$, then

(i) $\lambda \cup (\mu \cap \nu) = (\lambda \cup \mu) \cap (\lambda \cup \nu)$

(ii) $\lambda \cap (\mu \cup \nu) = (\lambda \cap \mu) \cup (\lambda \cap \nu)$

**Proof.** Straight forward. □

**Lemma 2.12.** Let $\lambda, \mu$ and $\nu$ be i-v fuzzy subsets of $S$. Then,

(i) $\lambda(\mu \cup \nu) = (\lambda \mu) \cup (\lambda \nu)$

(ii) $\lambda(\mu \cap \nu) \leq (\lambda \mu) \cap (\lambda \nu)$

**Proof.** Straight forward. □

**Lemma 2.13.** Let $\lambda, \mu$ and $\nu$ be i-v fuzzy subsets of $S$. If $\lambda \leq \mu$ then

$$\lambda \nu \leq \mu \nu \text{ and } \nu \lambda \leq \nu \mu.$$

**Proof.** Omitted as it is straight forward. □

**Proposition 2.14.** Let $A$ be a nonempty subset of a semigroup $S$. $A$ is a subsemigroup (resp. left ideal, right ideal, two-sided ideal) of $S$ if and only if $\chi_A$ is an i-v fuzzy subsemigroup (resp. left ideal, right ideal, two-sided ideal) of $S$.

**Proof.** Let $A$ be a subsemigroup of $S$.

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Let $a, b \in S$. Suppose $\chi_A(ab) < \min_i \{\chi_A(a), \chi_A(b)\}$, then $\chi_A(a) = \chi_A(b) = 1$ and $\chi_A(ab) = 0$. This implies that $a, b \in A$. Since $A$ is
a subsemigroup of $S$, $ab \in S$ and hence $\overline{\lambda}_A(ab) = 1$, a contradiction. Thus $\overline{\lambda}_A(ab) \geq \min^I\{\overline{\lambda}_A(a), \overline{\lambda}_A(b)\}$ for all $a, b \in S$.

Conversely, assume that $\overline{\lambda}_A$ is an i-v fuzzy subsemigroup of $S$. Let $a, b \in A$. Then $\overline{\lambda}_A(a) = 1 = \overline{\lambda}_A(b)$. As $\overline{\lambda}_A$ is an i-v fuzzy subsemigroup, $\min^I\{\overline{\lambda}_A(a), \overline{\lambda}_A(b)\} = 1 \leq \overline{\lambda}_A(ab)$. This implies that $\overline{\lambda}_A(ab) = 1$ and hence $ab \in A$. Thus $A$ is a subsemigroup of $S$. □

**Proposition 2.15.** Let $\overline{\lambda}$ be an i-v fuzzy subset of $S$. $\overline{\lambda}$ is an i-v fuzzy left ideal (resp. subsemigroup, right ideal) of $S$, if and only if $\overline{\lambda} \overline{\lambda} \leq \overline{\lambda}$ (resp. $\overline{\lambda} \leq \overline{\lambda} \overline{\lambda}$).

**Proof.** Let $x \in S$. Assume that $\overline{\lambda}$ is an i-v fuzzy left ideal of $S$. If $(\overline{\lambda} \overline{\lambda})(x) = \overline{0}$, then it is clear that $(\overline{\lambda} \overline{\lambda})(x) \leq \overline{\lambda}(x)$. Otherwise, there exist $a, b \in S$ such that $x = ab$. Then, since $\overline{\lambda}$ is an i-v fuzzy left ideal of $S$, we have

\[
(\overline{\lambda} \overline{\lambda})(x) = \sup_{x=ab} \{\min^I\{\overline{\lambda}(a), \overline{\lambda}(b)\}\}
= \sup_{x=ab} \{\min^I\{1, \overline{\lambda}(b)\}\}
= \sup_{x=ab} \{\overline{\lambda}(b)\}
\leq \sup_{x=ab} \{\overline{\lambda}(ab)\}
= \overline{\lambda}(x)
\]

and so $\overline{\lambda} \overline{\lambda} \leq \overline{\lambda}$.

Conversely, assume that $\overline{\lambda} \overline{\lambda} \leq \overline{\lambda}$, for any i-v fuzzy subset $\overline{\lambda}$ of $S$. Let $x, y, z \in S$ such that $z = xy$. Then we have

\[
\overline{\lambda}(xy) = \overline{\lambda}(z) \geq (\overline{\lambda} \overline{\lambda})(z)
= \sup_{z=xy} \{\min^I\{\overline{\lambda}(p), \overline{\lambda}(q)\}\}
\geq \min^I\{\overline{\lambda}(x), \overline{\lambda}(y)\}
= \min^I\{1, \overline{\lambda}(y)\}
= \overline{\lambda}(y).
\]

Hence $\overline{\lambda}$ is an i-v fuzzy left ideal of $S$. □

**Lemma 2.16.** Let $\overline{\lambda}$ and $\overline{\mu}$ be any i-v fuzzy subsemigroups (resp. right ideals, left ideals, two-sided ideals) of $S$. Then $\overline{\lambda} \cap \overline{\mu}$ is also an i-v fuzzy subsemigroup (resp. right ideal, left ideal, two-sided ideal) of $S$.

**Proof.** Let $\overline{\lambda}$ and $\overline{\mu}$ be any i-v fuzzy subsemigroups of $S$. Let $a, b \in S$. Then

\[
(\overline{\lambda} \cap \overline{\mu})(ab) = \min^I\{(\overline{\lambda}(ab), \overline{\mu}(ab)\}\}
\geq \min^I\{\min^I\{\overline{\lambda}(a), \overline{\lambda}(b)\}, \min^I\{\overline{\mu}(a), \overline{\mu}(b)\}\}\}
= \min^I\{\min^I\{\overline{\lambda}(a), \overline{\mu}(a)\}, \min^I\{\overline{\lambda}(b), \overline{\mu}(b)\}\}\}
\]

Thus $\overline{\lambda} \cap \overline{\mu}$ is an i-v fuzzy subsemigroup of $S$. □
The following lemma can easily be proved.

**Lemma 2.17.** Let $A$ and $B$ be nonempty subsets of $S$. Then the following properties hold.

(i) $\chi_A \cap \chi_B = \chi_{A \cap B}$.

(ii) $\chi_A \chi_B = \chi_{AB}$.

**Lemma 2.18.** If $\lambda$ is an i-v fuzzy right (left) ideal of $S$, then $\lambda \cup (S \lambda)$ is an i-v fuzzy ideal of $S$.

**Proof.** Suppose $\lambda$ is an i-v fuzzy right ideal of $S$. Then

\[
\overline{\lambda} \cup (S \lambda) = (S \overline{\lambda}) \cup (S \lambda) \text{ by Lemma 2.12 (i)}
\]

\[
\leq (S \overline{\lambda}) \cup (S \lambda) = S \overline{\lambda}
\]

Thus $\overline{\lambda} \cup (S \lambda)$ is an i-v fuzzy left ideal of $S$ by Proposition 2.15. Also

\[
(\overline{\lambda} \cup (S \lambda)) S = (\overline{\lambda} S) \cup (S \lambda S) \text{ by Lemma 2.12 (i)}
\]

\[
\leq (\overline{\lambda} S) \cup (S \lambda S)
\]

\[
\leq \overline{\lambda} \cup (S \lambda), \text{ since } \overline{\lambda} \text{ is an i-v fuzzy right ideal of } S.
\]

Hence $\overline{\lambda} \cup (S \lambda)$ is an i-v fuzzy right ideal of $S$. Therefore $\overline{\lambda} \cup (S \lambda)$ is an i-v fuzzy ideal of $S$. \qed

**Theorem 2.19.** Let $\overline{\lambda}$ be an i-v fuzzy subset of $S$. $\overline{\lambda} = [f^-, f^+]$ is an i-v fuzzy subsemigroup (resp. right ideal, left ideal, two-sided ideal) of $S$, if and only if $f^-$ and $f^+$ are fuzzy subsemigroups (resp. right ideal, left ideal, two-sided ideal) of $S$.

**Proof.** Assume that $\overline{\lambda}$ is an i-v fuzzy subsemigroup of $S$. For any $x, y \in S$, we have

\[
[f^-(xy), f^+(xy)] = \overline{\lambda}(xy)
\]

\[
\geq \min' \{\overline{\lambda}(x), \overline{\lambda}(y)\}
\]

\[
= \min' \{[f^-(x), f^+(x)], [f^-(y), f^+(y)]\}
\]

\[
= \min \{f^-(x), f^-(y)\}, \min \{f^+(x), f^+(y)\}.
\]

It follows that $f^-(xy) \geq \min \{f^-(x), f^-(y)\}$ and $f^+(xy) \geq \min \{f^+(x), f^+(y)\}$. Thus $f^-$ and $f^+$ are fuzzy subsemigroups of $S$.

Conversely, assume that $f^-$ and $f^+$ are fuzzy subsemigroups of $S$ and let $x, y \in S$. Then

\[
\overline{\lambda}(xy) = [f^-(xy), f^+(xy)]
\]

\[
\geq \min \{f^-(x), f^-(y)\}, \min \{f^+(x), f^+(y)\}
\]

\[
= \min' \{f^-(x), f^+(x)\}, \min' \{f^-(y), f^+(y)\}
\]

\[
= \min' \{\overline{\lambda}(x), \overline{\lambda}(y)\}.
\]
Thus $\overline{\lambda}$ is an i-v fuzzy subsemigroup of $S$.

**Theorem 2.20.** Let $\overline{\lambda}$ be an i-v fuzzy subset of $S$. $\overline{\lambda}$ is an i-v fuzzy subsemigroup (resp. right ideal, left ideal, two-sided ideal) of $S$ if and only if $\overline{U}(\overline{\lambda}: [r_1, r_2])$ is a subsemigroup (resp. right ideal, left ideal, two-sided ideal) of $S$.

**Proof.** Assume that $\overline{\lambda}$ is an i-v fuzzy subset of $S$ and let $[r_1, r_2] \in D[0, 1]$ such that $x, y \in \overline{U}(\overline{\lambda}: [r_1, r_2])$. Then

\[
\overline{\lambda}(xy) \geq \min^i(\overline{\lambda}(x), \overline{\lambda}(y)) \\
\geq \min^i([r_1, r_2], [r_1, r_2]) \\
= [r_1, r_2]
\]

Thus $xy \in \overline{U}(\overline{\lambda}: [r_1, r_2])$. Hence $\overline{U}(\overline{\lambda}: [r_1, r_2])$ is a subsemigroup of $S$.

Conversely, assume that $\overline{U}(\overline{\lambda}: [r_1, r_2])$ is a subsemigroup of $S$ for all $[r_1, r_2] \in D[0, 1]$. Let $x, y \in S$. Suppose $\overline{\lambda}(xy) < \min^i(\overline{\lambda}(x), \overline{\lambda}(y))$. Then there exists an interval $\overline{\sigma} = [a_1, a_2] \in D[0, 1]$ such that $\overline{\lambda}(xy) < [a_1, a_2] < \min^i(\overline{\lambda}(x), \overline{\lambda}(y))$. This implies that $\overline{\lambda}(x) > [a_1, a_2]$ and $\overline{\lambda}(y) > [a_1, a_2]$. Then we have $x, y \in \overline{U}(\overline{\lambda}: [a_1, a_2])$ and since $\overline{U}(\overline{\lambda}: [a_1, a_2])$ is a subsemigroup of $S$, $xy \in \overline{U}(\overline{\lambda}: [a_1, a_2])$. Hence, $\overline{\lambda}(xy) > [a_1, a_2]$, a contradiction. Thus $\overline{\lambda}(xy) \geq \min^i(\overline{\lambda}(x), \overline{\lambda}(y))$ for all $x, y \in S$. \hfill $\square$

**3. I-v fuzzy bi-ideals and I-v fuzzy quasi-ideals of a semigroup**

In this section we introduce i-v fuzzy bi-ideals and i-v fuzzy quasi-ideals in a semigroup $S$. We also give a characterization for semigroups.

**Definition 3.1.** An i-v fuzzy subsemigroup $\overline{\lambda}$ of $S$ is called an i-v fuzzy bi-ideal of $S$ if $\overline{\lambda}(xyz) \geq \min^i(\overline{\lambda}(x), \overline{\lambda}(z))$ for all $x, y, z \in S$.

**Proposition 3.2.** Let $A$ be a nonempty subset of $S$. $A$ is a bi-ideal of $S$ if and only if $\overline{\lambda}_A$ is an i-v fuzzy bi-ideal of $S$.

**Proof.** Assume that $A$ is a bi-ideal of $S$. By Proposition 2.14 $\overline{\lambda}_A$ is an i-v fuzzy subsemigroup of $S$. Suppose that $\overline{\lambda}_A(xyz) < \min^i(\overline{\lambda}_A(x), \overline{\lambda}_A(z))$ for some $x, y, z \in S$. Then $\overline{\lambda}_A(x) = \top$ and $\overline{\lambda}_A(z) = \top$. This implies that $x, z \in A$ and since $A$ is a bi-ideal of $S$, $xyz \in ASA \subseteq A$. Thus $\overline{\lambda}_A(xyz) = \top$, a contradiction. Hence $\overline{\lambda}_A(xyz) \geq \min^i(\overline{\lambda}_A(x), \overline{\lambda}_A(z))$ for all $x, y, z \in S$. Therefore $\overline{\lambda}_A$ is an i-v fuzzy bi-ideal of $S$.

Conversely, assume that $\overline{\lambda}_A$ is an i-v fuzzy bi-ideal of $S$. Then by Proposition 2.14 $A$ is a subsemigroup of $S$. Let $a = xyz \in ASA$ such that $x, z \in A$. Then we have

\[
\overline{\lambda}_A(a) = \overline{\lambda}_A(xyz) \\
\geq \min^i(\overline{\lambda}_A(x), \overline{\lambda}_A(z)) \\
= \min^i(\top, \top) \\
= \top.
\]
Hence $\chi_A(a) = 1$ and so $a = xyz \in A$. Thus $ASA \subseteq A$ which implies that $A$ is a bi-ideal of $S$. □

**Proposition 3.3.** Let $\lambda$ be an i-v fuzzy subsemigroup of $S$. $\lambda$ is an i-v fuzzy bi-ideal of $S$ if and only if $\lambda S \lambda \leq \lambda$.

**Proof.** Assume that $\lambda$ is an i-v fuzzy bi-ideal of $S$ and let $a \in S$. In the case when $(\lambda S \lambda)(a) = 0 \leq \lambda(a)$, otherwise there exist $x, y, p, q \in S$ such that $a = xy$ and $x = pq$. Since $\lambda$ is an i-v fuzzy bi-ideal of $S$, we have $\lambda(pqy) \geq \min^i \{\lambda(p), \lambda(y)\}$. Therefore

$$(\lambda S \lambda)(a) = \sup_{a=xy} \{\min^i \{\lambda S \lambda(x), \lambda(y)\}\}
= \sup_{a=xy} \{\min^i \{\sup_{x=pq} \{\min^i \{\lambda S \lambda(p), \lambda(y)\}\}\}\}
= \sup_{a=xy} \{\min^i \{\sup_{x=pq} \{\min^i \{\lambda S \lambda(x), \lambda(y)\}\}\}\}
= \sup_{a=xy} \{\min^i \{\lambda S \lambda(x), \lambda(y)\}\}
\leq \sup_{a=xy} \{\lambda S \lambda(x), \lambda(y)\}
= \lambda(a)$$

and so we have $\lambda S \lambda \leq \lambda$.

Conversely, assume that $\lambda S \lambda \leq \lambda$ holds for any i-v fuzzy subsemigroup $\lambda$. Let $a, x, y, z \in S$ such that $a = xyz$. Then we have

$$\lambda(xyz) = \lambda(x)
\geq (\lambda S \lambda)(a)
= \sup_{a=bc} \{\min^i \{\lambda S \lambda(b), \lambda(c)\}\}
\geq \min^i \{\lambda(x), \lambda(y), \lambda(z)\}
= \min^i \{\lambda(x), \lambda(z)\}.$$

Thus $\lambda(xyz) \geq \min^i \{\lambda(x), \lambda(z)\}$ for all $x, y, z \in S$. Hence $\lambda$ is an i-v fuzzy bi-ideal of $S$. □

**Lemma 3.4.** Let $\lambda$ and $\mu$ be any i-v fuzzy subset and i-v fuzzy bi-ideal of $S$ respectively. Then the products $\lambda \mu$ and $\mu \lambda$ are i-v fuzzy bi-ideals of $S$.

**Proof.** Since $\mu$ is an i-v fuzzy bi-ideal of $S$, by Proposition 3.3 we have

$$(\lambda \mu) (\lambda \mu) = \lambda(\mu S \lambda \mu)
\leq \lambda (\mu S \mu)
\leq \lambda.$$

Hence, it follows from Proposition 2.15 that $(\lambda S \mu)$ is an i-v fuzzy subsemigroup of $S$. We have
\[(\overline{\lambda} \mu) \overline{S} (\overline{\lambda} \mu) = \overline{\lambda} \mu (\overline{S} \overline{\lambda}) \mu \leq \overline{\lambda} \mu (\overline{S} \overline{S}) \mu \leq \overline{\lambda} (\overline{\mu} \overline{S} \mu) \leq \overline{\lambda} \mu \overline{S} \mu.
\]

Thus it follows from Proposition 3.3 that \(\overline{\lambda} \mu\) is an i-v fuzzy bi-ideal of \(S\). Similarly, it can be shown that \(\overline{\mu} \overline{\lambda}\) is an i-v fuzzy bi-ideal of \(S\). \(\square\)

**Lemma 3.5.** Let \(\overline{\lambda}\) and \(\overline{\mu}\) be two i-v fuzzy bi-ideals of \(S\). Then \(\overline{\lambda} \cap \overline{\mu}\) is an i-v fuzzy bi-ideal of \(S\).

**Proof.** Let \(a, b\) and \(x\) be elements of \(S\). Then
\[
(\overline{\lambda} \cap \overline{\mu})(ab) = \min^I(\overline{\lambda}(ab), \overline{\mu}(ab))
\]
\[
\geq \min^I(\min^I(\overline{\lambda}(a), \overline{\lambda}(b)), \min^I(\overline{\mu}(a), \overline{\mu}(b)))
\]
\[
= \min^I(\min^I(\overline{\lambda}(a), \overline{\lambda}(b)), \min^I(\overline{\mu}(a), \overline{\mu}(b)))
\]
\[
= \min^I(\min^I(\overline{\lambda}(a), \overline{\lambda}(b)), \min^I(\overline{\mu}(a), \overline{\mu}(b)))
\]
and
\[
(\overline{\lambda} \cap \overline{\mu})(axb) = \min^I(\overline{\lambda}(axb), \overline{\mu}(axb))
\]
\[
\geq \min^I(\min^I(\overline{\lambda}(a), \overline{\lambda}(b)), \min^I(\overline{\mu}(a), \overline{\mu}(b)))
\]
\[
= \min^I(\min^I(\overline{\lambda}(a), \overline{\lambda}(b)), \min^I(\overline{\mu}(a), \overline{\mu}(b)))
\]
\[
= \min^I(\min^I(\overline{\lambda}(a), \overline{\lambda}(b)), \min^I(\overline{\mu}(a), \overline{\mu}(b))).
\]

Hence \(\overline{\lambda} \cap \overline{\mu}\) is an i-v fuzzy bi-ideal of \(S\). \(\square\)

We now introduce the notion of i-v fuzzy interior ideal of \(S\).

**Definition 3.6.** An i-v fuzzy subset \(\overline{A}\) of \(S\) is called an i-v fuzzy interior ideal of \(S\) if \(\overline{\lambda}(xay) \geq \overline{\lambda}(a)\) for all \(x, a, y \in S\).

**Proposition 3.7.** Let \(A\) be a nonempty subset of \(S\). Then \(A\) is an interior ideal of \(S\) if and only if \(\overline{\lambda}_A\) is an i-v fuzzy interior ideal of \(S\).

**Proof.** Let \(A\) be an interior ideal of \(S\). Suppose \(\overline{\lambda}_A(xay) < \overline{\lambda}_A(a)\) for some \(x, a, y \in S\). Then \(\overline{\lambda}_A(a) = 1\) and \(\overline{\lambda}_A(xay) = 0\). Since \(\overline{\lambda}_A(a) = 1\), \(a \in A\) and \(A\) is an interior ideal of \(S\), \(xay \in SAS \subseteq A\). Thus \(\overline{\lambda}_A(xay) = 1\), a contradiction. Hence \(\overline{\lambda}_A(xay) \geq \overline{\lambda}_A(a)\) for all \(x, a, y \in S\).

Conversely, assume that \(\overline{\lambda}_A\) is an i-v fuzzy interior ideal of \(S\). Let \(z = xay\) such that \(x, a, y \in S\) and \(a \in A\). Then \(\overline{\lambda}_A(xay) \geq \overline{\lambda}_A(a) = 1\). This implies that \(\overline{\lambda}_A(xay) = 1\) and so \(xay \in A\). Hence we have \(SAS \subseteq A\) and so \(A\) is an interior ideal of \(S\). \(\square\)

**Proposition 3.8.** Let \(\overline{A}\) be an i-v fuzzy subset of \(S\). \(\overline{A}\) is an i-v fuzzy interior ideal of \(S\) if and only if \(\overline{S} \overline{A} \overline{S} \overline{A} \leq \overline{A}\).
Proof. Assume that \( \lambda \) is an i-v fuzzy interior ideal of \( S \). Let \( z \in S \). If there exist elements \( x, y, u, v \in S \) such that \( z = xy \) and \( x = uv \), then since \( \lambda(yv) \geq \lambda(v) \), we have

\[
(S \lambda S)(z) = \sup_{z = xy} \{ \min_{z = xy} \{ (S \lambda)(x), \lambda(y) \} \} \\
= \sup_{z = xy} \{ \min_{z = xy} \{ \sup_{z = xy} \{ \min_{z = xy} \{ (S \lambda)(x), \lambda(y) \} \} \} \} \\
\leq \sup_{z = xy} \{ \min_{z = xy} \{ \sup_{z = xy} \{ \min_{z = xy} \{ (S \lambda)(x), \lambda(y) \} \} \} \} \\
= \sup_{z = xy} \{ \lambda(v) \} \\
\leq \sup_{z = uv} \{ \lambda(x) \} \\
= \lambda(z).
\]

In the other case, \( (S \lambda S)(z) = 0 \leq \lambda(z) \). Therefore \( S \lambda S \leq \lambda \).

Conversely, assume that \( S \lambda S \leq \lambda \) holds for any i-v fuzzy subset \( \lambda \) of \( S \).

Let \( x, a, y \in S \). Then we have

\[
\lambda(xay) \geq (S \lambda S)(xay) \\
= \sup_{xay = pq} \{ \min_{xay = pq} \{ (S \lambda)(p), \lambda(q) \} \} \\
\geq \min_{xay = pq} \{ (S \lambda)(xa), \lambda(y) \} \\
= (S \lambda)(xa) \\
= \sup_{xa = cd} \{ \min_{xa = cd} \{ (S \lambda)(c), \lambda(d) \} \} \\
\geq \min_{xa = cd} \{ (S \lambda)(x), \lambda(a) \} = \lambda(a).
\]

Consequently, \( \lambda \) is an i-v fuzzy interior ideal of \( S \). \( \square \)

**Lemma 3.9.** Every i-v fuzzy ideal of \( S \) is an i-v fuzzy interior ideal of \( S \).

**Proof.** Let \( \lambda \) be an i-v fuzzy ideal of \( S \). Consider

\[
S \lambda S = (S \lambda S) S \\
\leq \lambda S S, \text{ since } \lambda \text{ is an i-v fuzzy left ideal} \\
\leq \lambda, \text{ since } \lambda \text{ is an i-v fuzzy right ideal.}
\]

Thus \( \lambda \) is an i-v fuzzy interior ideal of \( S \). \( \square \)

The converse of the above lemma is not true in general as shown by the following example.

**Example 3.10.** Let \( S = \{ a, b, c, d \} \) be a semigroup with the multiplication table given below:

\[
\begin{array}{cccc}
\cdot & a & b & c & d \\
a & a & a & a & a \\
b & a & a & a & a \\
c & a & a & b & a \\
d & a & a & b & b \\
\end{array}
\]
Let $\mathcal{X}$ be an i-v fuzzy subset of $S$ such that $\mathcal{X}(a) = [0.6, 0.8]$, $\mathcal{X}(c) = [0.4, 0.5]$, $\mathcal{X}(b) = [0.1, 0.2]$. Then $\mathcal{X}$ is an i-v fuzzy interior ideal of $S$, but it is not an i-v fuzzy ideal $S$. In fact, $\mathcal{X}(xyz) = \mathcal{X}(a) = [0.6, 0.8] \geq \mathcal{X}(y)$ for all $x, y, z \in S$. Thus $\mathcal{X}$ is an i-v fuzzy interior ideal of $S$. But since $\mathcal{X}(dc) = \mathcal{X}(b) = [0.1, 0.2] < \mathcal{X}(c) = [0.4, 0.5]$, $\mathcal{X}$ is not an i-v fuzzy left ideal of $S$, that is, it is not an i-v fuzzy ideal of $S$.

Now we introduce the notion of i-v fuzzy quasi-ideal of $S$ and examine the conditions on semigroups through i-v fuzzy subsets.

**Definition 3.11.** An i-v fuzzy subset $\mathcal{X}$ of a semigroup $S$ is called an i-v fuzzy quasi-ideal of $S$ if $(\mathcal{X} \mathcal{S}) \cap (\mathcal{S} \mathcal{X}) \leq \mathcal{X}$.

**Lemma 3.12.** For any nonempty subset $A$ of $S$, $\mathcal{X}_A$ is an i-v fuzzy quasi-ideal of $S$ if and only if $A$ is a quasi-ideal of $S$.

**Proof.** Suppose $\mathcal{X}_A$ is an i-v fuzzy quasi-ideal of $S$. Let $x \in SA \cap AS$. Then $x \in SA$ and $x \in AS$. This implies that $x = sa$ and $x = a_1s_1$ for some $a, a_1 \in A$ and $s, s_1 \in S$. Now

$$[(\mathcal{S} \mathcal{X}_A) \cap (\mathcal{X}_A \mathcal{S})](x) = \min^i\{(\mathcal{S} \mathcal{X}_A)(x), (\mathcal{X}_A \mathcal{S})(x)\}$$

$$\geq \min^i\{\sup_{x=yz}\{\min^i\{\mathcal{S}(y), \mathcal{X}_A(z)\}\}, \sup_{x=pq}\{\min^i\{\mathcal{X}_A(y), \mathcal{S}(q)\}\}\}$$

$$\geq \min^i\{\min^i\{\mathcal{S}(s), \mathcal{X}_A(a)\}, \min^i\{\mathcal{X}_A(a_1), \mathcal{S}(s_1)\}\}$$

$$= \min^i\{\mathcal{X}_A(a), \mathcal{X}_A(a_1)\} = \min^i\{\mathcal{S}, \mathcal{X}_A(a_1)\} = \min^i\{\mathcal{S}, \mathcal{X}_A(a_1)\} = \mathcal{I}.$$

Since $\mathcal{X}_A$ is an i-v fuzzy quasi-ideal, $\mathcal{X}_A(x) \geq ((\mathcal{S} \mathcal{X}_A) \cap (\mathcal{X}_A \mathcal{S}))(x) \geq \mathcal{I}$. This implies that $\mathcal{X}_A(x) = \mathcal{I}$ and hence $x \in A$. Thus $SA \cap AS \subseteq A$ and so $A$ is a quasi-ideal of $S$.

Conversely, assume that $A$ is a quasi-ideal of $S$. Suppose $\mathcal{X}_A(x) < ((\mathcal{S} \mathcal{X}_A) \cap (\mathcal{X}_A \mathcal{S}))(x)$ for some $x \in S$. This means that $\mathcal{X}_A(x) = \mathcal{S}$ and $((\mathcal{S} \mathcal{X}_A) \cap (\mathcal{X}_A \mathcal{S}))(x) = \mathcal{I}$. This implies that $(\mathcal{X}_A \mathcal{S})(x) = \mathcal{I}$ and $(\mathcal{S} \mathcal{X}_A)(x) = \mathcal{I}$. By Lemma 2.17 $((\mathcal{X}_A \mathcal{S})(x) = \mathcal{I}$ and $(\mathcal{S} \mathcal{X}_A)(x) = \mathcal{I}$. Thus $x \in AS \cap SA$. Since $A$ is a quasi-ideal, $x \in AS \cap SA \subseteq A$ and hence $x \in A$, a contradiction. Therefore, $\mathcal{X}_A(x) \geq ((\mathcal{X}_A \mathcal{S}) \cap (\mathcal{S} \mathcal{X}_A))(x)$ for all $x \in S$. It follows that $\mathcal{X}_A$ is an i-v fuzzy quasi-ideal of $S$. \hfill \Box

**Lemma 3.13.** Every i-v fuzzy quasi-ideal of $S$ is an i-v fuzzy subsemigroup of $S$.

**Proof.** Let $\mathcal{X}$ be an i-v fuzzy quasi-ideal of $S$. Since $\mathcal{X} \subseteq \mathcal{S} \mathcal{X} \subseteq \mathcal{S} \mathcal{X} \subseteq \mathcal{X}$ and $\mathcal{X} \subseteq \mathcal{X} \mathcal{S} \subseteq \mathcal{X} \mathcal{S} \subseteq \mathcal{S} \mathcal{X}$. Hence $\mathcal{X} \leq ((\mathcal{S} \mathcal{X}) \cap (\mathcal{X} \mathcal{S})) \leq \mathcal{X}$, as $\mathcal{X}$ is an i-v fuzzy quasi-ideal of $S$. Thus $\mathcal{X}$ is an i-v fuzzy subsemigroup of $S$. \hfill \Box
Lemma 3.14. Every i-v fuzzy left (right, two-sided) ideal of \(S\) is an i-v fuzzy quasi-ideal of \(S\).

Proof. Let \(X\) be an i-v fuzzy left ideal of \(S\). Then we have \(S \cap X \leq X\) and \((S \cap X) \cap (X \cap S) = S \cap X \leq X\). This means that \(X\) is an i-v fuzzy quasi-ideal of \(S\). \(\square\)

However, the converse of the above lemma is not true in general, which is demonstrated by the following example.

Example 3.15. Let \(S\) and \(X\) be as in Example 2.10. Then \(X\) is an i-v fuzzy quasi-ideal of \(S\) and \(X\) is not an i-v fuzzy left (right, two-sided) ideal of \(S\). In fact, \(((S \cap X) \cap (X \cap S))(x) \leq X(x)\) for all \(x \in S\). Thus \(X\) is an i-v fuzzy quasi-ideal of \(S\). But since, 

\[
\begin{align*}
(X S)(2) &= \sup_{2=ab}\{\min_i(\lambda(a), S(b))\} \\
&= \min_i\{\lambda(1), S(2)\} \text{ as } 2 = 1.2 \\
&= \min\{0.6, 0.7\} \\
&= [0.6, 0.7] \\
&\subseteq \lambda(2) = [0.1, 0.2]
\end{align*}
\]

and

\[
\begin{align*}
(S X)(3) &= \min_i\{S(3), \lambda(1)\} \text{ as } 3 = 3.1 \\
&= [0.6, 0.7] \\
&\not\subseteq \lambda(3) = [0.3, 0.4].
\end{align*}
\]

Thus \(X\) is neither an i-v fuzzy right ideal nor an i-v fuzzy left ideal of \(S\). That is \(X\) is not an i-v fuzzy ideal of \(S\).

However in the case when \(S\) is regular, the converse of Lemma 3.14 is true, which is shown in Theorem 4.6.

Lemma 3.16. Let \(X\) and \(Y\) be any i-v fuzzy right ideal and any i-v fuzzy left ideal of \(S\), respectively. Then \(X \cap Y\) is an i-v fuzzy quasi-ideal of \(S\).

Proof. Since \(X\) is an i-v fuzzy right ideal and \(Y\) is an i-v fuzzy left ideal, we have \(X S \leq X\) and \(S Y \leq Y\). Now

\[
((X \cap Y) \cap (Y \cap X)) \leq (X S) \cap (Y Y), \text{ since } X \cap Y \leq X, Y
\]

Thus \(X \cap Y\) is an i-v fuzzy quasi-ideal of \(S\). \(\square\)

Corollary 3.17. For any nonempty i-v fuzzy subset \(X\) of \(S\), \((X \cup (S \cap X)) \cap (X \cup (X \cap S))\) is an i-v fuzzy quasi-ideal of \(S\).

Lemma 3.18. Every i-v fuzzy quasi-ideal of \(S\) is an i-v fuzzy bi-ideal of \(S\).

Proof. Let \(X\) be any i-v fuzzy quasi-ideal of \(S\). Then we have
\[ \text{Also} \]
\[ \text{We have} \]
\[ \text{Thus} \]
\[ \text{This implies that} \]
\[ \text{Example 3.19. Let} \]
\[ \begin{array}{c|cccc}
 \cdot & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 1 \\
3 & 0 & 0 & 1 & 2 \\
\end{array} \]
\[ \text{Define} \]
\[ \text{Thus} \]
\[ \text{Thus} \]
\[ \text{Thus} \]
\[ \text{Thus} \]
\[ \text{Thus} \]
\[ \text{Lemma 3.20. Suppose that} \]
\[ \text{Proof. Let} \]
\[ \text{Hence} \]
\[(\overline{\lambda} \overline{\mu}) (\overline{\lambda} \overline{\mu}) = (\overline{\lambda} \overline{\mu} \overline{\lambda}) \overline{\mu} \leq (\overline{\lambda} \overline{\mu}).\]

Also
\[(\overline{\lambda} \overline{\mu}) S (\overline{\lambda} \overline{\mu}) = (\overline{\lambda} (S \overline{\mu}) \overline{\lambda}) \overline{\mu} \leq (\overline{\lambda} S \overline{\mu} \overline{\lambda}) \overline{\mu} \leq (\lambda \overline{\mu}).\]

This implies that \(\overline{\lambda} \overline{\mu}\) is an i-v fuzzy bi-ideal of \(S\). \(\square\)

**Corollary 3.21.** The product of two i-v fuzzy quasi-ideals of \(S\) is an i-v fuzzy bi-ideal of \(S\).

### 4. Regular semigroups

In this section, we characterize a regular semigroup in terms of i-v fuzzy ideals, i-v fuzzy interior ideals, i-v fuzzy quasi-ideals and i-v fuzzy bi-ideals. We find the equivalent conditions on regular semigroups.

**Theorem 4.1.** For a semigroup \(S\), the following conditions are equivalent.

1. \(S\) is regular.
2. \(R \cap L = RL\) for every right ideal \(R\) of \(S\) and every left ideal \(L\) of \(S\).
3. \(Q = QSQ\) for every quasi-ideal \(Q\) of \(S\).

**Proof.** The equivalence of (1) and (2) is due to Iseki[6], Theorem 1 and the equivalence of (1) and (3) follows from Steinfeld[10], p.10. \(\square\)

**Theorem 4.2.** The following conditions are equivalent.

1. \(S\) is regular.
2. \(\overline{\lambda} \overline{\mu} = \overline{\lambda} \cap \overline{\mu}\) for every i-v fuzzy right ideal \(\overline{\lambda}\) and every i-v fuzzy left ideal \(\overline{\mu}\) of \(S\).

**Proof.** (1) \(\Rightarrow\) (2). Let \(\overline{\lambda}\) and \(\overline{\mu}\) be any i-v fuzzy right ideal and i-v fuzzy left ideal of a regular semigroup \(S\) respectively. Let \(x \in S\). Then
\[
(\overline{\lambda} \overline{\mu})(x) = \sup_{a=ab} \{\min_{b=ab} \{\overline{\lambda}(a), \overline{\mu}(b)\}\}
\leq \min_{a=ab} \{\overline{\lambda}(ab), \overline{\mu}(ab)\}
= (\overline{\lambda} \cap \overline{\mu})(x)
\]
and so \(\overline{\lambda} \overline{\mu} \leq \overline{\lambda} \cap \overline{\mu}\). Again let \(a \in S\). Then since \(S\) is regular, there exists \(x \in S\) such that \(a = axa\). Now
\[
(\overline{\lambda} \overline{\mu})(a) = \sup_{a=pq} \{\min_{b=pq} \{\overline{\lambda}(p), \overline{\mu}(q)\}\}
\geq \min_{a=pq} \{\overline{\lambda}(a), \overline{\mu}(xa)\}
\geq \min_{a=pq} \{\overline{\lambda}(a), \overline{\mu}(a)\}
= (\overline{\lambda} \cap \overline{\mu})(a)
\]
and hence $\overline{x} \cap \overline{y} \geq \overline{x \cap y}$. Thus $\overline{x \cap y} = \overline{x} \cap \overline{y}$.

(2) $\Rightarrow$ (1). Assume that (2) holds. Let $R$ and $L$ be the right ideal and left ideal of $S$ respectively. Since $\overline{x}_R$ and $\overline{x}_L$ are respectively $i$-$v$ fuzzy right ideal and $i$-$v$ fuzzy left ideal, $\overline{x}_R \cap \overline{x}_L = \overline{x}_R \overline{x}_L$. By Lemma 2.17, $\overline{x}_R \cap \overline{x}_L = \overline{x}_R \overline{x}_L$. Thus $RL = R \cap L$ and hence by Theorem 4.1, $S$ is regular. $\square$

**Theorem 4.3.** Let $\overline{x}$ be an $i$-$v$ fuzzy subset of a regular semigroup $S$. Then the following conditions are equivalent.

(1) $\overline{x}$ is an $i$-$v$ fuzzy ideal of $S$.

(2) $\overline{x}$ is an $i$-$v$ fuzzy interior ideal of $S$.

**Proof.** (1) $\Rightarrow$ (2) follows by Lemma 3.9.

(2) $\Rightarrow$ (1). Assume that (2) holds. Let $a, b \in S$. Then since $S$ is regular, there exist elements $x, y \in S$ such that $a = axa$ and $b = byb$. Thus we have,

$\overline{x}(ab) = \overline{x}(axab) = \overline{x}(axab) \geq \overline{x}(a)$ and

$\overline{x}(ab) = \overline{x}(a(byb)) = \overline{x}(ab(b)) \geq \overline{x}(b)$.

This implies that $\overline{x}$ is an $i$-$v$ fuzzy ideal of $S$. $\square$

**Theorem 4.4.** For a semigroup $S$, the following conditions are equivalent.

(1) $S$ is regular.

(2) $\overline{x} = \overline{x} S \overline{x}$ for every $i$-$v$ fuzzy bi-ideal $\overline{x}$ of $S$.

(3) $\overline{x} = \overline{x} S \overline{x}$ for every $i$-$v$ fuzzy quasi-ideal $\overline{x}$ of $S$.

**Proof.** (1) $\Rightarrow$ (2). Let $\overline{x}$ be any $i$-$v$ fuzzy bi-ideal of $S$ and $a \in S$. Since $S$ is regular, there exists elements $x, y \in S$ such that $a = axa$ and $b = byb$. Then we have

$\overline{x}(ab) = \overline{x}(axb) = \overline{x}(axb) \geq \overline{x}(a)$ and

$\overline{x}(ab) = \overline{x}(a(byb)) = \overline{x}(ab(byb)) \geq \overline{x}(b)$.

This implies that $\overline{x}$ is an $i$-$v$ fuzzy ideal of $S$. $\square$
Theorem 4.5. For a regular semigroup $S$, the following conditions are equivalent.

1. Every bi-ideal of $S$ is a right (left, two-sided) ideal of $S$.
2. Every i-v fuzzy bi-ideal of $S$ is an i-v fuzzy right (left, two-sided) ideal of $S$.

Proof. (1) $\Rightarrow$ (2). Let $\lambda$ be any i-v fuzzy bi-ideal of $S$ and $a, b \in S$. Since the set $aSa$ is a bi-ideal of $S$, by the assumption it is a right ideal of $S$. Since $S$ is regular, $ab \in (aSa)S \subseteq aSa$. Thus there exists an element $x \in S$ such that $ab = axa$. Since $\lambda$ is an i-v fuzzy bi-ideal of $S$,

$$\lambda(ab) = \lambda(axa) \geq \min_i \{\lambda(a), \lambda(a)\} = \lambda(a).$$

Thus $\lambda$ is an i-v fuzzy right ideal of $S$ and hence (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1). Let $A$ be any bi-ideal of $S$. By Proposition 3.2, $\chi_A$ is an i-v fuzzy bi-ideal of $S$. Hence by assumption $\chi_A$ is an i-v fuzzy right ideal of $S$. Thus by Proposition 2.14, $A$ is a right ideal of $S$. Hence (2) $\Rightarrow$ (1). $\Box$

Theorem 4.6. For any semigroup $S$, the following conditions are equivalent.

1. $S$ is regular.
2. $\lambda \cap \mu = \mu \lambda \mu$ for every i-v fuzzy ideal $\lambda$ and every i-v fuzzy bi-ideal $\mu$ of $S$.
3. $\lambda \cap \mu = \mu \lambda \mu$ for every i-v fuzzy ideal $\lambda$ and every i-v fuzzy quasi-ideal $\mu$ of $S$.

Proof. (1) $\Rightarrow$ (2). Let $\lambda$ and $\mu$ be an i-v fuzzy ideal and an i-v fuzzy bi-ideal of $S$ respectively. Then $\mu \lambda \mu \leq \mu \mu \leq \mu$. Again $\mu \lambda \mu \leq \lambda \mu \lambda \mu \leq \mu$. Thus $\mu \lambda \mu \leq \lambda \cap \mu$. On the other hand let $a \in S$. Then since $S$ is regular, there exists $x \in S$ such that $a = axa = axaxa$. As $\lambda$ is an i-v fuzzy ideal of $S$, $\lambda(xax) \geq \lambda(a)$. Then we have

$$\mu \lambda \mu(a) = \sup_{a = yz} \{\min_i \{\mu(y), \lambda \mu(z)\}\}$$

$$\geq \min_i \{\lambda(a), \mu(xax)\}$$

$$\geq \min_i \{\mu(a), \min \{\lambda(x), \mu(a)\}\}$$

$$\geq \min_i \{\mu(a), \lambda(a)\}$$

$$= \min_i \{\mu(a), \lambda(a)\}$$

and so $\mu \lambda \mu \geq \mu \cap \lambda$ and hence $\mu \lambda \mu = \mu \cap \lambda$. Thus (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1). Let (3) hold. Let $\mu$ be any i-v fuzzy quasi-ideal of $S$. $\overline{\mu}$ itself being an i-v fuzzy bi-ideal of $S$, $\overline{\mu} = \overline{\mu} \subseteq \overline{\lambda} \subseteq \overline{\mu}$. Thus by Theorem 4.4, $S$ is regular and so (3) $\Rightarrow$ (1). $\Box$

Theorem 4.7. Every i-v fuzzy ideal of a regular semigroup is idempotent.
Proof. Let $\mu$ be an i-v fuzzy ideal of a regular semigroup $S$. Then by Proposition 2.15, $\mu S S \mu \leq \mu$. Hence $\mu$ is an i-v fuzzy bi-ideal of $S$. Since $S$ is regular, by Theorem 4.4, we have $\mu = \mu S S \mu \leq \mu S S \mu = \mu$ and so $\mu = \mu S S \mu$. Thus $\mu$ is idempotent.

\[ \square \]

Theorem 4.8. For a semigroup $S$, the following conditions are equivalent.

(1) $S$ is regular.
(2) $\mu \cap \lambda = \mu \lambda \mu$ for every i-v fuzzy quasi-ideal $\mu$ and every fuzzy ideal $\lambda$ of $S$.
(3) $\mu \cap \lambda = \mu \lambda \mu$ for every i-v fuzzy quasi-ideal $\mu$ and every i-v fuzzy interior ideal $\lambda$ of $S$.
(4) $\mu \cap \lambda = \mu \lambda \mu$ for every i-v fuzzy bi-ideal $\mu$ and every i-v fuzzy ideal $\lambda$ of $S$.
(5) $\mu \cap \lambda = \mu \lambda \mu$ for every i-v fuzzy bi-ideal $\mu$ and every i-v fuzzy interior ideal $\lambda$ of $S$.

Proof. (1) $\Rightarrow$ (5). Assume that (1) holds. Let $\mu$ and $\lambda$ be any i-v fuzzy bi-ideal and any i-v fuzzy interior ideal of $S$ respectively. Then $\mu S S \mu \leq \mu \cap \lambda$. Let $a \in S$. Since $S$ is regular, there exists $x \in S$ such that $a = axa$ (or $axaaxa$). Then

\[
(\mu \lambda \mu)(a) = \sup_{a=yz} \{\min_{a=yz} \{\mu(y), (\mu \lambda \mu)(z)\}\}
\]

\[
= \min_{a=yz} \{\mu(a), (\mu \lambda \mu)(xax)\}
\]

\[
\geq \min_{a=yz} \{\mu(a), (\lambda \mu)(xax)\}
\]

\[
= \min_{a=yz} \{\mu(a), (\lambda \mu)(a)\},
\]

since $\lambda$ is an i-v fuzzy interior ideal, $(\lambda \mu)(xax) \geq (\lambda \mu)(a)$.

\[
= \min_{a=yz} \{\lambda(a), \mu(a)\}
\]

and so $\mu \lambda \mu \geq \mu \cap \lambda$. Hence $\mu \lambda \mu = \mu \cap \lambda$. Thus (1) $\Rightarrow$ (5).

By Theorem 4.6, (1), (2), (3) and (4) are equivalent. It is clear that (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2). $\square$

Theorem 4.9. For a semigroup $S$, the following conditions are equivalent.

(1) $S$ is regular.
(2) $\lambda \cap \mu \leq \lambda \mu \mu$ for every i-v fuzzy right ideal $\lambda$ and every i-v fuzzy bi-ideal $\mu$ of $S$.
(3) $\lambda \cap \mu \leq \lambda \mu \mu$ for every i-v fuzzy right ideal $\lambda$ and every i-v fuzzy quasi-ideal $\mu$ of $S$.
(4) $\mu \cap \nu \leq \mu \nu \mu$ for every i-v fuzzy left ideal $\nu$ and every i-v fuzzy bi-ideal $\mu$ of $S$.
(5) $\mu \cap \nu \leq \mu \nu \mu$ for every i-v fuzzy left ideal $\nu$ and every i-v fuzzy quasi-ideal $\mu$ of $S$. 

...
(6) $\lambda \cap \mu \cap \nu \leq \lambda \mu \nu$ for every i-v fuzzy right ideal $\lambda$, every i-v fuzzy left ideal $\nu$ and every i-v fuzzy bi-ideal $\mu$ of $S$.

(7) $\lambda \cap \mu \cap \nu \leq \lambda \mu \nu$ for every i-v fuzzy right ideal $\lambda$, every i-v fuzzy left ideal $\nu$ and every i-v fuzzy quasi-ideal $\mu$ of $S$.

Proof. (1) $\Rightarrow$ (2). Let $\lambda$ and $\mu$ be an i-v fuzzy right ideal and an i-v fuzzy bi-ideal of $S$ respectively. Suppose $a \in S$. Since $S$ is regular, there exists $x \in S$ such that $a=axa=axaxa$. Then
\[
(\lambda \mu \nu)(a) = \sup_{a=xy} \{ \min_{\mu \nu} \{ \lambda(x), \mu(y) \} \} \\
\geq \min_{\mu \nu} \{ \lambda(axa), \mu(a) \} \\
= \min_{\mu \nu} \{ \lambda(a(xax)), \mu(a) \} \\
\geq \min_{\mu \nu} \{ \lambda(a), \mu(a) \} = (\lambda \cap \mu \cap \nu)(a).
\]
Thus (1) $\Rightarrow$ (2). It can be seen in a similar way that (1) $\Rightarrow$ (4).

Clearly (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5).

Now assume that (3) holds. Let $\lambda$ be an i-v fuzzy right ideal and $\mu$ be an i-v fuzzy left ideal of $S$. Since every i-v fuzzy left ideal is an i-v fuzzy quasi-ideal of $S$, by (3), we have $\lambda \cap \mu \leq \lambda \mu$. Again since $\lambda$ is an i-v fuzzy right ideal and $\mu$ is an i-v fuzzy left ideal of $S$, we have $\lambda \mu \leq \lambda \cap \mu$. Thus $\lambda \mu = \lambda \cap \mu$.

By Theorem 4.2, $S$ is regular hence we obtain (3) $\Rightarrow$ (1). Similarly (4) $\Rightarrow$ (1).

Again assume that (1) holds. Let $\lambda, \mu$ and $\nu$ be an i-v fuzzy right ideal, an i-v fuzzy left ideal and an i-v fuzzy bi-ideal of $S$ respectively and let $a \in S$. Since $S$ is regular, there exists $x \in S$ such that $a=axa=axaxa$. Then
\[
(\lambda \mu \nu)(a) = \sup_{a=yz} \{ \min_{\mu \nu} \{ \lambda(x), \mu(y) \} \} \\
\geq \min_{\mu \nu} \{ \lambda(axa), \mu(a) \} \\
= \min_{\mu \nu} \{ \lambda(a), \mu(a), \nu(a) \} \\
= (\lambda \cap \mu \cap \nu)(a)
\]
and hence $\lambda \cap \mu \cap \nu \leq \lambda \mu \nu$ and so (1) $\Rightarrow$ (6). It is clear that (6) $\Rightarrow$ (7).

Finally, assume that (7) holds. Let $\lambda$ and $\mu$ respectively be an i-v fuzzy right ideal and an i-v fuzzy left ideal of $S$. Since $\mu$ is itself an i-v fuzzy quasi-ideal of $S$, we have
\[
(\lambda \cap \mu) = \lambda \cap S \cap \mu \leq \lambda S \mu \leq \lambda \mu,
\]
by Proposition 2.15.

Clearly $\lambda \mu \leq \lambda \cap \mu$, hence $\lambda \mu = \lambda \cap \mu$. It follows from Theorem 4.2, that $S$ is regular and (7) $\Rightarrow$ (1).

\[\Box\]

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