A LARGE-UPDATE INTERIOR POINT ALGORITHM FOR $P_*(\kappa)$ LCP BASED ON A NEW KERNEL FUNCTION

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Abstract. In this paper we generalize large-update primal-dual interior point methods for linear optimization problems in [2] to the $P_*(\kappa)$ linear complementarity problems based on a new kernel function which includes the kernel function in [2] as a special case. The kernel function is neither self-regular nor eligible. Furthermore, we improve the complexity result in [2] from $O(\sqrt{n}(\log n)^2 \log \frac{\mu_0}{\epsilon})$ to $O(\sqrt{n}(\log n) \log(\log n) \log \frac{\mu_0}{\epsilon})$.

1. Introduction

In this paper we propose a new large-update interior point algorithm for solving linear complementarity problem (LCP) as follows:

$$s = Mx + q, \quad xs = 0, \quad x \geq 0, \quad s \geq 0,$$

where $x, s, q \in \mathbb{R}^n$, $M \in \mathbb{R}^{n \times n}$ is a $P_*(\kappa)$ matrix, and $xs$ denotes the componentwise product of the vectors $x$ and $s$.

Primal-dual interior point method (IPM) is one of the most efficient numerical methods for various optimization problems. Linear complementarity problems (LCPs) have many applications in science, economics, and engineering ([5]).

It is generally agreed that the iteration complexity of the algorithm is an appropriate measure for its efficiency ([6]). Most of polynomial-time interior point algorithms are based on the logarithmic barrier function. Peng et al. ([11], [12], [13]) proposed a new variant of interior point methods (IPMs) based on self-regular barrier functions and achieved so far the best known complexity result for large-update methods with a specific self-regular barrier function. Roos et al. ([1], [2]) proposed new primal-dual IPMs for linear optimization (LO) problems based on eligible barrier functions and proposed the unified scheme for analyzing the algorithm based on four conditions on the kernel function ([2]). Cho et al. ([3], [4]) extended the algorithm for LO to $P_*(\kappa)$ LCPs.

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Motivated by their works, we introduce a new class of kernel functions which is the generalized form of the ones in [2] and is not eligible. We obtained $O\left(\frac{1+2\kappa}{\epsilon} n^{1+\tau} \log \log n \right)$ iteration complexity for large-update method. Taking $p = 1$ and $r = \frac{\log \log n}{\log n}$, we have $O\left((1+2\kappa)\sqrt{n} \log n \log(log(n)) \log \frac{n\mu}{\epsilon}\right)$ iteration complexity for $P_\kappa(\kappa)$ LCP which is better than the one in [2].

The paper is organized as follows. In Section 2 we recall the generic IPM and propose some basic concepts for LCP. In Section 3 we introduce a new class of kernel functions and its properties. In Section 4 we derive the complexity result for the algorithm based on a new kernel function.

We will make use of the following notations throughout the paper. $R^n_+$ and $R^n_{++}$ denote the set of $n$-dimensional nonnegative vectors and positive vectors, respectively. For $x \in R^n$, $x_{\min}$ denotes the smallest component of the vector $x$. We denote $X$ and $S$ the diagonal matrices from a vector $x$ and $s$, respectively, i.e. $X = \text{diag}(x)$ and $S = \text{diag}(s)$. $e$ and $E$ denote the $n$-dimensional vector of ones and the identity matrix, respectively. For $f(t), g(t) : R_{++} \rightarrow R_{++}$, $f(t) = O(g(t))$ if $f(t) \leq c_1 g(t)$ for some positive constant $c_1$ and $f(t) = \Theta(g(t))$ if $c_2 g(t) \leq f(t) \leq c_3 g(t)$ for some positive constants $c_2$ and $c_3$. $I$ denotes the index set, e.g. $I = \{1, 2, \cdots, n\}$. $\log$ denotes the natural logarithmic function.

2. Preliminaries

In this section, we recall the generic IPM and introduce basic concepts.

Definition 1. [8] Let $\kappa \geq 0$. $P_\kappa(\kappa)$ is the class of matrices $M$ satisfying

$$(1 + 4\kappa) \sum_{i \in I_+(\xi)} \xi_i |M\xi|_i + \sum_{i \in I_-(\xi)} \xi_i |M\xi|_i \geq 0,$$

where $\xi \in \mathbb{R}^n$, $|M\xi|_i$ denotes the $i$-th component of the vector $M\xi$ and

$$I_+(\xi) = \{i \in I : \xi_i |M\xi|_i \geq 0\}, \quad I_-(\xi) = \{i \in I : \xi_i |M\xi|_i < 0\}.$$

Lemma 2.1. [8] If $M \in R^{n \times n}$ is a $P_\kappa(\kappa)$ matrix, then

$$M' = \begin{pmatrix} -M & E \\ S & X \end{pmatrix}$$

is a nonsingular matrix for any positive diagonal matrices $X, S \in R^{n \times n}$.

Corollary 2.2. Let $M \in R^{n \times n}$ be a $P_\kappa(\kappa)$ matrix and $x, s \in R^n_{++}$. Then for all $c \in R^n$ the system

$$-M \Delta x + \Delta s = 0, \quad S \Delta x + X \Delta s = c$$

has a unique solution $(\Delta x, \Delta s)$.

The basic idea of primal-dual IPMs is to replace the second equation in (1) by the parameterized equation $xs = \mu e, \mu > 0$. Now we consider the following system:

$$s = Mx + q, \quad xs = \mu e, \quad x > 0, \quad s > 0.$$

(2)
Without loss of generality, we assume that (1) has a strictly feasible point, i.e., there exists \((x^0, s^0) > 0\) such that \(s^0 = Mx^0 + q\). For this, the reader refers to [8]. Since \(M\) is a \(P(\kappa)\) matrix and (1) is strictly feasible, the system (2) has a unique solution for each \(\mu > 0\). We denote the solution \((x(\mu), s(\mu))\) for each \(\mu > 0\). We call it the \(\mu\)-center. The set of \(\mu\)-centers \((\mu > 0)\) is called the central path of (1). The limit of this central path (as \(\mu\) goes to zero) exists and since the limit point satisfies (1), it yields an optimal solution for (1) ([8]). IPMs follow this central path approximately and approach the solution of (1) as \(\mu\) goes to zero.

For given \((x, s) := (x^0, s^0)\) by applying Newton method to the system (2) we have the following Newton system:

\[-M\Delta x + \Delta s = 0,\]

\[S\Delta x + X\Delta s = \mu e - xs,\] (3)

By Corollary 2.2, the system (3) has a unique search direction \((\Delta x, \Delta s)\).

By taking a step along the search direction \((\Delta x, \Delta s)\), one constructs a new positive iterate \((x_+, s_+)\), where

\[x_+ = x + \alpha \Delta x, \quad s_+ = s + \alpha \Delta s,\]

for some \(\alpha \geq 0\). To have the motivation of new algorithm we define the following scaled vectors:

\[v := \sqrt{\frac{x s}{\mu}}, \quad d := \sqrt{\frac{x}{s}}, \quad d_x := \frac{v\Delta x}{x}, \quad d_s := \frac{v\Delta s}{s},\] (4)

whose \(i\)th components are \(\sqrt{x_i s_i / \mu}, \sqrt{x_i / s_i}, v_i [\Delta x]_i / x_i, \) and \(v_i [\Delta s]_i / s_i,\) respectively. Using (4), we can rewrite the system (3) as follows:

\[-\bar{M}d_x + d_s = 0, \quad d_x + d_s = v^{-1} - v,\] (5)

where \(\bar{M} := DMD\) and \(D := \text{diag}(d)\). Note that the right side of the second equation in (5) equals the negative gradient of the logarithmic barrier function \(\Psi_l(v)\), i.e.,

\[d_x + d_s = -\nabla \Psi_l(v),\] (6)

where

\[\Psi_l(v) := \sum_{i=1}^{n} \psi_l(v_i), \quad \psi_l(t) = \frac{t^2 - 1}{2} - \log t, \quad t > 0.\]

We call \(\psi_l\) the kernel function of the logarithmic barrier function \(\Psi_l(v)\).

The generic interior point algorithm works as follows. Assume that we are given a strictly feasible point \((x, s)\) which is in a \(\tau\)-neighborhood of the given \(\mu\)-center. Then we decrease \(\mu\) to \(\mu_+ := (1 - \theta)\mu\), for some fixed \(\theta \in (0, 1)\) and solve the Newton system (3) to obtain the unique search direction. The positivity condition of a new iterate is ensured with the right choice of the step size \(\alpha\) which is defined by some line search rule. This procedure is repeated until we find a new iterate \((x_+, s_+)\) that is in a \(\tau\)-neighborhood of the \(\mu_+\)-center and then we let \(\mu := \mu_+\) and \((x, s) := (x_+, s_+)\). Then \(\mu\) is again reduced by the
factor $1 - \theta$ and we solve the Newton system targeting at the new $\mu_+$-center, and so on. This process is repeated until $\mu$ is small enough, say until $n\mu < \varepsilon$.

### Generic Primal-Dual Algorithm

Input:
- a threshold parameter $\tau > 0$;
- an accuracy parameter $\varepsilon > 0$;
- a fixed barrier update parameter $\theta$, $0 < \theta < 1$;
- $(x^0, s^0)$ and $\mu^0 > 0$ such that $\Psi_l(x^0, s^0, \mu^0) \leq \tau$.

begin
  $x := x^0; s := s^0; \mu := \mu^0$;
  while $n\mu \geq \varepsilon$ do
    begin
      $\mu := (1 - \theta)\mu$;
      while $\Psi_l(v) > \tau$ do
        begin
          solve the system (3) for $\Delta x$ and $\Delta s$;
          determine a step size $\alpha$;
          $x := x + \alpha \Delta x$;
          $s := s + \alpha \Delta s$;
          $v := \sqrt{xs}$;
        end
      end
    end
end

When the barrier update parameter $\theta$ is independent of $n$, we call the algorithm a large-update method.

### 3. New kernel function

In this section we define a new class of kernel functions and its properties.

**Definition 2.** The function $\psi: R_+ \rightarrow R_+$ is called a kernel function if $\psi$ is twice differentiable and satisfies the following conditions:

(a) $\psi'(1) = \psi(1) = 0$, (b) $\psi''(t) > 0$, $t > 0$, (c) $\lim_{t \to 0} \psi(t) = \lim_{t \to \infty} \psi(t) = \infty$.

Now we define a new class of kernel functions with parameters $p$ and $r$ as follows:

$$\psi(t) := \frac{t^{p+1} - 1}{p + 1} + r(e^{t^{\frac{1}{r}} - 1} - 1), \quad 0 \leq p \leq 1, \quad 0 < r \leq 1, \quad t > 0. \quad (7)$$
Note that $\psi(t)$ includes the kernel function defined in [2] as a special case. For $\psi(t)$ we have the following:

$$
\psi'(t) = t^p - t^{-\frac{2}{r}} - 1, \\
\psi''(t) = pt^{p-1} + \left(\frac{1}{r} + \frac{1}{r} + 1\right)t^{r-\frac{2}{r}} - 2 - 1, \\
\psi'''(t) = p(p-1)t^{p-2} - \left(\frac{1}{r^2} + \frac{1}{r} + 1\right)t^{r-\frac{2}{r}} + \left(\frac{1}{r} + 1\right)(\frac{1}{r} + 2)t^{r-\frac{2}{r}} - 3 - 1. 
$$

(8)

In this paper, we replace the function $\Psi(v)$ in (6) with the function $\Psi(v)$ as follows:

$$
d_x + d_s = -\nabla \Psi(v),
$$

(9)

where $\Psi(v) = \sum_{i=1}^{n} \psi(v_i)$ and $\psi(t)$ is defined in (7). Hence the new search direction $(\Delta x, \Delta s)$ is obtained by solving the following modified Newton-system:

$$
-M \Delta x + \Delta s = 0, \quad S \Delta x + X \Delta s = -\mu v \nabla \Psi(v).
$$

(10)

Since $\Psi(v)$ is strictly convex and minimal at $v = e$, we have

$$
\Psi(v) = 0 \iff v = e \iff x = x(\mu), \quad s = s(\mu).
$$

We use $\Psi(v)$ as the proximity function to measure the distance between the current iterate and the $\mu$-center. Also, we define the norm-based proximity measure $\delta(v)$ as follows:

$$
\delta(v) := \frac{1}{2}||\nabla \Psi(v)|| = \frac{1}{2}||d_x + d_s||.
$$

(11)

In the following we give properties of $\psi(t)$ which are essential to the complexity analysis.

**Lemma 3.1.** Let $\psi(t)$ be as defined in (7). Then we have the following:

(i) $\psi(t)$ is exponentially convex, $t > 0$.

(ii) $\psi''(t)$ is monotonically decreasing, $t > 0$.

*Proof.* For (i), by Lemma 2.1.2 in [13], it suffices to show the function $\psi(t)$ satisfies $t\psi''(t) + \psi'(t) \geq 0$ for all $t > 0$. Using (8), we have

$$
t\psi''(t) + \psi'(t) = (p+1)t^p + \frac{1}{r}t^{r-1}e^{-\frac{2}{r}} - 1 + \frac{1}{r}t^{r-1}e^{\frac{2}{r}} - 1 \geq 0, \quad t > 0.
$$

For (ii), from (8), $\psi''''(t) < 0$. This completes the proof. \hfill \Box

**Remark 1.** Recall that the function $\psi : R_{++} \rightarrow R_+$ is eligible if $\psi$ is three times differentiable and satisfies the following conditions([2]):

(a) $t\psi''(t) + \psi'(t) > 0$, $t < 1$,

(b) $t\psi''(t) - \psi'(t) > 0$, $t > 1$,

(c) $\psi'''(t) < 0$, $t > 0$,

(d) $2\psi''(t)^2 - \psi'(t)\psi'''(t) > 0$, $t < 1$. 
Using (8), we have
\[ t\psi''(t) - \psi'(t) = (p - 1)t^p + \frac{1}{r}t^{r^{-\frac{1}{2}} - 1}e^{t^{r^{-\frac{1}{2}} - 1}} + \left(\frac{1}{r} + 2\right)t^{r^{-\frac{1}{2}} - 1}e^{t^{r^{-\frac{1}{2}} - 1}}. \]

Since \( t\psi''(t) - \psi'(t) < -7 < 0 \) for \( p = \frac{1}{2}, r = \frac{1}{4}, t = 2^8 \),
condition (b) is not satisfied. Hence \( \psi(t) \) is not eligible. Note that the kernel function in [2] is eligible.

**Lemma 3.2.** For \( \psi(t) \) and \( p \in [0, 1] \), we have
\[ \frac{1}{p + 1} \sum_{i=1}^{n} v_i^{p+1} \leq \Psi(v) + \frac{(pr + r + 1)n}{p + 1}. \]

**Proof.** Since \( re^{t^{\frac{1}{2}} - 1} > 0 \), we have
\[ \psi(t) = \frac{t^{p+1}}{p+1} - \frac{1}{p + 1} + re^{t^{\frac{1}{2}} - 1} - r \geq \frac{t^{p+1}}{p+1} - \frac{1}{p + 1} - r. \]
So we have \( \frac{t^{p+1}}{p+1} \leq \psi(t) + \frac{pr+r+1}{p+1} \). Hence we have
\[ \frac{1}{p + 1} \sum_{i=1}^{n} v_i^{p+1} \leq \Psi(v) + \frac{(pr + r + 1)n}{p + 1}. \]
This completes the proof. \( \square \)

Define \( \psi_b(t) := r(e^{t^{\frac{1}{2}} - 1} - 1) \). Then we have \( \psi(t) := \frac{t^{p+1}}{p+1} + \psi_b(t) \). Since \( \psi_b'(t) = -t^{\frac{1}{2}} e^{t^{\frac{1}{2}} - 1} < 0 \), \( \psi_b(t) \) is monotonically decreasing in \( t \).

**Lemma 3.3.** Let \( \beta \geq 1 \). Then \( \psi(\beta t) \leq \psi(t) + \frac{\beta^p + 1}{p+1}(\beta^{p+1} - 1) \).

**Proof.** Since \( \psi_b(t) \) is monotonically decreasing in \( t \), \( \psi_b(\beta t) - \psi_b(t) \leq 0 \) for \( \beta \geq 1 \). Hence we have
\[
\psi(\beta t) = \frac{(\beta t)^{p+1} - 1}{p+1} + \psi_b(\beta t)
= \frac{t^{p+1} - 1}{p+1} + \psi_b(t) + \frac{1}{p+1}(\beta^{p+1} t^{p+1} - t^{p+1}) + \psi_b(\beta t) - \psi_b(t)
= \psi(t) + \frac{t^{p+1}}{p+1}(\beta^{p+1} - 1) + \psi_b(\beta t) - \psi_b(t)
\leq \psi(t) + \frac{t^{p+1}}{p+1}(\beta^{p+1} - 1).
\]
This completes the proof. \( \square \)

In the following we obtain an estimate for the effect of a \( \mu \)-update on the value of \( \Psi(v) \).
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Theorem 3.4. Let $0 \leq \theta < 1$ and $v_+ = \frac{v}{\sqrt{1-\theta}}$. Then we have

\[ \Psi(v_+) \leq \Psi(v) + \frac{\theta}{(1-\theta)^{1+p}} \left( \Psi(v) + \frac{(pr + r + 1)n}{p+1} \right). \]

Proof. Using Lemma 3.3 with $\beta = \frac{1}{\sqrt{1-\theta}}$ and Lemma 3.2, we have

\[ \Psi(v_+) = \Psi(\beta v) = \sum_{i=1}^{n} \left( \psi(\beta v_i) + \frac{1}{p+1}(\beta^{p+1} - 1)v_i^{p+1} \right) \]
\[ \leq \Psi(v) + \frac{1}{(1-\theta)^{1+p}} \left( \Psi(v) + \frac{(pr + r + 1)n}{p+1} \right). \]

Since $1 - (1-\theta)^{1+p} \leq \theta$ for $0 \leq \theta < 1$,
\[ \Psi(v_+) \leq \Psi(v) + \frac{\theta}{(1-\theta)^{1+p}} \left( \Psi(v) + \frac{(pr + r + 1)n}{p+1} \right). \]

This completes the proof. \hfill \Box

Note that at the start of outer iteration of the algorithm, i.e., just before the update of $\mu$ with the factor $1 - \theta$, we have $\Psi(v) \leq \tau$. During the inner iteration we have

\[ \Psi(v_+) \leq \Psi(v) + \frac{\theta}{(1-\theta)^{1+p}} \left( \tau + \frac{(pr + r + 1)n}{p+1} \right). \]

Each subsequent inner iteration will rise to a decrease of the value of $\Psi(v)$. Denote

\[ \tilde{\Psi}_0 := \tau + \frac{\theta}{(1-\theta)^{1+p}} \left( \tau + \frac{(pr + r + 1)n}{p+1} \right). \]

Define $\Psi_0$ the value of $\Psi(v)$ after the $\mu$-update. Then $\Psi_0 \leq \tilde{\Psi}_0$.

Lemma 3.5. Define $\varphi : [0, \infty) \to [1, \infty)$ be the inverse function of $\psi(t)$ for $t \geq 1$. For $0 \leq p \leq 1$ and $u \geq 0$ we have

\[ \varphi(u) \geq (1 + (p+1)u)^{\frac{1}{p+1}}. \]

Proof. Let $u = \psi(t)$, $t \geq 1$. Since $\psi_b(t)$ is monotonically decreasing in $t$ and $\psi_b(1) = 0$, $\psi_u(t) < 0$ for $t > 1$. Hence $u = \psi(t) = \frac{t^{p+1}-1}{p+1} + \psi_b(t) \leq \frac{t^{p+1}-1}{p+1}$, $t \geq 1$. This implies $(p+1)u + 1 \leq \varphi^{p+1}$. By the definition of $\varphi$, $\varphi(u) = t \geq (1 + (p+1)u)^{\frac{1}{p+1}}$. This completes the proof. \hfill \Box

From Lemma 3.1 (ii), we cite the following lemma in [2] without proof.
Lemma 3.6. (Theorem 4.9 in [2]) Let \( \delta(v) \) be as defined in (11). Then we have
\[
\delta(v) \geq \frac{1}{2} \psi'(\varrho(\Psi(v))).
\]
For notational convenience we denote \( \delta := \delta(v) \) and \( \Psi := \Psi(v) \).

Lemma 3.7. Let \( \delta \) be as defined in (11). Then for all \( \Psi \geq 1 \) and \( 0 \leq p \leq 1 \) we have
\[
\delta \geq \frac{1}{4} ((p + 1)\Psi) ^{\frac{p}{p+1}}.
\]
Proof. By Lemma 3.6, Lemma 3.5, and \( \psi''(t) > 0 \),
\[
\delta \geq \frac{1}{2} \psi'(\varrho(\Psi)) \geq \frac{1}{2} \psi'(\left(1 + (p + 1)\Psi\right) ^{\frac{p}{p+1}})
= \frac{1}{2} \left(1 + (p + 1)\Psi\right) ^{\frac{p}{p+1}} - e^{(1+(p+1)\Psi)^{\frac{1}{p+1}}-1} \frac{1}{(1 + (p + 1)\Psi) ^{\frac{p}{p+1}}}
\geq \frac{1}{2} \left(1 + (p + 1)\Psi\right) ^{\frac{p}{p+1}} - \frac{1}{(1 + (p + 1)\Psi) ^{\frac{p}{p+1}}}
\geq \frac{1}{2} \left(1 + (p + 1)\Psi\right) ^{\frac{p}{p+1}} - \frac{1}{(1 + (p + 1)\Psi) ^{\frac{p}{p+1}}}
= \frac{1}{2} \left(1 + (p + 1)\Psi\right) ^{\frac{p}{p+1}} \geq \frac{1}{4} ((p + 1)\Psi) ^{\frac{p}{p+1}},
\]
where the third inequality is satisfied from \( e^{(1+(p+1)\Psi)^{\frac{1}{p+1}}-1} \leq 1 \) and the last inequality from the fact \( 1 \leq (p + 1)\Psi \). This completes the proof. \( \Box \)

4. Complexity result

In this section we compute a feasible step size and the decrease of the proximity function during an inner iteration and give the complexity result of the algorithm. For fixed \( \mu \) if we take a step size \( \alpha \), then we have new iterates \( x_+ = x + \alpha \Delta x \), \( s_+ = s + \alpha \Delta s \). Using (4), we have
\[
x_+ = x \left( e + \alpha \frac{\Delta x}{x} \right) = x \left( e + \alpha \frac{d x}{v} \right) = \frac{x}{v}(v + \alpha d x)
\]
and
\[
s_+ = s \left( e + \alpha \frac{\Delta s}{s} \right) = s \left( e + \alpha \frac{d s}{v} \right) = \frac{s}{v}(v + \alpha d s).
\]
Thus we have
\[
v_+ = \sqrt{\frac{x_+ s_+}{\mu}} = \sqrt{(v + \alpha d x)(v + \alpha d s)}.
\]
Define for $\alpha > 0$, $f(\alpha) = \Psi(v_+) - \Psi(v)$. Then $f(\alpha)$ is the difference between proximities of a new iterate and a current iterate for fixed $\mu$. Using Lemma 3.1 (i), we have

$$\Psi(v_+) = \Psi(\sqrt{(v + \alpha d_s)(v + \alpha d_s)}) \leq \frac{1}{2} (\Psi(v + \alpha d_s) + \Psi(v + \alpha d_s)).$$

Hence we have $f(\alpha) \leq f_1(\alpha)$, where

$$f_1(\alpha) := \frac{1}{2} (\Psi(v + \alpha d_s) + \Psi(v + \alpha d_s)) - \Psi(v).$$

We have $f(0) = f_1(0) = 0$. Taking the derivative of $f_1(\alpha)$ with respect to $\alpha$, we have

$$f_1'(\alpha) = \frac{1}{2} \sum_{i=1}^{n} (\psi'(v_i + \alpha [d_s])d_s[i])^2 + \psi'(v_i + \alpha [d_s])d_s[i]).$$

where $[d_s[i]$ and $[d_s]$ denote the $i$-th components of the vectors $d_s$ and $d_s$, respectively. Using (9) and (11), we have

$$f_1'(0) = \frac{1}{2} \nabla \Psi(v)^T (d_s + d_s) = -\frac{1}{2} \nabla \Psi(v)^T \nabla \Psi(v) = -2\delta(v)^2.$$

Differentiating $f_1'(\alpha)$ with respect to $\alpha$, we have

$$f_1''(\alpha) = \frac{1}{2} \sum_{i=1}^{n} (\psi''(v_i + \alpha [d_s])d_s[i])^2 + \psi''(v_i + \alpha [d_s])d_s[i]).$$

Since $f_1''(\alpha) > 0$, $f_1(\alpha)$ is strictly convex in $\alpha$ unless $d_s = d_s = 0$. Since $M$ is a $P_o(\kappa)$ matrix and $M \Delta x = \Delta s$ from (10), for $\Delta x \in \mathbb{R}^n$,

$$(1 + 4\kappa) \sum_{i \in I_+} |\Delta x[i] | |\Delta s[i] | + \sum_{i \in I_-} |\Delta x[i] | |\Delta s[i] | \geq 0,$$

where $I_+ = \{ i \in I : |\Delta x[i] | \geq 0 \}$, $I_- = I - I_+$. Since $d_s = d_s = \frac{v^T \Delta x \Delta s}{\mu} = \Delta x \Delta s$ and $\mu > 0$, we have

$$(1 + 4\kappa) \sum_{i \in I_+} |d_s[i] | [d_s[i] | + \sum_{i \in I_-} |d_s[i] | [d_s[i] | \geq 0.$$

For convenience we denote $\sigma_+ := \sum_{i \in I_+} [d_s[i] | [d_s[i] |$ and $\sigma_- := -\sum_{i \in I_-} [d_s[i] | [d_s[i] |$. In the following we cite some lemmas in [4] without proof.

**Lemma 4.1.** (Modification of Lemma 4.1 in [4]) $\sigma_+ \leq \delta^2$ and $\sigma_- \leq (1 + 4\kappa)\delta^2$.

**Lemma 4.2.** (Modification of Lemma 4.2 in [4]) $\sum_{i=1}^{n} ([d_s]^2 + [d_s]^2) \leq 4(1 + 2\kappa)\delta^2$, $\|d_s\| \leq 2\delta \sqrt{1 + 2\kappa}$, and $\|d_s\| \leq 2\delta \sqrt{1 + 2\kappa}$.

**Lemma 4.3.** (Modification of lemma 4.3 in [4]) $f_1''(\alpha) \leq 2(1 + 2\kappa) \delta^2 \psi''(v_{min} - 2\alpha \delta \sqrt{1 + 2\kappa}).$
Lemma 4.4. (Modification of lemma 4.4 in [4]) \( f'_1(\alpha) \leq 0 \) if \( \alpha \) is satisfying
\[
-\psi'(v_{\text{min}} - 2\alpha \delta \sqrt{1 + 2\kappa}) + \psi'(v_{\text{min}}) \leq \frac{2\delta}{\sqrt{1 + 2\kappa}}. \tag{13}
\]

Lemma 4.5. (Modification of lemma 4.5 in [4]) Define \( \rho : [0, \infty) \to (0, 1] \) be the inverse function of \(-\frac{1}{2} \psi'(t)\) for \( 0 < t \leq 1 \) and \( a := 1 + \frac{1}{\sqrt{1 + 2\kappa}} \). Then the largest step size \( \alpha \) satisfying (13) is given by
\[
\hat{\alpha} := \frac{1}{2\delta \sqrt{1 + 2\kappa}} \left( \rho(\delta) - \rho(a\delta) \right).
\]

Lemma 4.6. (Modification of lemma 4.6 in [4]) Let \( \rho \) and \( \hat{\alpha} \) be as defined in Lemma 4.5. Then
\[
\hat{\alpha} \geq \frac{1}{(1 + 2\kappa)\psi''(\rho(a\delta))}.
\]

Define
\[
\bar{\alpha} := \frac{1}{(1 + 2\kappa)\psi''(\rho(\alpha))}. \tag{14}
\]

Then we have \( \hat{\alpha} \leq \bar{\alpha} \).

Lemma 4.7. Let \( \bar{\alpha} \) be as defined in (14). Then for \( a = 1 + \frac{1}{\sqrt{1 + 2\kappa}} \) and \( \kappa \geq 0 \), we have
\[
\bar{\alpha} \geq \frac{1}{(1 + 2\kappa)\left( p + (2a\delta + 1)(\frac{1}{p} + 1)(1 + \log(2a\delta + 1))^{1+r} \right)}.
\]

Proof. Using the definition of \( \rho \), we have \(-\frac{1}{2} \psi'(\rho(a\delta)) = a\delta \). Let \( z = \rho(a\delta) \).
Then \(-\psi'(z) = 2a\delta \) and \( 0 < z \leq 1 \). From (8), we have \(-z^p + z^{-\frac{1}{p}} - 1 = 2a\delta \). Then for \( 0 < z \leq 1 \),
\[
z^{-\frac{1}{p}} + z^{1 - \frac{1}{p}} - 1 = 2a\delta + z^p \leq 2a\delta + 1. \tag{15}
\]

By taking the natural logarithmic function on both sides of (15), we have
\[
z^{-\frac{1}{p}} - 1 - \left( \frac{1}{p} + 1 \right) \log z \leq \log(2a\delta + 1). \tag{16}
\]

Using (16) and \( 0 < z \leq 1 \), we obtain
\[
z^{-\frac{1}{p}} \leq 1 + \log(2a\delta + 1) + \left( \frac{1}{p} + 1 \right) \log z \leq 1 + \log(2a\delta + 1).
\]

This implies
\[
z^{p-1} \leq (1 + \log(2a\delta + 1))^{r(1-p)} \leq (1 + \log(2a\delta + 1))^{1+r},
z^{-1} \leq (1 + \log(2a\delta + 1))^{1} \leq (1 + \log(2a\delta + 1))^{1+r},
z^{-\frac{1}{p}} \leq (1 + \log(2a\delta + 1))^{1+r}. \tag{17}
\]
From (14), we have for $0 < z \leq 1$ and $0 \leq p \leq 1$,
\[
\bar{\alpha} = \frac{1}{(1 + 2\kappa)\psi'(\rho(a\delta))} = \frac{1}{(1 + 2\kappa)\psi'(z)}
\]
\[
\geq \frac{1}{(1 + 2\kappa)\left(pz^{p-1} + \frac{1}{r}z^{-\frac{p}{r}}e^{-\frac{1}{r}z^{1-r}}\right)}
\]
\[
\geq \frac{1}{(1 + 2\kappa)\left(pz^{p-1} + \frac{1}{r}(2a\delta + 1)\left(\frac{1}{r}z^{-\frac{p}{r}} + \frac{1}{2}\right)\right)}
\]
\[
\geq \frac{1}{(1 + 2\kappa)(p + (2a\delta + 1)(\frac{2}{r} + 1))(1 + \log(2a\delta + 1))^{1+\tau}}.
\]
where the first inequality follows from (15) and the last inequality from (17). This proves the lemma. \(\Box\)

Define
\[
\tilde{\alpha} = \frac{1}{(1 + 2\kappa)(p + (2a\delta + 1)(\frac{2}{r} + 1))(1 + \log(2a\delta + 1))^{1+\tau}}. \tag{18}
\]
Note that $\tilde{\alpha} \leq \bar{\alpha}$. We will use $\tilde{\alpha}$ as the default step size.

**Lemma 4.8.** (Lemma 1.3.3 in [13]) Suppose that $h(t)$ is a twice differentiable convex function with $h(0) = 0$ and $h'(0) < 0$ and $h(t)$ attains its global minimum at $t^* > 0$ and $h''(t)$ is increasing with respect to $t$. Then for any $t \in [0, t^*]$,
\[
h(t) \leq \frac{th'(0)}{2}.
\]

**Lemma 4.9.** (Modification of lemma 4.8 in [4]) If the step size $\alpha$ is such that $\alpha \leq \tilde{\alpha}$, then
\[
f(\alpha) \leq -\alpha \delta^2.
\]
In our algorithm we assume that $\tau \geq 1$. Using Lemma 3.7 and the fact $\Psi \geq \tau$, we have
\[
\delta \geq \frac{1}{\tau}((p + 1)\Psi)^{\frac{1}{1+\tau}} \geq \frac{1}{4}. \tag{19}
\]

**Lemma 4.10.** For $0 < r \leq 1$ the function
\[
g(\delta) = -\frac{\delta}{(1 + \log(2a\delta + 1))^{1+\tau}}
\]
is monotonically decreasing in $\delta$. 

Proof. It suffices to show that the function \(-g(\delta)\) is monotonically increasing in \(\delta\). If we differentiate \(-g(\delta)\) with respect to \(\delta\), we have

\[
-g'(\delta) = \frac{(2a\delta + 1)(1 + \log(2a\delta + 1))^{1+r} - 2a\delta(1+r)(1+\log(2a\delta + 1))^{r}}{(2a\delta + 1)(1 + \log(2a\delta + 1))^{2(1+r)}}.
\]

Since the denominator is strictly positive, it is enough to show that numerator is positive. The numerator is

\[
(2a\delta + 1)(1 + \log(2a\delta + 1))^{1+r} - 2a\delta(1+r)(1+\log(2a\delta + 1))^{r}
\]

\[
= (1 + \log(2a\delta + 1))^{r}(1 + \log(2a\delta + 1)(2a\delta + 1) - 2a\delta(1+r))
\]

\[
= (1 + \log(2a\delta + 1))^{r}(1 + (2a\delta + 1)\log(2a\delta + 1) - 2a\delta r).
\]

Let \(\tilde{g}(\delta) := 1 + (2a\delta + 1)\log(2a\delta + 1) - 2a\delta r\). Then \(\tilde{g}'(\delta) = 2a\log(2a\delta + 1) + 2a - 2ar\). From \(1 < a \leq 2\) and (19), we have \(\tilde{g}'(\delta) > 0\). Since \(\tilde{g}(\frac{1}{4}) > 0\), \(\tilde{g}(\delta) > 0\) for \(\delta \geq \frac{1}{4}\). Hence, \(-g(\delta)\) is monotonically increasing in \(\delta\). This completes the proof. \(\square\)

Theorem 4.11. Let \(\tilde{\alpha}\) be as defined in (18). Then

\[
f(\tilde{\alpha}) \leq -\frac{(p + 1)\Psi}{16(1 + 2\kappa)(p + \frac{4}{r} + 2)\left(1 + \log\left(\frac{a}{2}((p + 1)\Psi)\frac{r}{p} + 1\right)\right)^{1+r}}.
\]

Proof. Using (18) and Lemma 4.9, we have

\[
f(\tilde{\alpha}) \leq -\frac{\delta^2}{(1 + 2\kappa)(p + (2a\delta + 1)(\frac{2}{r} + 1))(1 + \log(2a\delta + 1))^{1+r}}
\]

\[
\leq -\frac{\delta^2}{(1 + 2\kappa)(4p\delta + \delta(2a + 4)(\frac{2}{r} + 1))(1 + \log(2a\delta + 1))^{1+r}}
\]

\[
= -\frac{\delta^2}{2(1 + 2\kappa)(2p + \frac{2a}{r} + \frac{4}{r} + a + 2)(1 + \log(2a\delta + 1))^{1+r}}
\]

\[
\leq -\frac{\delta^2}{4(1 + 2\kappa)(p + \frac{4}{r} + 2)(1 + \log(2a\delta + 1))^{1+r}}
\]

\[
\leq -\frac{\delta^2}{4(1 + 2\kappa)(p + \frac{4}{r} + 2)\left(1 + \log\left(\frac{a}{2}((p + 1)\Psi)\frac{r}{p} + 1\right)\right)^{1+r}}
\]

\[
= -\frac{(p + 1)\Psi}{16(1 + 2\kappa)(p + \frac{4}{r} + 2)\left(1 + \log\left(\frac{a}{2}((p + 1)\Psi)\frac{r}{p} + 1\right)\right)^{1+r}}
\]

\[
\leq -\frac{(p + 1)\Psi}{16(1 + 2\kappa)(p + \frac{4}{r} + 2)\left(1 + \log\left(\frac{a}{2}((p + 1)\Psi)\frac{r}{p} + 1\right)\right)^{1+r}}.
\]
where the second inequality is satisfied from (19), third inequality from $1 < a \leq 2$, the fourth inequality from Lemma 3.7 and Lemma 4.10, and the last inequality from the definition of $\Psi_0$. This completes the proof. □

**Lemma 4.12.** (Lemma 1.3.2 in [13]) Let $t_0, t_1, \cdots, t_J$ be a sequence of positive numbers such that

$$t_{j+1} \leq t_j - \gamma t_j^{1-\lambda}, \quad j = 0, 1, \cdots, J - 1,$$

where $\gamma > 0$ and $0 < \lambda \leq 1$. Then $J \leq \lfloor \frac{t_J}{\gamma \lambda} \rfloor$.

We define the value of $\Psi(\nu)$ after the $\mu$-update as $\Psi_0$ and the subsequent values in the same outer iteration $\Psi_k$, $k = 1, 2, \cdots$. Let $K$ denote the total number of inner iterations in the outer iteration. Then we have

$$\Psi_{K-1} > \tau, \quad 0 \leq \Psi_K \leq \tau.$$

**Lemma 4.13.** Let $\tilde{\Psi}_0$ be as defined in (12) and $K$ be the total number of inner iterations in the outer iteration. Then we have

$$K \leq 16(1 + 2\kappa)(p + \frac{4}{r} + 2)(p + 1)^{1+r} \left(1 + \log((p + 1)\tilde{\Psi}_0)\right)^{1+r} \tilde{\Psi}_0^{\frac{1}{1+r}}.$$

**Proof.** By Theorem 4.11 with $\gamma = \frac{(p+1)^{1+r}}{16(1+2\kappa)(p+\frac{4}{r}+2)(1+\log((p+1)\tilde{\Psi}_0)^{\frac{1}{1+r}})}$ and $\lambda = \frac{1}{1+r}$, we have

$$K \leq 16(1 + 2\kappa)(p + \frac{4}{r} + 2) \left(1 + \log\left(\frac{q}{2} \left((p+1)\Psi_0 \right)^{\frac{1}{1+r}} \right) \right)^{1+r} \tilde{\Psi}_0^{\frac{1}{1+r}}.$$

Since $\Psi_0 \leq \tilde{\Psi}_0$ and $1 < a \leq 2$, we have the result. □

**Theorem 4.14.** Let a $P_\kappa(\kappa)$ LCP be given and $\tau \geq 1$. Then the total number of iterations to have an approximate solution with $n\mu < \epsilon$ is bounded by

$$\left\lceil 16(1 + 2\kappa)(p + \frac{4}{r} + 2)(p + 1)^{1+r} \left(1 + \log((p + 1)\tilde{\Psi}_0)\right)^{1+r} \tilde{\Psi}_0^{\frac{1}{1+r}} \right\rceil \left\lceil \frac{1}{\theta} \log \frac{n\mu_0}{\epsilon} \right\rceil,$$

where $\epsilon > 0$ is the desired accuracy, $\mu_0 > 0$ is given, and $\theta, 0 < \theta < 1$, is the given barrier update parameter.

**Proof.** If the central path parameter $\mu$ has the initial value $\mu_0 > 0$ and is updated by multiplying $1 - \theta$ with $0 < \theta < 1$, then after at most

$$\left\lceil \frac{1}{\theta} \log \frac{n\mu_0}{\epsilon} \right\rceil$$
iterations we have \( n\mu < \epsilon \) ([14]). For the total number of iterations, we multiply the number of inner iterations by that of outer iterations. i.e.,

\[
\left[ 16(1 + 2\kappa)(p + \frac{4}{r} + 2)(p + 1)^{1+r} \Psi_0(1 + \log((p + 1)\Psi_0))^{1+r} \Psi_0^{1+r} \right] \cdot \left[ \frac{1}{\theta} \log \frac{n\mu_0}{\epsilon} \right].
\]

This completes the proof. □

Remark 2. Taking \( \tau = \mathcal{O}(n) \) and \( \theta = \Theta(1) \), the large-update algorithm has

\[
\mathcal{O}\left( \frac{(1 + 2\kappa)}{r}n^{1+r} \log n \frac{n\mu_0}{\epsilon} \right)
\]

iteration complexity. In particular, for \( r = \frac{1 + \epsilon}{\log \log n} \) with a sufficiently small \( \epsilon > 0 \), we have \( \frac{1}{r} (\log n)^{1+r} = \frac{\epsilon^{1+r}}{1+r} (\log n) \log(\log n) \). So we have \( \mathcal{O}\left( (1 + 2\kappa)\sqrt{n} \log(\log n) \right) \) iteration complexity with \( p = 1 \) and \( r = \frac{1 + \epsilon}{\log \log n} \). This complexity result improves the one in [2].

References


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