VECTOR $F$-COMPLEMENTARITY PROBLEMS WITH $g$-DEMI-PSEUDOMONOTONE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, a class of $g$-demi-pseudomonotone mappings is introduced and the solvability of a class of generalized vector $F$-complementarity problems with the mappings in Banach spaces is considered.

1. Introduction and Preliminaries

In the past years, many important generalizations of monotonicity such as quasi monotonicity, pseudo-monotonicity, dense-pseudomonotonicity and semi-monotonicity have been introduced to study the various classes of variational inequalities and complementarity problems [7, 9, 11-14].


On the other hand, Fang and Huang [4] also considered the vector $F$-complementarity problems with demi-pseudomonotone single-valued mappings, which are vector demicontinuous in the first variable and pseudomonotone in the second variable.

In this paper, we consider the generalized vector $F$-complementarity problems which generalize the vector $F$-complementarity problems considered by Fang and Huang, by adding a continuous convex mapping $g$ as finding $u \in K$
such that
\[ \langle A(u, u), g(u) \rangle + F(g(u)) \neq 0 \]
\[ \langle A(u, u), g(v) \rangle + F(g(v)) \neq 0, \text{ for } v \in K, \]
where \( A : K \times K \to L(X, Y) \), \( F : K \to Y \) and \( g : K \to K \) are mappings for a subset \( K \) of a reflexive Banach space \( X \), an ordered Banach space \((Y, \preceq)\) and a collection \( L(X, Y) \) of continuous linear mappings from \( X \) into \( Y \).

**Definition 1.1.** Let \((Y, C)\) be an ordered Banach space, where \( C \) is a pointed (i.e., \( C \cap \{-C\} = \{0\}\)) closed convex cone with a nonempty interior \( \text{int} C \).

With \( C \) we define the order relations \( \geq, \preceq, < \) and \( \not< \) as follows;
\[ x \geq y \iff x - y \in C, \]
\[ x \preceq y \iff x - y \not\in C, \]
\[ x < y \iff y - x \in \text{int} C, \]
\[ x \not< y \iff y - x \not\in \text{int} C \text{ for } x, y \in Y. \]

**Definition 1.2.** A mapping \( T : K \to L(X, Y) \) is said to be hemicontinuous if for any fixed \( x, y \in K \), the mapping \( t \to \langle T(x + t(y - x)), y - x \rangle \) is continuous at \( 0^+ \).

**Definition 1.3.** Let \( g : K \to K \) be a single-valued mapping, \( T : K \to L(X, Y) \) and \( F : K \to Y \) two nonlinear mappings. \( T \) is said to be \( g \)-pseudo-monotone with respect to \( F \) if for \( x, y \in K \),
\[ (T(x), g(y) - g(x)) + F(g(y)) - F(g(x)) \not\leq 0 \]
implies \( (T(y), g(y) - g(x)) + F(g(y)) - F(g(x)) \geq 0 \).

**Definition 1.4.** A mapping \( G : K \subset X \to 2^X \) is said to be a KKM mapping if for any finite set \( \{x_1, x_2, \ldots, x_n\} \subset K \),  \( \text{Co}\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{i=1}^{n} G(x_i) \), where \( 2^X \) denotes the family of all nonempty subsets of \( X \).

**Definition 1.5.** A mapping \( f : K \to Y \) is said to be convex if \( f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \) for \( x, y \in K \) and \( t \in [0, 1] \).

**F-KKM Theorem ([3]).** Let \( M \) be a nonempty subset of a Hausdorff topological vector space \( E \) and \( G : M \to 2^E \) be a KKM mapping. If \( G(x) \) is closed in \( E \) for every \( x \in M \) and compact for some \( x \in M \) then
\[ \bigcap_{x \in M} G(x) \neq \emptyset. \]

**Lemma 1.1.** ([1]) Let \((Y, \preceq)\) be an ordered Banach space induced by a pointed closed convex cone \( C \) with nonempty \( \text{int} C \). For \( a, b, c \in Y \), the following unifications hold:
\[ c \not< a \text{ and } a \geq b \implies b \not< c, \]
\[ c \not< a \text{ and } a \leq b \implies b \not< c. \]
2. Main results

First we consider the equivalence of Stampacchia-type of $g$-pseudomonotone vector variational inequalities and Minty-type of $g$-pseudomonotone vector variational inequalities, and then the existences of solutions to them mentioned.

Next we consider the existences of solutions to the more generalized vector $F$-complementarity problems with $g$-demi-pseudomonotone mappings.

In this paper, $K$ is a bounded closed and convex subset of a real reflexive Banach space, $(Y, \leq)$ an ordered Banach space induced by a pointed closed convex cone $C$ with $\text{int} C \neq \emptyset$ and $L(X,Y)$ the space of all the continuous linear mappings from $X$ into $Y$.

**Theorem 2.1.** Let $T : K \to L(X,Y)$ be a hemicontinuous mapping, $g : K \to K$ and $F : K \to Y$ two convex mappings. Suppose that $T$ is $g$-pseudomonotone with respect to $F$. Then for any given point $x_0 \in K$, the following are equivalent

(i) $\langle T(x_0), g(x) - g(x_0) \rangle + F(g(x)) - F(g(x_0)) \leq 0$ for $x \in K$;
(ii) $\langle T(x), g(x) - g(x_0) \rangle + F(g(x)) - F(g(x_0)) \geq 0$ for $x \in K$.

**Proof.** We only prove that (ii) implies (i), the converse is obvious by Definition 1.3.

Suppose that (ii) holds. For any given $x \in K$ and $t \in (0,1)$, let $x_t = x_0 + t(x - x_0)$ then it follows from the convexities of $g$ and $F$ that

\[
t(\langle T(x_0 + t(x - x_0)), g(x) - g(x_0) \rangle + t(F(g(x)) - F(g(x_0))) \\
\geq \langle T(x_0 + t(x - x_0)), t(g(x) - g(x_0)) \rangle + F(g(tx + (1 - t)x_0)) - F(g(x_0)) \\
\geq 0.
\]

Hence

\[
\langle T(x_0 + t(x - x_0)), g(x) - g(x_0) \rangle + F(g(x)) - F(g(x_0)) \geq 0.
\]

Since $T$ is hemicontinuous and $C$ is closed, letting $t \to 0^+$ in the above inequality, we have

\[
\langle T(x_0), g(x) - g(x_0) \rangle + F(g(x)) - F(g(x_0)) \geq 0.
\]

Hence

\[
\langle T(x_0), g(x) - g(x_0) \rangle + F(g(x)) - F(g(x_0)) \leq 0 \text{ for } x \in K.
\]

**Theorem 2.2.** Let $g : K \to K$, $F : K \to Y$ be continuous convex mappings and $T : K \to L(X,Y)$ a hemicontinuous mapping.

If $T$ is $g$-pseudomonotone with respect to $F$, then there exists $x \in K$ such that

\[
\langle T(x), g(y) - g(x) \rangle + F(g(y)) - F(g(x)) \neq 0 \text{ for } y \in K.
\]

**Proof.** Define two set-valued mappings $G_1, G_2 : K \to 2^K$ as follows:

\[
G_1(z) = \{ x \in K : \langle T(x), g(z) - g(x) \rangle + F(g(z)) - F(g(x)) \neq 0 \}
\]

and

\[
G_2(z) = \{ x \in K : \langle T(x), g(x) - g(z) \rangle + F(g(x)) - F(g(z)) \neq 0 \}
\]

Then $G_1$ and $G_2$ are demiconotone mappings.
and
\[ G_2(z) = \{ x \in K : (T(z), g(z) - g(x)) + F(g(z)) - F(g(x)) \geq 0 \}. \]

Then \( G_1 \) is a KKM mapping. In fact, if it is not, then there exist \( \{ x_1, \ldots, x_n \} \subset K, x = \sum_{i=1}^{n} t_i x_i \) with \( t_i > 0 \) and \( \sum_{i=1}^{n} t_i = 1 \) such that \( x \not\in \bigcup_{i=1}^{n} G_1(x_i) \). It follows that
\[
(T(x), g(x_i) - g(x)) + F(g(x_i)) - F(g(x)) < 0, \quad i = 1, \ldots, n.
\]

By the convexities of \( F \) and \( g \), we have
\[
0 = \langle T(x), g(x) - g(x) \rangle + F(g(x)) - F(g(x)) \\
\leq \sum_{i=1}^{n} t_i \langle T(x), g(x_i) - g(x) \rangle + \sum_{i=1}^{n} t_i F(g(x_i)) - F(g(x)) \\
= \sum_{i=1}^{n} t_i \left[ (T(x), g(x_i) - g(x)) + F(g(x_i)) - F(g(x)) \right] \\
< 0.
\]

Hence \( 0 \in \text{int} C \), which derives a contradiction. Thus \( G_1 \) is a KKM mapping.

On the other hand, since \( T \) is \( g \)-pseudomonotone with respect to \( F \), \( G_1(z) \subset G_2(z) \) for \( z \in K \) and so \( G_2 \) is also a KKM mapping. Also since \( K \) is bounded closed and convex, \( K \) is weakly compact. Furthermore, it is easy to check that \( G_2(z) \subset K \) is closed and convex because \( F \) and \( g \) are continuous and convex. Hence \( G_2(z) \) is weakly compact for each \( z \in K \). It follows from F-KKM Theorem and Theorem 2.1 that
\[
\bigcap_{z \in K} G_1(z) = \bigcap_{z \in K} G_2(z) \neq \emptyset.
\]

Thus there exists \( x \in K \) such that
\[
\langle T(x), g(y) - g(x) \rangle + F(g(y)) - F(g(x)) \not< 0 \quad \text{for } y \in K.
\]
\[
\square
\]

**Definition 2.1.** Let \( g : K \to K \) be a single-valued mapping, \( A : K \times K \to L(X,Y) \) and \( F : K \to Y \) two nonlinear mappings. \( A \) is said to be \( g \)-demi-pseudomonotone with respect to \( F \) if the following two conditions hold:

\( (a) \) for each fixed \( u \in K \), \( A(u, \cdot) \) is \( g \)-pseudomonotone with respect to \( F \).

That is,
\[
\langle A(u, x), g(y) - g(x) \rangle + F(g(y)) - F(g(x)) \not< 0
\]

implies
\[
\langle A(u, y), g(y) - g(x) \rangle + F(g(y)) - F(g(x)) \geq 0 \quad \text{for } x, y \in K.
\]
(b) for each fixed \( v \in K \), \( A(v, \cdot) \) is vector demicontinuous, that is, for any net \( \{ u_\alpha \} \subset K \) and \( w \in X \), \( \{ u_\alpha \} \) converges to \( u_0 \) in the weak topology of \( X \) implies that \( \langle A(u_\alpha, v), w \rangle \) converges to \( \langle A(u_0, v), w \rangle \) in the norm topology of \( Y \).

**Definition 2.2.** A mapping \( F : K \to Y \) is said to be completely continuous if for any net \( \{ u_\alpha \} \subset K \), \( \{ u_\alpha \} \) converges to \( u_0 \) in the weak topology implies that \( F(u_\alpha) \) converges to \( F(u_0) \) in the norm topology.

**Theorem 2.3.** Let \( K \subset X \) be a nonempty bounded closed and convex set, \( F : K \to Y \) a completely continuous and convex mapping and \( g : K \to K \) a continuous and convex mapping. Suppose that

\[
\langle A(u, u), g(v) - g(u) \rangle + F(g(v)) - F(g(u)) \neq 0 \quad \text{for } v \in K.
\]

**Proof.** Let \( D \subset X \) be a finite-dimensional subspace with \( K_D = D \cap K \neq \emptyset \). For each \( w \in K \), consider the following problem:

Find \( u_0 \in K_D \) such that

\[
\langle A(w, u_0), g(v) - g(u_0) \rangle + F(g(v)) - F(g(u_0)) \neq 0 \quad \text{for } v \in K_D. \tag{2.1}
\]

Since \( K_D \subset D \) is bounded closed and convex, \( A(w, \cdot) \) is continuous on \( K_D \) and \( g \)-demi-pseudomonotone with respect to \( F \) for each fixed \( w \in K \), from Theorem 2.2, we know that problem (2.1) has a solution \( u_0 \in K_D \).

Now we define a set-valued mapping \( T : K_D \to 2^{K_D} \) as follows:

\[
T(w) = \{ u \in K_D : \langle A(w, u), g(v) - g(u) \rangle + F(g(v)) - F(g(u)) \neq 0 \quad \text{for } v \in K_D \},
\]

for \( w \in K_D \).

By Theorem 2.1, for each fixed \( w \in K_D \),

\[
\{ u \in K_D : \langle A(w, v), g(v) - g(u) \rangle + F(g(v)) - F(g(u)) \neq 0 \quad \text{for } v \in K_D \}
\]

\[
= \{ u \in K_D : \langle A(w, v), g(v) - g(u) \rangle + F(g(v)) - F(g(u)) \geq 0 \quad \text{for } v \in K_D \}.
\]

Since \( F \) is completely continuous and convex, it follows that \( T : K_D \to 2^{K_D} \) has nonempty bounded closed and convex values. We also know that \( T \) is upper semicontinuous by the vector demicontinuity of \( A(\cdot, u) \). By using the Glicksberg fixed point theorem [6], \( T \) has a fixed point \( w_0 \in K_D \), i.e.,

\[
\langle A(w_0, w_0), g(v) - g(w_0) \rangle + F(g(v)) - F(g(w_0)) \neq 0 \quad \text{for } v \in K_D. \tag{2.2}
\]

Let \( D = \{ D \subset X : D \) is a finite-dimensional subspace with \( D \cap K \neq \emptyset \} \) and \( W_D = \{ u \in K : \langle A(u, v), g(v) - g(u) \rangle + F(g(v)) - F(g(u)) \geq 0 \quad \text{for } v \in K_D \} \) for \( D \in D \).
By (2.2) and Theorem 2.1, we know that \( W_D \) is nonempty and bounded. Then the weak closure \( cl(W_D) \) of \( W_D \) is weakly compact in \( D \).

For any \( D_i \in \mathcal{D}, i = 1, 2, \ldots, n \), we know that \( W_{\bigcup D_i} \subset \bigcap W_{D_i} \). So \( \{ cl(W_D) : D \in \mathcal{D} \} \) has the finite intersection property. It follows that

\[
\bigcap_{D \in \mathcal{D}} cl(W_D) \neq \emptyset.
\]

Let \( u \in \bigcap_{D \in \mathcal{D}} cl(W_D) \). We claim that

\[
\langle A(u, u), g(v) - g(u) \rangle + F(g(v)) - F(g(u)) \neq 0 \quad \text{for } v \in K.
\]

Indeed, for each \( v \in K \), let \( D \in \mathcal{D} \) be such that \( v \in K_D \) and \( u \in K_D \). Since \( W_D \) is weakly closed there exists a net \( \{ u_\alpha \} \subset W_D \) such that \( \{ u_\alpha \} \) converges to \( u \) with respect to the weak topology of \( X \). It follows that

\[
\langle A(u_\alpha, v), g(v) - g(u_\alpha) \rangle + F(g(v)) - F(g(u_\alpha)) \geq 0.
\]

It follows that

\[
\langle A(u, v), g(v) - g(u) \rangle + F(g(v)) - F(g(u)) \neq 0 \quad \text{for } v \in K,
\]

by the vector demicontinuity of \( A(\cdot, v) \) and the continuities of \( F \) and \( g \). By Theorem 2.1, we know

\[
\langle A(u, u), g(v) - g(u) \rangle + F(g(v)) - F(g(u)) \neq 0 \quad \text{for } v \in K.
\]

\[\square\]

**Theorem 2.4**. Suppose that \( K \) is a nonempty closed convex cone and all the conditions of Theorem 2.3 hold. Furthermore, if \( g(0) = 0 \) and \( F(0) = 0 \), then there exists \( u \in K \) such that

\[
\langle A(u, u), g(u) \rangle + F(g(u)) \neq 0 \quad \text{and} \quad \langle A(u, u), g(v) \rangle + F(g(v)) \neq 0 \quad \text{for } v \in K.
\]

**Proof.** By Theorem 2.3, there exists \( u \in K \) such that

\[
\langle A(u, u), g(v) - g(u) \rangle + F(g(v)) - F(g(u)) \neq 0, \quad \text{for } v \in K. \tag{2.3}
\]

Since \( g(0) = 0 \) and \( F(0) = 0 \), we have

\[
\langle A(u, u), g(u) \rangle + F(g(u)) \neq 0.
\]

On the other hand, any \( w \in K \), substituting \( v = u + w \) into (2.3), we have

\[
\langle A(u, u), g(u + w) - g(u) \rangle + F(g(u + w)) - F(g(u)) \neq 0.
\]

Since \( g \) and \( F \) are convex,

\[
g(u + w) \leq g(u) + g(w)
\]

and

\[
F(g(u + w)) \leq F(g(u) + g(w)) \leq F(g(u)) + F(g(w))
\]

It follows Lemma 1.1, that

\[
\langle A(u, u), g(w) \rangle + F(g(w)) \neq 0 \quad \text{for } w \in K.
\]

\[\square\]
Remark 2.1. By putting $g = I$, the identity in Theorems 2.1, 2.2, 2.3 and 2.4, we obtain the corresponding results in Fang and Huang [4].

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