STRONG CONVERGENCE THEOREM OF COMMON ELEMENTS FOR EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS

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Abstract. In this paper, we introduce an iterative method for finding a common element of the set of solutions of an equilibrium problem, the set of common fixed points of an asymptotically strictly pseudocontractive mapping in a Hilbert space. We show that the iterative sequence converges strongly to a common element of the two sets.

1. Introduction

Let $H$ be a Hilbert space and $C$ be a nonempty closed convex subset of $H$. Let $f$ be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $f : C \times C \rightarrow \mathbb{R}$ is to find $x \in C$ such that
\[
 f(x, y) \geq 0, \forall y \in C.
\] (1.1)
The set of solutions of (1.1) is denoted by $EP(f)$. Given a mapping $T : C \rightarrow H$, let $f(x, y) = \langle Tx, y - x \rangle$ for all $x, y \in C$. Then, $\hat{x} \in EP(f)$ if and only if $\langle T\hat{x}, y - \hat{x} \rangle \geq 0$ for all $y \in C$, i.e., $\hat{x}$ is a solution of the variational inequality.

A mapping $T : C \rightarrow C$ is said to be asymptotically $\lambda$-strictly pseudocontractive if there exist $\lambda \in [0,1)$ and a sequence $\{k_n\}$ with $k_n \geq 1$ for all $n$ and $\lim_{n \to \infty} k_n = 1$ and such that
\[
 \|T^n x - T^n y\|^2 \leq k_n \|x - y\|^2 + \lambda \|(I - T^n)x - (I - T^n)y\|^2
\] for all $n \geq 1$ and $x, y \in C$. This class of mappings has been studied by several authors, and it includes the important class of asymptotically nonexpansive maps ($\lambda = 0$). It is well known that if $T$ is asymptotically strictly pseudocontractive, then $T$ is uniformly $L$-Lipschitzian, i.e., $\|T^n x - T^n y\| \leq L\|x - y\|$. 

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see [2]. A point \( x \in C \) is a fixed point of \( T \) provided \( Tx = x \). Denoted by \( F(T) \) the set of fixed points of \( T \).

Recently, many authors studied the problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem; for instance [3]. Inspired and motivated by these facts, we prove strong convergence theorems for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of an asymptotically strictly pseudocontractive mapping.

2. Preliminaries

Let \( H \) be a real Hilbert space with inner product \( \langle \cdot , \cdot \rangle \) and norm \( \| \cdot \| \). \( \{x_n\} \) is a sequence in \( H \), \( x_n \rightharpoonup x \) implies that \( \{x_n\} \) converges weakly to \( x \) and \( x_n \rightarrow x \) means the strong convergence. In a real Hilbert space \( H \), we have

\[
\| \lambda x + (1 - \lambda)y \| = \lambda \| x \| + (1 - \lambda)\| y \| - \lambda(1 - \lambda) \| x - y \|^2,
\]

for all \( x,y \in H \) and \( \lambda \in [0,1] \).

Such a \( P_C \) is called the metric projection of \( H \) onto \( C \). We know that \( P_C \) is nonexpansive. Further, for any \( x \in H \) and \( z \in C \), \( z = P_C x \iff \langle x - z, z - y \rangle \geq 0 \) for all \( y \in C \). We also know that for any sequence \( \{x_n\} \subset H \) with \( x_n \rightharpoonup x \), the inequality

\[
\lim \inf_{n \rightarrow \infty} \|x_n - x\| < \lim \inf_{n \rightarrow \infty} \|x_n - y\|
\]

holds for every \( y \in H \) with \( x \neq y \).

For solving the equilibrium problem for a bifunction \( f : C \times C \rightarrow \mathbb{R} \) satisfying (A1)-(A4), we assume that \( f \) satisfies the following conditions:

(A1) \( f(x,x) = 0 \) for all \( x \in C \);

(A2) \( f \) is monotone, i.e., \( f(x,y) + f(y,x) \leq 0 \) for all \( x,y \in C \);

(A3) for each \( x,y,z \in C \),

\[
\lim_{t \downarrow 0} f(tz + (1-t)x,y) \leq f(x,y);
\]

(A4) for each \( x \in C \), \( y \mapsto f(x,y) \) is convex and lower semicontinuous.

**Lemma 2.1.** ([4]) Let \( C \) be a nonempty closed subset of \( H \) and \( f \) be a bifunction of \( C \times C \) into \( \mathbb{R} \) satisfying (A1)-(A4). Let \( r > 0 \) and \( x \in H \). Then, there exists \( z \in C \) such that

\[
f(z,y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C.
\]

**Lemma 2.2.** ([5]) Assume that \( f : C \times C \rightarrow \mathbb{R} \) satisfies (A1)-(A4). For \( r > 0 \) and \( x \in H \), define a mapping \( \Phi_r : H \rightarrow C \) as follows:

\[
\Phi_r(x) = \{ z \in C : f(z,y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \}.
\]
for all \( x \in H \). Then the following hold:

1. \( \Phi_r \) is single-valued;
2. \( \Phi_r \) is firmly nonexpansive, i.e., for any \( x, y \in H \),
   \[ ||\Phi_r(x) - \Phi_r(y)||^2 \leq (\Phi_r(x) - \Phi_r(y), x - y); \]
3. \( F(\Phi_r) = EP(f) \);
4. \( EP(f) \) is closed and convex.

**Lemma 2.3.** ([7]) Let \( E \) be a real \( q \)-uniformly smooth Banach space which is also uniformly convex. Let \( C \) be a nonempty closed convex subset of \( E \) and \( T : C \to C \) an asymptotically \( k \)-strictly pseudocontractive mapping with a nonempty fixed point set. Then \((I - T)\) is demiclosed at zero.

**Lemma 2.4.** ([7]) Let \( H \) be a real Hilbert space. Given a closed convex subset \( C \subset H \) and points \( x, y, z \in H \). Given also a real number \( a \in R \). The set \( D := \{ v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v \rangle + a \} \) is convex and closed.

3. Main result

**Theorem 3.1.** Let \( C \) be a bounded closed convex subset of a real Hilbert space \( H \). Let \( T : C \to C \) be a nonempty closed convex subset of \( E \) and \( T : C \to C \) is an asymptotically \( k \)-strict pseudocontraction mapping. Let \( f \) be a bifunction from \( C \times C \) into \( R \) satisfying (A1)-(A4). Assume that \( \{\alpha_n\} \) is a sequence in \((0, 1)\) satisfying the condition: \( 0 < a + \lambda \leq \alpha_n \leq 1 - b, \forall n \geq 0 \) and for some \( a, b, \in (0, 1), \{r_n\} \subset [m, \infty) \) for some \( m > 0 \). If \( F := F(T) \cap EP(f) \neq \emptyset \), then the sequence \( \{x_n\} \) generated by

\[
\begin{cases}
x_0 \in C, \\
y_n = \alpha_n x_n + (1 - \alpha_n)T^n x_n, \\
u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - y_n \rangle \geq 0, \forall y \in C, \\
C_{n+1} = \{ z \in C_n : \|u_n - z\|^2 \leq \|x_n - z\|^2 - (1 - \alpha_n)(\alpha_n - \lambda)\|x_n - T^n x_n\|^2 + \theta_n \}, \\
x_{n+1} = P_{C_{n+1}} x_0, 
\end{cases}
\]

where \( \theta_n = (1 - \alpha_n)(k_n - 1)(\text{diam} C)^2 \to 0 \) as \( n \to \infty \), converges in norm to \( Pf x_0 \).

**Proof.** Firstly, We observe that \( C_n \) is convex by Lemma 2.4.

Next observe that \( F \subset C_n \) for all \( n \). Indeed, for all \( p \in F \), we have

\[
\begin{align*}
\|u_n - p\|^2 &= \|p - \Phi_{r_n} y_n\|^2 \\
&\leq \|y_n - p\|^2 \\
&\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n)\|T^n x_n - p\|^2 - \alpha_n (1 - \alpha_n)\|x_n - T^n x_n\|^2 \\
&\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n)k_n \|x_n - p\|^2 + (1 - \alpha_n)\lambda \|x_n - T^n x_n\|^2 \\
&\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n)k_n \|x_n - p\|^2 + (1 - \alpha_n)\lambda \|x_n - T^n x_n\|^2.
\end{align*}
\]
\[- \alpha_n (1 - \alpha_n) \|x_n - T^n x_n\|^2 \leq [1 + (1 - \alpha_n) (k_n - 1)] \|x_n - p\|^2 - (1 - \alpha_n) (\alpha_n - \lambda) \|x_n - T^n x_n\|^2 \leq \|x_n - p\|^2 - (1 - \alpha_n) (\alpha_n - \lambda) \|x_n - T^n x_n\|^2 + \theta_n.\]

So \( p \in C_{k+1} \). This implies that \( F \subset C_n \) for all \( n \).

From \( x_n = P_{C_n} x_0 \), we have \( \langle x_0 - x_n, x_n - y \rangle \geq 0 \) for all \( y \in C_n \). Using \( F \subset C_n \), we also have \( \langle x_0 - x_n, x_n - p \rangle \geq 0 \) for all \( p \in F \).

So, for \( p \in F \) we have

\[
0 \leq \langle x_0 - x_n, x_n - p \rangle = \langle x_0 - x_n, x_n - x_0 + x_0 - p \rangle \leq -\|x_0 - x_n\|^2 + \|x_0 - p\|\|x_0 - x_n\|.
\]

This implies that

\[
\|x_0 - x_n\| \leq \|x_0 - p\|.
\]

From \( x_n = P_{C_n} x_0 \) and \( x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1} \subset C_n \), we also have \( \langle x_{n+1} - x_n, x_n - x_0 \rangle \geq 0 \), from the above inequality, we have for all \( n \),

\[
0 \leq \langle x_0 - x_n, x_n - x_{n+1} \rangle = \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1} \rangle \leq -\|x_0 - x_n\|^2 + \|x_0 - x_{n+1}\|\|x_0 - x_n\|
\]

and hence

\[
\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|.
\]

Since \( \{\|x_0 - x_n\|\} \) is bounded, \( \lim_{n \to \infty} \|x_0 - x_n\| \) exists. Next, we show that \( \lim_{n \to \infty} \|x_{n+1} - x_n\| = 0 \). In fact that \( x_n = P_{C_n} x_0 \) and \( x_{n+1} \in C_n \) which imply that

\[
\|x_{n+1} - x_n\|^2 = \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 = \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0 \rangle \leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2.
\]

So we have \( \lim_{n \to \infty} \|x_{n+1} - x_n\| = 0 \).

Since \( x_{n+1} \in C_{n+1} \subset C_n \), we have

\[
\|u_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 - (1 - \alpha_n) (\alpha_n - \lambda) \|x_n - T^n x_n\|^2 + \theta_n \leq \|x_n - x_{n+1}\|^2 + \theta_n.
\]

So we have \( \lim_{n \to \infty} \|x_{n+1} - u_n\| = 0 \) and \( \lim_{n \to \infty} \|x_n - u_n\| = 0 \). Observe that

\[
\|y_n - x_n\|^2 = (1 - \alpha_n)^2 \|x_n - T^n x_n\|^2 \leq \|y_n - u_n\| + \|u_n - x_n\\|^2 \leq \|y_n - u_n\|^2 + 2\|y_n - u_n\|\|u_n - x_n\| + \|u_n - x_n\|^2.
\]
Since $\Phi_r$ is firmly nonexpansive, for all $p \in F$, we have
\[
\|u_n - p\|^2 = \|\Phi_{r_n}y_n - \Phi_{r_n}p\|^2 \leq \langle \Phi_{r_n}y_n - \Phi_{r_n}p, y_n - p \rangle \\
= \langle u_n - p, y_n - p \rangle = \frac{1}{2}(\|u_n - p\|^2 + \|y_n - p\|^2 - \|u_n - y_n\|^2),
\]
and hence
\[
\|u_n - y_n\|^2 \leq \|y_n - p\|^2 - \|u_n - p\|^2 \\
\leq \|x_n - p\|^2 - \|u_n - p\|^2 + \|u_n - x_n\|^2.
\]
So we have
\[
(1 - \alpha_n)^2\|x_n - T^n x_n\|^2 \\
\leq \|x_n - p\|^2 - \|u_n - p\|^2 + \|u_n - x_n\|^2 \\
+ 2\|y_n - u_n\|\|u_n - x_n\| + \|u_n - x_n\|^2.
\]
Thus
\[
b(1 - \lambda)\|x_n - T^n x_n\|^2 \\
\leq \|u_n - x_n\|\|u_n - p\| + \|x_n - p\| + \|u_n - x_n\| + \|u_n - x_n\|^2.
\]
Hence $\lim_{n \to \infty} \|x_n - T^n x_n\| = 0$, $\lim_{n \to \infty} \|x_n - y_n\| = 0$, $\lim_{n \to \infty}\|u_n - y_n\| = 0$.

Observing that
\[
\|x_n - T x_n\| \\
\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1} x_{n+1}\| + \|T^{n+1} x_{n+1} - T^{n+1} x_n\| \\
+ \|T^{n+1} x_n - T x_n\| \\
\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1} x_{n+1}\| + L\|x_{n+1} - x_n\| + L\|T^n x_n - x_n\|.
\]
Hence $\lim_{n \to \infty} \|x_n - T x_n\| = 0$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x}$. By Lemma 2.3, we have that $\hat{x} \in F(T)$. From $x_{n_k} \rightharpoonup \hat{x}$, $\|u_n - y_n\| \to 0$ and $\|u_n - x_n\| \to 0$, we have $y_{n_k} \rightharpoonup \hat{x}$ and $u_{n_k} \rightharpoonup \hat{x}$. From $r_n \geq m$, we have $\lim_{n \to \infty} \|u_n - y_n\|/r_n = 0$. By $u_n = \Phi_{r_n}y_n$, we have
\[
f(u_n, y) + \frac{1}{r_n}(y - u_n, u_n - y_n) \geq 0, \forall y \in C.
\]
Replacening $n$ by $n_k$, we have from (A2) that
\[
\frac{1}{r_{n_k}}(y - u_{n_k}, u_{n_k} - y_{n_k}) \geq -f(u_{n_k}, y) \geq f(y, u_{n_k}), \forall y \in C.
\]
Letting $k \to \infty$, we have from (A4) that $f(y, \hat{x}) \leq 0, \forall y \in C$. For $0 < t < 1$ and $y \in C$, define $y_t = ty + (1 - t)\hat{x}$. Since $y \in C$ and $\hat{x} \in C$, we have $y_t \in C$ and hence $f(y_t, \hat{x}) \leq 0$. So, from (A1) we have
\[
0 = f(y_t, y_t) \leq tf(y, y) + (1 - t)f(y, \hat{x}) \\
\leq tf(y, y).
\]
Dividing by $t$, we have

$$f(y, y) \geq 0, \forall y \in C.$$  

Letting $t \downarrow 0$, from (A3) we have

$$f(\hat{x}, y) \geq 0, \forall y \in C.$$  

Therefore, $\hat{x} \in EP(f)$.

Let $w = P_Fx_0$. From $x_n = P_{C_n}x_0$ and $w \in F \subset C_n$, we have

$$\|x_0 - x_n\| \leq \|x_0 - w\|.$$  

Since the norm is weakly lower semicontinuous, we have

$$\|x_0 - w\| \leq \|x_0 - \hat{x}\| \leq \liminf_{k \to \infty} \|x_0 - x_{n_k}\| \leq \limsup_{k \to \infty} \|x_0 - x_{n_k}\| \leq \|x_0 - w\|.$$  

This implies that $\|x_0 - w\| = \|x_0 - \hat{x}\|$ and $\|x_0 - x_{n_k}\| \to \|x_0 - w\|$. It follows that $w = \hat{x}$ and $x_{n_k} \to w$. Therefore $\{x_n\}$ converges strongly to $w$.  

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