A DIFFERENCE SET METHOD FOR CIRCULANT DECOMpositions OF COMPLETE PARTITE GRAPHS INTO GREGARIOUS 4-CYCLES

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Abstract. The complete multipartite graph $K_{n(m)}$ with $n \geq 4$ partite sets of size $m$ is shown to have a decomposition into 4-cycles in such a way that vertices of each cycle belong to distinct partite sets of $K_{n(m)}$ if 4 divides the number of edges. Such cycles are called gregarious, and were introduced by Billington and Hoffman ([2]) and redefined in [3]. We independently came up with the result of [3] by using a difference set method, and improved the result so that the composition is circulant, in the sense that it is invariant under the cyclic permutation of partite sets. The composition is then used to construct gregarious 4-cycle decompositions when one partite set of the graph has different cardinality than that of others. Some results on joins of decomposable complete multipartite graphs are also presented.

1. Introduction

Edge disjoint decompositions of graphs into cycles has been considered in a number of ways. Necessary and sufficient conditions for a complete graph of odd order or a complete graph of even order minus an 1-factor to have a decomposition into cycles of some fixed length are known (see [1], [9] and [10] as well as their references). Some authors have also considered cycle decompositions with special properties such as resolvable cycle decompositions (see [4], [8], [7]). Although much work has gone into the decomposition of complete graphs into cycles, less attention has been paid to the same problem for complete multipartite graphs. Sotteau [11] showed that the complete bipartite graph $K_{m,n}$ has a decomposition into $2^k$-cycles if and only if $m,n$ are even and $2k$ divides $mn$. Note that the two obvious conditions turned out to be sufficient conditions. This result has been used as a powerful tool by later authors for various decompositions of multipartite graphs into cycles, as was done in [5].

Billington and Hoffman [2] introduced the notion of a gregarious cycle, meaning a cycle having at least one vertex from each partite set, and then changed
the definition slightly in [3]. There, a *gregarious cycle* is a cycle having at most one vertex from any given partite set. They gave a necessary and sufficient condition for an equipartite graph to have a decomposition into gregarious 4-cycles. An equipartite graph is a complete multipartite graph all of whose partite sets have the same size. They also considered complete multipartite graphs with one partite set of different size.

Throughout the paper, $K(A_1, A_2, \ldots, A_n)$ will denote the complete multipartite graph with partite sets $A_1, A_2, \ldots, A_n$, which we sometimes denote by $K(m_1, m_2, \ldots, m_n)$ when $|A_i| = m_i$ for $i = 1, 2, \ldots, n$. When all $m_i$ are equal to $m$, we use the notation $K_n(m)$. This notation is standard and can be found in [6] amongst other places.

In this paper, we use the new definition of gregarious cycles. The purpose of this paper is, using difference sets, to show that a complete graph $K_n$ or an equipartite graph $K_n(m)$ has a decompositions into gregarious 4-cycles if $n$ is at least 4 and the number of edges is divisible by 4. Note that the sufficient condition is an obvious necessary condition. Improving a result in [3], we show that the decomposition can be circulant if $n \geq 5$, in the sense that the decomposition is invariant under the cyclic permutations of the partite sets. The decompositions of graphs obtained so far by other authors heavily relied on successively taking joins of graphs, and so the decompositions were not circulant at all. Some of our results overlaps with some results in [3], but we want make it clear that the results are obtained independently and use a different method from theirs.

The following is informally mentioned in many papers, and formally proved in [5] when the graph is decomposable into cycles of a fixed even length. However, it also applies when graph is decomposable into cycles of arbitrary lengths.

**Lemma 1.1.** If $K(m_1, m_2, \ldots, m_n)$ is decomposable into cycles, then all $m_1, m_2, \ldots, m_n$ have the same parity, and furthermore $n$ must be odd if the parity of each $m_i$ is odd.

**Proof.** For any $i$, the degree of any vertex in $A_i$ is $\sum_{k \neq i} m_k$, which must be even. Thus, for any pair $i$ and $j$, $m_j - m_i = \sum_{k \neq i} m_k - \sum_{k \neq j} m_k$ is also even. Thus, all $m_1, m_2, \ldots, m_n$ have the same parity. If this parity is odd, since $\sum_{k=1}^{n-1} m_k$ is even, $n - 1$ must be even, i.e., $n$ is odd. □

For simplicity, we will call a gregarious 4-cycle a $\gamma_4$-cycle. A graph will be called $\gamma_4$-*decomposable* if it is decomposable into $\gamma_4$-cycles, and a decomposition into $\gamma_4$-cycles will be called a $\gamma_4$-*decomposition*.

The following lemma is proved in [5] for decompositions into arbitrary (non-necessarily gregarious) cycles, by the standard “expanding points method”. However, exactly the same method can be applied for decompositions into gregarious cycles.
Lemma 1.2. If \( K(m_1, m_2, \ldots, m_n) \) is decomposable into gregarious \( k \)-cycles for an even integer \( k \), then so is \( K(m_1t, m_2t, \ldots, m_nt) \) for every integer \( t \geq 1 \).

Put \( D_n = \{ \pm 1, \pm 2, \ldots, \pm \frac{n-1}{2} \} \) if \( n \) is odd and \( D_n = \{ \pm 1, \pm 2, \ldots, \pm \frac{n-2}{2}, \frac{n-1}{2} \} \) if \( n \) is even. Then, \( D_n \) is a complete set of differences of pairs of distinct integers in \( \mathbb{Z}_n = \{ 0, 1, \ldots, n-1 \} \).

A sequence \( (r_1, r_2, r_3, r_4) \) of differences in \( D_n \) is called a feasible sequence, or an \( f \)-sequence for simplicity, if \( r_1 + r_2 + r_3 + r_4 = 0 \) and \( r_i + r_{i+1} \neq 0 \) for \( i = 1, 2, 3 \), where the arithmetic is done in \( \mathbb{Z}_n \). The sequence of initial sums, or the \( s \)-sequence for short, corresponding to an \( f \)-sequence \( \rho = (r_1, r_2, r_3, r_4) \) is the sequence \( \sigma_\rho = (s_0, s_1, s_2, s_3) \) of elements in \( \mathbb{Z}_n \) such that \( s_0 = 0 \) and \( s_i = \sum_{j=1}^{i} r_j \) for \( i = 1, 2, 3 \). Note that, \( s_i = s_{i-1} + r_i \) for each \( i = 1, 2, 3 \) and \( s_3 + r_4 = s_0 \). Here, all the arithmetic is done in \( \mathbb{Z}_n \).

Intuitively, an \( s \)-sequence is the sequence of partite sets of a multipartite graph or the sequence of vertices of a complete graph by which a 4-cycle traverses, and the feasibility of the \( f \)-sequence guarantees that the 4-cycle is proper and gregarious.

In the next section, we give circulant \( \gamma_4 \)-decompositions of equipartite graphs with odd partite sets. In Section 3, we show that \( K_{n(2t)} \) has a \( \gamma_4 \)-decomposition for every \( n \geq 4 \) and \( t \geq 1 \), which is circulant if \( n \geq 5 \). In Section 4, we study \( \gamma_4 \)-decomposition of \( K_{h,n(m)} \) when \( h \neq m \). In Section 5, we consider joins of \( \gamma_4 \)-decomposable graphs.

Before we continue to next section, we mention the following lemma.

Lemma 1.3. If \( K(m_1, m_2, m_3, m_4) \) is \( \gamma_4 \)-decomposable, then \( m_1 = m_2 = m_3 = m_4 \) and this number is an even integer.

Proof. Let \( A_i \) be the partite sets with \( |A_i| = m_i \) for \( i = 1, 2, 3, 4 \). Without loss of generality, we may assume that \( m_1 \geq m_2 \geq m_3 \geq m_4 \). Suppose \( C \) is a \( \gamma_4 \)-decomposition of \( K(m_1, m_2, m_3, m_4) \). Let \( a, b, c \) be the numbers of gregarious 4-cycles in \( C \) traversing the partite sets by the orders \((A_1, A_2, A_3, A_4)\), \((A_1, A_2, A_1, A_3)\), and \((A_1, A_4, A_2, A_3)\), respectively. Counting edges used by each type of gregarious 4-cycle, we have equations
\[ m_1m_2 = a + b = m_3m_4, \quad m_1m_3 = b + c = m_2m_4, \quad m_1m_4 = a + c = m_2m_3. \]

However, since \( m_1 \geq m_2 \geq m_3 \geq m_4 \), the first equation implies \( m_1 = m_2 = m_3 = m_4 \). Lemma 1.1 says that this number must be even. \( \Box \)

2. Equipartite graphs with partite sets of odd size

Let \( \tau : \mathbb{Z}_n \to \mathbb{Z}_n \) be the mapping defined by \( \tau(i) = i + 1 \) for all \( i \) in \( \mathbb{Z}_n \), i.e., the circulant permutation \((0, 1, 2, \ldots, n - 1)\) on \( \mathbb{Z}_n \). We can extend \( \tau \) to a mapping \( \tau_* : \mathbb{Z}^4 \to \mathbb{Z}^4 \) by defining \( \tau_* (s_0, s_1, s_2, s_3) = (\tau(s_0), \tau(s_1), \tau(s_2), \tau(s_3)) \). Then,
Lemma 2.1. (i) $K_n$ is decomposable into $4$-cycles if and only if $n \equiv 1 \pmod{8}$.
(ii) If $n$ is odd and $K_{n(m)}$ is decomposable into $4$-cycles, then $n \equiv 1 \pmod{8}$.

Proof. (i) is obtained from [1]. For (ii), note that $n$ is odd by Lemma 1, and $4$ must divide the number $\frac{n(n-1)m^2}{2}$ of edges in $K_{n(m)}$. \hfill \Box

Note that cycle decompositions of $K_n$ obtained by the method in [1] are not circulant at all. Here, we want to give a circulant decomposition.

Let $n \equiv 1 \pmod{8}$, say $n = 8k + 1$. Then $\mathcal{D}_n = \{\pm 1, \pm 2, \ldots, \pm 4k\}$. We use $\mathbb{Z}_n$ as the vertex set of $K_n$ as well. An edge $ij$ of $K_n$ will be called an edge of distance $d$ if $|i - j| = d$ for $d = 1, 2, \ldots, 4k$.

Consider the $f$-sequences $\rho_i = (4i + 1, -(4i + 2), -(4i + 3), 4i + 3)$ for $i = 0, 1, \ldots, k - 1$. For each $i$, the corresponding $s$-sequence is $\sigma_{\rho_i} = (0, 4i + 1, 1, -4i + 4)$, and the set

$$C_i = \{\tau_i^j(\sigma_{\rho_i}) \mid 0 \leq j \leq n-1\} = \{(j, 4i+1+j, -1+j, -(4i+4)+j) \mid 0 \leq j \leq n-1\}$$

consists of disjoint $4$-cycles and these cycles exhaust all and only edges of distance $4i+1, 4i+2, 4i+3$ or $4i+4$. Thus, the $4$-cycles in the set $\bigcup_{i=0}^{k-1} C_i$ are disjoint and exhaust all edges of every distance in $K_n$. Furthermore, this set is clearly invariant under $\tau_n$ as each $C_i$ is. By this, we have the following lemma, which extends Lemma 2.1(i) so that the decomposition is circulant.

Lemma 2.2. Let $n \equiv 1 \pmod{8}$, say $n = 8k + 1$. Then, $K_n$ is decomposable into $4$-cycles and the set

$$\mathcal{C} = \bigcup_{i=0}^{k-1} C_i = \{(j, 4i+1+j, -1+j, -(4i+4)+j) \mid 0 \leq i \leq k-1, 0 \leq j \leq n-1\}$$

is a circulant decomposition of $K_n$ into $4$-cycles, in the sense that $\mathcal{C}$ is invariant under $\tau_n$.

If we regard $K_n$ as $K(1,1,\ldots,1)$, then $K_{n(m)}$ has a $\gamma_4$-decomposition for any integers $n$ with $n \equiv 1 \pmod{8}$ and $m$ by Lemmas 1.2 and 2.2. However, we want to be more elaborate with the “expanding points method” in [5] to see that the decomposition can be circulant.

Let $n \equiv 1 \pmod{8}$ and $m$ be a positive integer. Expand each vertex $i$ of $K_n$ to the set $A_i = \{i_1, i_2, \ldots, i_m\}$, and let the partite sets of $K_{n(m)}$ be $A_i$ for $i = 0, 1, \ldots, n-1$. We extend the mapping $\tau$ to a mapping $\tau_{\ast\ast}$ of $4$-cycles of $K_{n(m)}$ by defining

$$\tau_{\ast\ast}(a_j, b_k, c_p, d_q) = \langle \tau(a)_j, \tau(b)_k, \tau(c)_p, \tau(d)_q \rangle = \langle (a+1)_j, (b+1)_k, (c+1)_p, (d+1)_q \rangle$$
for all \( j, k, p, q \) in \( \{1, 2, \ldots, m\} \). Let \( \mathcal{C} \) be the circulant decomposition of \( K_n \) in Lemma 2.2. Then, the set
\[
\mathcal{C}^* = \{ \langle a_j, b_k, c_j, d_k \rangle \mid \langle a, b, c, d \rangle \in \mathcal{C}, \ 1 \leq j \leq m, \ 1 \leq k \leq m \}.
\]
is a \( \gamma_4 \)-decomposition of \( K_{n(m)} \). Since \( \mathcal{C} \) is invariant under \( \tau_s \), if \( \langle a_j, b_k, c_j, d_k \rangle \) is in \( \mathcal{C}^* \), then \( \tau_s \langle a_j, b_k, c_j, d_k \rangle = \langle \tau(a)_j, \tau(b)_k, \tau(c)_j, \tau(d)_k \rangle \) also belongs to \( \mathcal{C}^* \). That is, \( \mathcal{C}^* \) is invariant under \( \tau_{ss} \).

Summarizing the above discussion, we have the following lemma.

**Lemma 2.3.** Let \( n \equiv 1 \pmod{8} \) and \( m \) be any integer, then \( K_{n(m)} \) has a circulant \( \gamma_4 \)-decomposition, in the sense that it is invariant under \( \tau_{ss} \).

Combining the discussions in this section, we have the following theorem.

**Theorem 2.4.** Let \( m \) be an odd integer. Then, \( K_{n(m)} \) has a circulant \( \gamma_4 \)-decomposition, in the sense that it is invariant under \( \tau_{ss} \) if and only if \( n \equiv 1 \pmod{8} \).

We remark that Billington and Hoffman ([3]) independently showed Theorem 2.1 using a different method, except that their decomposition is not circulant.

3. **Multipartite graphs with partite sets of even size**

Let \( A_i = \{i, i+7\}, i = 0, 1, \ldots, n-1 \), be the partite sets of \( K_{n(2)} \). Thus, the elements in \( \mathbb{Z}_n \) are used as indices of the partite sets and as vertices of the graph as well. The next lemma, which appears in [4], also serves as an example. However, the decomposition is not circulant.

**Lemma 3.1.** \( K_{4(2)} \) is \( \gamma_4 \)-decomposable.

**Proof.** The six 4-cycles
\[
\langle 0, 1, 2, 3 \rangle, \ \langle 0, 1, 2, 3 \rangle, \ \langle 0, 1, 2, 3 \rangle, \ \langle 0, 1, 2, 3 \rangle, \ \langle 0, 1, 2, 3 \rangle, \ \langle 0, 1, 2, 3 \rangle
\]
constitute a \( \gamma_4 \)-decomposition of \( K_{4(2)} \). \( \Box \)

An edge joining a vertex in \( A_i \) to a vertex in \( A_j \) is called an edge of distance \( d \) if \( i-j \equiv \pm d \pmod{n} \), where \( \pm d \) is in \( D_n \). In particular, if \( n \) is even and \( d = \pm 0 \) then an edge of distance \( d \) is called a diagonal edge. For example, the edges \( 01, 73, 72 \) and \( 82 \) are all edges of distance 4 in \( K_{4(2)} \), while the edges \( 00 \) and \( 55 \) are diagonal edges of \( K_{10(2)} \). If \( \langle a_1, a_2, a_3, a_4 \rangle \) is a 4-cycle, then the edges \( a_1a_2, a_2a_3, a_3a_4 \) and \( a_4a_1 \) will be called the first, second, third and fourth edges of the cycle, respectively.

Let \( \phi^+ \) and \( \phi^- \) be the mappings of \( \mathbb{Z}_n \) into \( \bigcup_{i=0}^{n-1} A_i \) defined by \( \phi^+(i) = i \) and \( \phi^-(i) = i+7 \) for all \( i \) in \( \mathbb{Z}_n \). A flag is a sequence \( \phi^* = \langle \phi_0, \phi_1, \phi_2, \phi_3 \rangle \) each of whose entries is \( \phi^+ \) or \( \phi^- \). Given such a flag \( \phi^* \), we also use the same notation
\( \phi^* \) to denote the mapping defined by \( \phi^*(a, b, c, d) = (\phi_0(a), \phi_1(b), \phi_2(c), \phi_3(d)) \) for every sequence \( (a, b, c, d) \) of numbers in \( \mathbb{Z}_n \). So, \( \phi^*(a, b, c, d) \) is a \( \gamma_4 \)-cycle of \( K_{n(2)} \) if \( a, b, c, d \) are distinct, and every \( \gamma_4 \)-cycle of \( K_{n(2)} \) is in this form.

Recall that \( \tau \) is the cyclic permutation \( (0, 1, \ldots, n-1) \) on \( \mathbb{Z}_n \). We extend \( \tau \) to a mapping \( \tau_4 \) on the \( \gamma_4 \)-cycles of \( K_{n(2)} \) by defining

\[
\tau_4(\phi^*(a, b, c, d)) = (\phi_0(\tau(a)), \phi_1(\tau(b)), \phi_2(\tau(c)), \phi_3(\tau(d)))
= (\phi_0(a+1), \phi_1(b+1), \phi_2(c+1), \phi_3(d+1)).
\]

Suppose we are given an \( f \)-sequence \( \rho = (r_1, r_2, r_3, r_4) \) and a flag \( \phi^* = (\phi_0, \phi_1, \phi_2, \phi_3) \). With the \( s \)-sequence \( \sigma_\rho = (s_0, s_1, s_2, s_3) \) corresponding to \( \rho \), we can generate a class \( \{ \tau_j^*(\phi^*(\sigma_\rho)) \mid 0 \leq j \leq n-1 \} \) of \( \gamma_4 \)-cycles of \( K_{n(2)} \). This class is called the \( n \)-class generated by \( \phi^*(\sigma_\rho) \), and \( \phi^*(\sigma_\rho) \) is called the \emph{starter cycle} of the class. It is clear that such an \( n \)-class is invariant under \( \tau \). For example, if \( (\phi_0, \phi_1, \phi_2, \phi_3) = (\phi^+, \phi^-, \phi^-, \phi^+) \), the \( \gamma_4 \)-cycles in the class are as below:

\[
\begin{align*}
\tau_4^0(\phi^*(\sigma_\rho)) &= (0, \overline{s_1}, \overline{s_2}, s_3), \\
\tau_4^1(\phi^*(\sigma_\rho)) &= (1, s_1+1, s_2+1, s_3+1), \\
&\vdots \\
\tau_4^j(\phi^*(\sigma_\rho)) &= (j, s_1+j, s_2+j, s_3+j), \\
&\vdots \\
\tau_4^{n-1}(\phi^*(\sigma_\rho)) &= (n-1, s_1-1, s_2-1, s_3-1).
\end{align*}
\]

Note that, every column of vertices above has one vertex from each partite set. Thus, the edge \( p\overline{q} \) appears as the first edge of a \( \gamma_4 \)-cycle above if \( q-p = s_1 = r_1 \). The edge \( p\overline{q} \) appears as the second edge of a \( \gamma_4 \)-cycle above if \( q-p = s_2 - s_1 = r_2 \). Similarly, the edge \( pq \) with \( q-p = r_3 \) and the edge \( pq \) with \( q-p = r_4 \) appear in the above \( \gamma_4 \)-cycles.

If \( \phi^*(\sigma_\rho) \) has a diagonal edge of the form \( pq \) or \( p\overline{q} \) and we generate an \( n \)-class with it, the same diagonal edge would appear twice in the \( n \)-class. For example, if \( n = 10 \) and \( r_2 = 5 \), then the diagonal edge \( T \overline{U} \) appears twice, once in the form \( T \overline{U} \) and once in the form \( \overline{U} T \). Thus, to avoid such double appearances of the same 4-cycle, we need to generate a class \( \{ \tau_j^*(\phi^*(\sigma_\rho)) \mid j = 0, 1, \ldots, \frac{n-1}{2} \} \) with only \( \frac{n}{2} \) \( \gamma_4 \)-cycles. Such a class is called an \( \frac{2}{2} \)-class. When we need an \( \frac{2}{2} \)-class, we will find one which is invariant under \( \tau \).

The above procedure is the basic method we adopt to produce a circulant \( \gamma_4 \)-decomposition of \( K_{n(2)} \). We note that each of the above classes is invariant under \( \tau \). As one may noticed already, the main problem then is how to choose \( f \)-sequences and flags so that, in the \( \gamma_4 \)-cycles produced by them, each edge of distance \( d \) appears exactly once for every possible distance \( d \) of the graph. We do not have a method to produce all such \( f \)-sequences and flags. However, we can present at least one such a pair for every \( n \geq 5 \).
Since the number of edges of $K_{n(2)}$ is $2n(n-1)$, we need to produce $\frac{n(n-1)}{2}$ disjoint $\gamma_4$-cycles. We will produce $\frac{2n-1}{2}$ $n$-classes if $n$ is odd, and one $\frac{n}{2}$-class and $\frac{n}{2}$ $n$-classes if $n$ is even.

Let $\Phi_1^* = (\phi^+, \phi^-, \phi^+, \phi^-)$ and $\Phi_2^* = (\phi^+, \phi^+, \phi^+, \phi^+)$ be two special flags. Suppose $i, j$ are differences in $\mathcal{D}_n$ such that $i \neq j$ and $i + j \not\equiv 0 \pmod{n}$, and put $\eta = (i, j, -i, -j)$. Then $\eta$ is an $f$-sequence and $\sigma_\eta = (0, i, i + j, j)$. We generate the following two $n$-classes from $\Phi_1^*(\sigma_\eta)$ and $\Phi_2^*(\sigma_\eta)$, respectively.

\[
\begin{align*}
\langle 0, i, i+1, j \rangle, & \quad \langle 0, i, i+j, j \rangle, \\
\langle 1, i+1, i+j+1, j+1 \rangle, & \quad \langle 1, i+1, i+j+1, j+1 \rangle, \\
\langle 2, i+2, i+j+2, j+2 \rangle, & \quad \langle 2, i+2, i+j+2, j+2 \rangle, \\
\vdots & \quad \vdots \\
\langle n-2, i-2, i+j-2, j-2 \rangle, & \quad \langle n-2, i-2, i+j-2, j-2 \rangle, \\
\langle n-1, i-1, i+j-1, j-1 \rangle. & \quad \langle n-1, i-1, i+j-1, j-1 \rangle.
\end{align*}
\]

Then, each edge of distance $i$ or $j$ appears exactly once in the cycles above. We call these classes the standard $n$-classes produced from $\eta$.

We assume $n \geq 5$, and divide the decomposition problem of $K_{n(2)}$ into four cases depending on $n$ modulo 4.

**Case (1).** Suppose $n = 4k + 1$ with $k \geq 1$. Then $\frac{n-1}{2} = 2k$ and $\mathcal{D}_n = \{\pm 1, \pm 2, \ldots, \pm 2k\}$. Partition $\mathcal{D}_n$ into $k$ subsets $\{\pm 1, \pm 2\}, \{\pm 3, \pm 4\}, \ldots$, and $\{\pm (2k-1), \pm 2k\}$, and take $f$-sequences $\eta_1 = (1, 2, -1, -2), \eta_2 = (3, 4, -3, -4), \ldots$, and $\eta_k = (2k-1, 2k, -(2k-1), -2k)$ for the respective subsets of the partition. Now, we produce two standard $n$-classes from each $\eta$, and let $C$ be the union of all these standard $n$-classes. Clearly, two distinct $f$-sequences generate disjoint classes, and so every edge of every distance appears exactly once in the $\gamma_4$-cycles of $C$. Thus, $C$ is a circulant $\gamma_4$-decomposition of $K_{n(2)}$, and is invariant under $\tau_\sigma$ as is each $n$-class.

Note that if $n = 8k+1$ then Lemma 2.3 applies. However, The decomposition obtained here is different from the one obtained by the method in the preceding section.

**Example 3.1** Let $n = 4k+1 = 9$. Then $k = 2$ and $D_9 = \{\pm 1, \pm 2, \pm 3, \pm 4\}$. Following the procedure in Case (1), we take $\eta_1 = (1, 2, -1, -2)$ and $\eta_2 = (3, 4, -3, -4)$. Then $\sigma_{\eta_1} = (0, 1, 3, 2)$ and $\sigma_{\eta_2} = (0, 3, 7, 4)$. The circulant $\gamma_4$-decomposition of $K_{9(2)}$ produced by the above method is as below:

\[
\begin{align*}
\langle 0, 1, 3, 2, 3 \rangle, & \quad \langle 0, 3, 7, 3 \rangle, & \quad \langle 3, 7, 4 \rangle, \\
\langle 1, 2, 4, 3 \rangle, & \quad \langle 1, 3, 5, 3 \rangle, & \quad \langle 3, 4, 8, 5 \rangle, \\
\langle 2, 3, 5, 3 \rangle, & \quad \langle 2, 5, 6, 3 \rangle, & \quad \langle 5, 6, 0, 6 \rangle, \\
\langle 3, 4, 5, 5 \rangle, & \quad \langle 3, 6, 1, 7 \rangle, & \quad \langle 6, 1, 7 \rangle, \\
\langle 4, 5, 7, 7 \rangle, & \quad \langle 4, 7, 2, 5 \rangle, & \quad \langle 7, 2, 8 \rangle, \\
\langle 5, 6, 8, 7 \rangle, & \quad \langle 5, 8, 3, 0 \rangle. & \quad \langle 8, 3, 0 \rangle.
\end{align*}
\]

\[
\begin{align*}
\langle 6, 7, 0, 8 \rangle, & \quad \langle 6, 0, 4, 1 \rangle, \\
\langle 7, 8, 1, 9 \rangle, & \quad \langle 7, 1, 5, 2 \rangle, \\
\langle 8, 2, 2, 1 \rangle, & \quad \langle 8, 2, 6, 3 \rangle.
\end{align*}
\]
Case (2). Suppose \( n = 4k + 3 \) with \( k \geq 1 \). Then we have \( \frac{n-1}{2} = 2k + 1 \) and \( D_n = \{ \pm 1, \pm 2, \ldots, \pm (2k+1) \} \). With differences \( \pm 1, \pm 2, \) and \( \pm 3, \) take \( f \)-sequences \( \rho_1 = (1,1,1,-3), \rho_2 = (1,3,-2,-2), \rho_3 = (3,2,-3,-2), \) and we obtain the corresponding \( s \)-sequences \( \sigma_{\rho_1} = (0,1,2,3), \sigma_{\rho_2} = (0,1,4,2), \sigma_{\rho_3} = (0,3,5,2) \). Take tags \( \phi_1^s = (\phi^-, \phi^+, \phi^-, \phi^-), \phi_2^s = (\phi^-, \phi^-, \phi^+, \phi^-), \) and \( \phi_3^s = (\phi^+, \phi^+, \phi^-). \) Then, the \( n \)-classes generated from the pairs \( \langle \phi_1^s(\sigma_{\rho_i}), \phi_2^s(\sigma_{\rho_i}) \rangle, \) \( i = 1, 2, 3, \) are as below.

\[
\begin{align*}
&\langle 0, 1, 2, 3 \rangle, \quad \langle 0, 1, 4, 2 \rangle, \quad \langle 0, 3, 5, 2 \rangle, \\
&\langle 1, 2, 3, 4 \rangle, \quad \langle 1, 2, 5, 3 \rangle, \quad \langle 1, 4, 6, 3 \rangle, \\
&\langle 2, 5, 4 \rangle, \quad \langle 2, 5, 7, 4 \rangle, \quad \langle n-2, n-1, 0 \rangle, \\
&\langle n-2, n-1, 0 \rangle, \quad \langle n-2, 1, 3, 0 \rangle, \\
&\langle n-3, 1, 2, 4, 1 \rangle.
\end{align*}
\]

Then, as explained earlier, it can be check that each edge of distance \( d \) appears exactly once in the \( \gamma_d \)-cycles above for \( d = 1, 2, 3 \).

If \( k \geq 2, D_n \setminus \{ \pm 1, \pm 2, \pm 3 \} = \{ \pm 4, \pm 5, \ldots, \pm (2k+1) \} \) and this set is nonempty. Partition this set into \( k-1 \) subsets \( \{ \pm 4, \pm 5 \}, \{ \pm 6, \pm 7 \}, \ldots, \{ \pm 2k, \pm (2k+1) \} \). With each subset \( \{ 2i, 2i+1 \} \) for \( i = 1, 2, \ldots, k, \) take \( f \)-sequence \( \eta_i = (2i, 2i+1, -2i, -(2i+1)) \). As in Case (1), if we generate the two standard \( n \)-classes from each of these \( \eta_i \), then every edge of distance \( d \) appears exactly once in the \( 4 \)-cycles of these \( 2(k-1) \) standard \( n \)-classes, for \( d = 4, 5, \ldots, 2k+1 \). If \( k = 1, \) we do not have these cycles.

Let \( C \) be the union of these \( 2(k-1) \) standard \( n \)-classes and the three classes above. Then, \( C \) is a circulant \( \gamma_d \)-decomposition of \( K_n \), invariant under \( \tau \).

Case (3). Suppose \( n = 4k + 2 \) with \( k \geq 1 \). Then \( \frac{n}{2} = 2k + 1 \) and \( D_k = \{ \pm 1, \pm 2, \ldots, \pm 2k, 2k+1 \} \). With differences \( \pm 1, \pm (k+1) \) and \( 2k+1, \) take the \( f \)-sequences \( \rho_1 = (1,2k+1,-1,2k+1), \rho_2 = (1,k+1,-1,-(k+1)), \rho_3 = (2k+1,k+1,1,-k+1), \) and obtain \( s \)-sequences \( \sigma_{\rho_1} = (0,1,2k+2,2k+1), \sigma_{\rho_2} = (0,1,k+2,2k+1), \sigma_{\rho_3} = (0,2k+1,3k+2,3k+1) \). Take tags \( \phi_1^s = (\phi^+, \phi^-, \phi^-, \phi^+), \phi_2^s = (\phi^+, \phi^+, \phi^-, \phi^-), \) and \( \phi_3^s = (\phi^-, \phi^+, \phi^+, \phi^-) \). From \( \phi_1^s(\sigma_{\rho_i}) \), we generate the \( \frac{n}{2} \)-class \( C_1 = \{ \tau^j(\phi_1^s(\sigma_{\rho_i})) \mid j = 0, 1, \ldots, \frac{n}{2} - 1 \} \). From \( \phi_2^s(\sigma_{\rho_2}) \) and \( \phi_3^s(\sigma_{\rho_3}) \), we generate \( n \)-classes \( C_2 \) and \( C_3 \), respectively. We recall that \( \frac{n}{2} - 1 = 2k \) and \( n-1 = 4k+1 \). These classes are listed in three columns as below:

\[
\begin{align*}
&\langle 0, 1, 2k+2, 2k+1 \rangle, \quad \langle 0, 1, 2k+2, 2k+1 \rangle, \\
&\langle 1, 2k+3, 2k+2 \rangle, \quad \langle 1, 2k+3, 2k+2 \rangle, \\
&\langle 2k+1, 2k+1 \rangle, \quad \langle 4k+1, k \rangle, \quad \langle 4k+1, k \rangle, \\
&\langle 4k+1, 4k+1, 3k+1, 3k+2 \rangle, \quad \langle 4k+1, 4k+2, k+1, k \rangle, \\
&\langle 4k+1, 4k+2, k+1, k \rangle.
\end{align*}
\]
We have the following observations for diagonal edges. Put $p-q = \frac{q}{2} = 2k+1$.

(i) The diagonal edge $pq$ appears as the fourth edge of a cycle in $C_1$, once for each $p = 0, 1, \ldots, 2k$.

(ii) The diagonal edge $p\overline{q}$ appears as the second edge of a cycle in $C_1$, once for each $p = 0, 1, \ldots, 2k$.

(iii) The diagonal edge $p\overline{q}$ appears as the first edge of a cycle in $C_3$, once for each $p = 0, 1, \ldots, 4k, 4k+1$. Note that the edge $p\overline{q}$ can be written as $\overline{p'}q'$, where $p' = p+2k+1$ and $q' = q+2k+1$.

If $q-p = 1$, then the edge $p\overline{q}$ appears exactly once in a cycle in $C_1$, in the form $\overline{p}\overline{q}$ as the first edge of a cycle for $p = 0, 1, \ldots, 2k$, and in the form $p\overline{q}$ as the third edge of a cycle for $p = 2k+1, \ldots, 4k+1$. In a similar way as before, it can be checked that each edge of distance 1 or distance $k+1$ appears exactly once in the cycles of $C_2$ or $C_3$. Furthermore, all these classes including the $\frac{2}{2}$-class are invariant under $\tau_\ast$. In fact, we see that $\tau_\ast$ maps the last cycle of $C_1$ to the first cycle of $C_1$, except that the order of the vertices is reversed.

If $k \geq 2$ then $D_n \setminus \{\pm 1, \pm (k+1), \pm (2k+1)\}$ and this set is nonempty. In this case, we partition this set into $k-1$ subsets $\{\pm \alpha_i, \pm \beta_i\}$ so that $\alpha_i + \beta_i \neq 0$ (mod $n$) for $i = 1, 2, \ldots, k-1$. This is always possible since we can take $\alpha_i$ and $\beta_i$ in such a way that $2 \leq \alpha_i \leq 2k$, and $2 \leq \beta_i \leq 2k$. Now, put $\eta_i = (\alpha_i, \beta_i, -\alpha_i, -\beta_i)$ for $i = 1, 2, \ldots, k-1$, and generate $2(k-1)$ standard $n$-classes from these $\eta_i$ as in Case (1). Then each edge of distance $d$ appears exactly once in these 4-cycles if $d$ is in the set $\{\alpha_i, \beta_i \mid 1 \leq i \leq k-1\}$, that is, if $d \neq 1, k+1, 2k+1$. We already know that these classes are invariant under $\tau_\ast$.

Thus, if we let $C$ be the union of these $2(k-1)$ standard $n$-classes and the classes $C_1, C_2$ and $C_3$ above, then $C$ is a circulant $\gamma_4$-decomposition of $K_{n(2)}$, invariant under $\tau_\ast$.

**Example 3.2.** Let $n = 4k+2 = 10$. Then $k = 2$ and $D_{10} = \{\pm 1, \pm 2, \pm 3, \pm 4, 5\}$.

According to Case (3), we take $\rho_1 = (1, 5, -1, 5), \rho_2 = (1, 3, -1, -3), \rho_3 = (5, 3, -1, 3)$ and $\eta_1 = (2, 4, -2, 4)$. Then $\sigma_{\rho_1} = (0, 1, 6, 5), \sigma_{\rho_2} = (0, 1, 4, 3), \sigma_{\rho_3} = (0, 5, 8, 7)$, and $\sigma_{\eta_1} = (0, 2, 6, 4)$. The classes generated by the method in Case (3) are as follows:

\[
\begin{align*}
&\{0, \overline{7}, 5, 5\}, \quad \{0, \overline{7}, 6, 7\}, \quad \{0, \overline{7}, 6, 8\}, \quad \{0, \overline{7}, 7, 5\}, \quad \{0, \overline{7}, 7, 6\}, \\
&(0, \overline{7}, 8, 7), \quad (0, \overline{7}, 9, 2, 1), \quad (0, \overline{7}, 9, 2, 3), \quad (0, \overline{7}, 9, 3, 1), \quad (0, \overline{7}, 9, 3, 2), \\
&(0, \overline{7}, 9, 4, 2), \quad (0, \overline{7}, 9, 5, 3). \\
\end{align*}
\]
Case (4). Suppose \( n = 4k \) with \( k \geq 2 \). Then \( \frac{n}{2} = 2k \) and \( \mathcal{D}_n = \{ \pm 1, \pm 2, \ldots, \pm (2k - 1), 2k \} \). With differences \( \pm 1, \pm 3, \pm (2k - 2) \) and \( \pm 2k \), take the \( f \)-sequences \( \rho_1 = (2k - 2, 2k, -(2k - 2), 2k) \), \( \rho_2 = (1, 3, -1, -3) \), \( \rho_3 = (3, 2k - 2, 2k-2, 1) \) and \( \rho_4 = (2k, -3, -(2k-2), 1) \), and obtain \( s \)-sequences \( \sigma_{\rho_1} = (0, 2k-2, 4k-2, 2k) \), \( \sigma_{\rho_2} = (0, 1, 4, 3) \), \( \sigma_{\rho_3} = (0, 3, 2k+1, 4k-1) \) and \( \sigma_{\rho_4} = (0, 2k, 2k-3, 4k-1) \). Then, we take flags \( \phi_1 = (\phi^+, \phi^-, \phi^-, \phi^+) \), \( \phi_2 = (\phi^-, \phi^-, \phi^+, \phi^-) \), \( \phi_3 = (\phi^+, \phi^-, \phi^-, \phi^+) \), and \( \phi_4 = (\phi^-, \phi^+, \phi^+, \phi^-) \). From \( \phi_i(\sigma_{\rho_i}) \), we generate the \( \frac{n}{2} \)-class \( \{ \tau_j^i(\phi_i(\sigma_{\rho_i})) \mid j = 0, 1, \ldots, \frac{n}{2} - 1 \} \), and generate an \( n \)-class from \( \phi_i(\sigma_{\rho_i}) \) for each \( i = 2, 3, 4 \). As in Case (3), it can be checked that, in the \( \gamma_4 \)-cycles of these classes, every edge of distance \( d \) appears exactly once for \( d = 1, 3, 2k-2, 2k \), and that these classes are invariant under \( \tau_* \).

If \( k \geq 3 \), the set \( \mathcal{D}_n \setminus \{ \pm 1, \pm 3, \pm (2k - 2), 2k \} \) is nonempty. We partition this set into \( k-2 \) subsets \( \{ \pm \alpha_i, \pm \beta_i \} \) such that \( \alpha_i \neq \beta_i \) (mod \( n \)) for \( i = 1, 2, \ldots, k-2 \). As in Case (1), generate \( 2(k-2) \) standard \( n \)-classes from \( \rho_i = (\alpha_i, \beta_i, -\alpha_i, -\beta_i) \) for \( i = 1, 2, \ldots, k-2 \). Then, each edge of distance \( d \) appears exactly once in the \( 4 \)-cycles of these classes if \( d \neq 1, 3, 2k-2, 2k \). These classes are already invariant under \( \tau_* \).

If we let \( C \) be the union of these \( 2(k-2) \) standard \( n \)-classes and the \( 4 \) classes above, \( C \) is a circulant \( \gamma_4 \)-decomposition of \( K_{n(2)} \), invariant under \( \tau_* \).

**Example 3.3.** Let \( n = 4k = 8 \). Then \( k = 2 \) and \( \mathcal{D}_8 = \{ \pm 1, \pm 2, \pm 3, 4 \} \). By Case (4), we have \( \rho_1 = (2, 4, -2, 4) \), \( \rho_2 = (1, 3, -1, -3) \), \( \rho_3 = (3, 2, 2, 1) \) and \( \rho_4 = (4, -3, -2, 1) \), and \( \sigma_{\rho_1} = (0, 2, 6, 4) \), \( \sigma_{\rho_2} = (0, 1, 4, 3) \), \( \sigma_{\rho_3} = (0, 3, 5, 7) \), and \( \sigma_{\rho_4} = (0, 4, 1, 7) \). The classes generated by the method in Case (4) are as follows:

\[
\begin{align*}
(0, & 2, 5, 4), \\
(1, & 3, 7, 5), \\
(2, & 4, 5, 6), \\
(3, & 5, 7, 7), \\
(5, & 6, 1, 5), \\
(6, & 7, 3, 7).
\end{align*}
\]

Summarizing the discussion in this section and combining with Lemma 3.1, we have the following lemma.

**Lemma 3.2.** For every integer \( n \geq 4 \), \( K_{n(2)} \) has a \( \gamma_4 \)-decomposition, and the decomposition can be circulant if \( n \geq 5 \), in the sense that it is invariant under \( \tau_* \).

Let \( n \geq 5 \) and \( t \) be any positive integer. For each \( i = 1, 2, \ldots, t \), blow up the partite set \( A_i = \{ i, \bar{i} \} \) of \( K_n(2) \) to \( A'_i = \{ i_1, i_2, \ldots, i_t, \bar{i}_1, \bar{i}_2, \ldots, \bar{i}_t \} \), and use these sets as the partite sets of \( K_{n(2t)} \). A gregarious 4-cycle of \( K_{n(2t)} \) is then of the form \( \langle \phi_0(a)_j, \phi_1(b)_k, \phi_2(c)_j, \phi_3(d)_k \rangle \) for some gregarious 4-cycle \( \langle \phi_0(a), \phi_1(b), \phi_2(c), \phi_3(d) \rangle \) of \( K_{n(2)} \) and \( j, k \) in \( \{ 1, 2, \ldots, t \} \). We extend the
For every integer \( n \geq 4 \) and \( t \geq 1 \), \( K_{n(t)} \) has a \( \gamma_4 \)-decomposition, and the decomposition can be circulant if \( n \geq 5 \), in the sense that it is invariant under \( \tau_* \).

The notations preceding Theorem 3.1 will be needed in the next section.

4. Multipartite graphs with one partite set of different size

Let \( K_{h,n(m)} \) denote \( K(h,m,m,\ldots,m) \) with \( m \) repeated \( n \) times. We assume \( h \) is even, and then \( m \) is also even by Lemma 1.1. So let \( m = 2t \). In the light of Theorem 3.1, we may assume that \( h \neq m \), and then we may assume \( n \geq 4 \) due to Lemma 1.3.

Let the partite sets of \( K_{h,n(2)} \) be \( A_i = \{ i_1, i_2, \ldots, i_t, \bar{i}_1, \bar{i}_2, \ldots, \bar{i}_t \} \) for \( i = 0, 1, \ldots, n - 1 \) and \( B = \{ a_1, b_1, a_2, b_2, \ldots, a_{h/2}, b_{h/2} \} \). An edge of \( K_{h,n(2)} \) is called an \( \alpha \)-edge if it joins two vertices in \( \bigcup_{i=0}^{n-1} A_i \), and a \( \beta \)-edge if it involves a vertex in \( B \). A gregarious 4-cycle of \( K_{h,n(2)} \) is said to be of type I if it involves \( \alpha \)-edges only, and of type II if it involves \( \beta \)-edges.
The graph $K_{h,n(2t)}$ has $\frac{n(n-1)(2t)^2}{2} = 2n(n-1)t^2$ $\alpha$-edges and $2hn+t$ $\beta$-edges. Note that a 4-cycle of type II consists of two $\alpha$-edges and two $\beta$-edges. Thus, for $K_{h,n(2t)}$ to have a $\gamma_4$-decomposition, we need to have $2hn+t \leq 2n(n-1)t^2$, and so $h \leq (n-1)t$. Regard $K_{n(2t)}$ as a subgraph of $K_{h,n(2t)}$. Thus, a 4-cycle in $K_{n(2t)}$ is a 4-cycle of type II in $K_{h,n(2t)}$.

Suppose $n \geq 5$ and let $h$ is odd such that $h \leq (n-1)t$. Assume $n$ is odd. Let $\mathcal{C}^*$ be the $\gamma_4$-decomposition of $K_{n(2t)}$ obtained from $\mathcal{C}$ as in the preceding section. Using notations there, we have

$$\mathcal{C}^* = \bigcup \{C^{(i)}_{jk} \mid 1 \leq i \leq \frac{n-1}{2}, 1 \leq j \leq t, 1 \leq k \leq t\}.$$ 

Let $E = \{(i, k) \mid 1 \leq i \leq \frac{n-1}{2}, 1 \leq k \leq t\}$, and let $F$ be a subset of $E$ with $\frac{b}{2}$ elements. This is possible since $\frac{b}{2} \leq \frac{(n-1)t}{2} = |E|$. Since $|B| = h$, we can partition $B$ into $\frac{b}{2}$ subsets $\{a_{(i,k)}, b_{(i,k)}\}$ for $(i,k) \in F$, using elements of $F$ as indices for the vertices in $B$. For each $(i,k) \in F$ and $j = 1, 2, \ldots, t$, assume

$$C^{(i)}_{jk} = \{(\phi_0(s_0+r)_j, \phi_1(s_1+r)_k, \phi_2(s_2+r)_j, \phi_3(s_3+r)_k) \mid 0 \leq r \leq n-1\},$$

and split each 4-cycle in $C^{(i)}_{jk}$ into two paths of length 2 and join the end vertices of the paths to $a_{(i,k)}$ and $b_{(i,k)}$, to produce two 4-cycles

$$\langle a_{(i,k)}, \phi_0(s_0+r)_j, \phi_1(s_1+r)_k, \phi_2(s_2+r)_j \rangle, \quad \langle b_{(i,k)}, \phi_2(s_2+r)_j, \phi_3(s_3+r)_k, \phi_0(s_0+r)_j \rangle$$

type II. The following figure depicts this process.

Let $C^{(i)*}_{jk}$ be the set of these 4-cycles of type II obtained from 4-cycles in $C^{(i)}_{jk}$ and put $\mathcal{F} = \bigcup \{C^{(i)*}_{jk} \mid (i,k) \in F, 1 \leq j \leq t\}$. By the edges of cycles in $\mathcal{F}$, the vertex $a_{(i,k)}$ is joined to each vertex of the set

$$N_{a_{(i,k)}} = \{\phi_0(r)_j, \phi_2(r)_j \mid 0 \leq r \leq n-1, 1 \leq j \leq t\}.$$ 

Note that, at the constructions of $\gamma_4$-cycles in Section 3, we always have $(\phi_0, \phi_2) = (\phi^+, \phi^-)$ or $(\phi_0, \phi_2) = (\phi^-, \phi^+)$. Thus, $N_{a_{(i,k)}} = \{r_j, t_j \mid r = 0, 1, \ldots, n-1, j = 1, 2, \ldots, t\} = \bigcup_{r=0}^{n-1} A_r$. That is, every $\alpha$-edge involving $a_{(i,k)}$ is used by 4-cycles in $\mathcal{F}$. The same is true for the $\beta$-edges involving $b_{(i,k)}$. Note that, we keep the 4-cycles in $C^{(i)*}_{jk}$ untouched if $(i,k) \notin F$. Thus, the class

$$\mathcal{C}^{**} = \left( \bigcup \{C^{(i)*}_{jk} \mid (i,k) \in E \setminus F, 1 \leq j \leq t\} \right) \cup \mathcal{F}.$$
is a $\gamma_4$-decomposition of $K_{h,n(2t)}$.

If $n$ is even, it gets more complicated. In this case, we have

$$C^* = \bigcup \{ C_{jk}^{(i)} \mid 1 \leq i \leq \frac{n}{2}, 1 \leq j \leq t, 1 \leq k \leq t \},$$

where $C_{jk}^{(i)}$ are $n$-classes and $C_{jk}^{(i)}$ are $\frac{n}{2}$-classes for $i = 2, 3, \ldots, \frac{n}{2}$. Put $E = \{ (i, k) \mid 2 \leq i \leq \frac{n}{2}, 1 \leq k \leq t \}$. Recall that $\frac{h}{2} \leq \frac{(n-1)t}{2}$. Assume $\frac{h}{2} \leq \frac{(n-1)t}{2}$.

Let $F$ be a subset of $E$ with $\frac{n}{2}$ elements, and repeat the above procedure except that we put

$$C^{**} = \left( \bigcup \{ C_{jk}^{(i)} \mid (i, k) \in E \setminus F, 1 \leq j \leq t \} \right) \cup \left( \bigcup \{ C_{jk}^{(i)} \mid 1 \leq j \leq t, 1 \leq k \leq t \} \right) \cup F.$$

Then, $C^{**}$ is a $\gamma_4$-decomposition of $K_{h,n(2t)}$.

Summarizing the discussion above, we have the following theorem.

**Theorem 4.1.** Suppose $n \geq 5$ and $h$ is even with $h \leq (n-1)t$. If $n$ is odd or $n$ is even with $h \leq (n-2)t$, then $K_{h,n(2t)}$ is $\gamma_4$-decomposable.

**Corollary 4.2.** Suppose $n \geq 5$ and $h$ is even with $h \leq n-1$. Then, $K_{h,n(2t)}$ is $\gamma_4$-decomposable for every $t \geq 1$.

**Proof.** Since $h$ is even, $h \leq n-1$ implies $h \leq n-2$ if $n$ is even. By the above theorem, $K_{h,n(2t)}$ is $\gamma_4$-decomposable, and then, by Lemma 1.2, $K_{h,n(2t)}$ is also $\gamma_4$-decomposable for every integer $t \geq 1$. □

**Example 4.1.** Consider $K_{4,8(2)}$ with partite sets $B = \{ a_1, b_1, a_2, b_2 \}$ and $A_i = \{ i, 7 \}$ for $i = 0, 1, \ldots, 7$. We have a $\gamma_4$-decomposition of $K_{4,8(2)}$ in Example 3.3. Take the last two $n$-classes of them and proceed as explained above. Then, we obtain four $n$-classes of 4-cycles of type II in $K_{4,8(2)}$, listed at the last four columns in the following table. These four $n$-classes of $K_{4,8(2)}$ together with the $\frac{n}{2}$-class and the unused $n$-class above yields a $\gamma_4$-decomposition of $K_{4,8(2)}$ as below.

<table>
<thead>
<tr>
<th>${0, 7, 6, 4}$</th>
<th>${1, 7, 6, 4}$</th>
<th>${1, 6, 3, 5}$</th>
<th>${1, 2, 5, 7}$</th>
<th>${0, 5, 7, 0}$</th>
<th>${a_2, 5, 4, 1}$</th>
<th>${b_1, 1, 7, 0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${0, 7, 6, 4}$</td>
<td>${1, 7, 6, 4}$</td>
<td>${1, 6, 3, 5}$</td>
<td>${1, 2, 5, 7}$</td>
<td>${0, 5, 7, 0}$</td>
<td>${a_2, 5, 4, 1}$</td>
<td>${b_1, 1, 7, 0}$</td>
</tr>
<tr>
<td>${3, 5, 7, 7}$</td>
<td>${3, 5, 7, 7}$</td>
<td>${3, 5, 7, 7}$</td>
<td>${3, 5, 7, 7}$</td>
<td>${3, 5, 7, 7}$</td>
<td>${3, 5, 7, 7}$</td>
<td>${3, 5, 7, 7}$</td>
</tr>
</tbody>
</table>

We remark that no $\gamma_4$-decompositions of graphs in this section are circulant since the graphs are not regular, that is, the graphs have vertices of distinct degrees.
5. Join of $\gamma_4$-decomposable graphs

Throughout this section, let $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_n$ be pairwise disjoint sets, and put $h_i = |A_i|$ for $i = 1, 2, \ldots, m$ and $k_j = |B_j|$ for $j = 1, 2, \ldots, n$.

Lemma 5.1. If $K(A_1, A_2, \ldots, A_n)$ is $\gamma_4$-decomposable, then $h_p \leq \sum_{i \neq p} h_i$ for each $p = 1, 2, \ldots, n$.

Proof. There are $h_p : (\sum_{i \neq p} h_i)$ edges which join a vertex in $A_p$ to a vertex in $\bigcup_{i \neq p} A_i$. We say that these edges are of type $I_p$. There are $\sum_{1 \leq i < j \leq m, i, j \neq p} h_i h_j$ edges which join vertices in $\bigcup_{i \neq p} A_i$ among themselves. We say that these edges are of type $I_p$. Any gregarious 4-cycle having a vertex in $A_p$ involves two edges of type $I_p$ and two edges of type $I_p$. Thus, to have a $\gamma_4$-decomposition, we need more edges of type $I_p$ than edges of type $I_p$. Therefore, we have

$$h_p(\sum_{i \neq p} h_i) \leq \sum_{1 \leq i < j \leq m, i, j \neq p} h_i h_j \leq (\sum_{i \neq p} h_i)^2,$$

which yields the desired inequality. \hfill \Box

The conclusion of the above lemma is not true if we allow arbitrary 4-cycles in the decomposition. For example, $K(6, 2, 2)$ and $K(10, 2, 2, 2, 2)$ are decomposable into (not necessarily gregarious) 4-cycles, but they do not satisfy the conclusion in the above lemma.

The following lemma is a simple restatement of a well-known result on perfect matchings in multipartite graphs.

Lemma 5.2. Let $m \geq 2$ and $\sum_{i=1}^{m} h_i$ be even. Then, $\bigcup_{i=1}^{m} A_i$ can be partitioned into subsets consisting of of two vertices from distinct $A_i$ and $A_j$ if and only if $h_p \leq \sum_{i \neq p} h_i$ for each $p = 1, 2, \ldots, m$.

For graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the join $G_1 + G_2$ of the two graphs is the graph $G = (V, E)$ where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup \{v_1 v_2 \mid v_1 \in V_1, v_2 \in V_2\}$. With this notation, we have that $K(h_1, h_2, \ldots, h_m) = K(h_{i_1}, \ldots, h_{i_{n+1}}) + K(h_{i_{n+2}}, \ldots, h_{i_{2n}})$ for any partition $\{i_1, \ldots, i_n\}$ of $\{1, 2, \ldots, m\}$. The union $G_1 \cup G_2$ of graphs $G_1$ and $G_2$ is the graph $G = (V, E)$ where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. Thus, we have that $G_1 + G_2 = G_1 \cup G_2 \cup K(V_1, V_2)$.

Lemma 5.3. Let $\sum_{i=1}^{m} h_i$ and $\sum_{j=1}^{n} k_j$ be even with $m \geq 2$ and $n \geq 2$. If

$$h_p \leq \sum_{i \neq p} h_i \quad (p = 1, 2, \ldots, m) \quad \text{and} \quad k_q \leq \sum_{j \neq q} k_j \quad (q = 1, 2, \ldots, n),$$

then the graph $K(\bigcup_{i=1}^{m} A_i, \bigcup_{j=1}^{n} B_j)$ has a decomposition into 4-cycles such that no two vertices of a 4-cycle belong to the same subset $A_i$ or $B_j$. 
Proof. By the above lemma, $\bigcup_{i=1}^{m} A_i$ has a partition $\Phi$ into subsets consisting of two elements from distinct $A_i$ and $A_j$, and $\bigcup_{i=1}^{n} B_i$ has a partition $\Psi$ into subsets consisting of two elements from distinct $B_i$ and $B_j$. Note that 

$K(\bigcup_{i=1}^{m} A_i, \bigcup_{i=1}^{n} B_i) = \bigcup \left\{ K(\{a_1, a_2\}, \{b_1, b_2\}) \mid \{a_1, a_2\} \in \Phi, \{b_1, b_2\} \in \Psi \right\}.$

However, since each $K(\{a_1, a_2\}, \{b_1, b_2\})$ is simply the 4-cycle $\langle a_1, b_1, a_2, b_2 \rangle$, these 4-cycles constitute a desired decomposition. □

Theorem 5.4. Let $\sum_{i=1}^{m} h_i$ and $\sum_{j=1}^{n} k_j$ be even. If $K(A_1, \ldots, A_m)$ and $K(B_1, \ldots, B_n)$ are $\gamma_4$-decomposable, then so is $K(A_1, \ldots, A_m, B_1, \ldots, B_n)$.

Proof. Because $K(A_1, \ldots, A_m, B_1, \ldots, B_n) = K(A_1, \ldots, A_m) \cup K(B_1, \ldots, B_n) \cup K(\bigcup_{i=1}^{m} A_i, \bigcup_{i=1}^{n} B_i)$, we only need to show that $K(\bigcup_{i=1}^{m} A_i, \bigcup_{i=1}^{n} B_i)$ is $\gamma_4$-decomposable. This follows immediately from Lemmas 5.1 and 5.3. Thus, the theorem is proved. □

With the above theorem, we can build up various $\gamma_4$-decomposable graphs from known $\gamma_4$-decomposable graphs.

Note. Only after producing the main results of this paper, the authors became aware of the fact that Billington and Hoffman [3] have also considered gregarious 4-cycle decompositions (see [3]) and much of our results overlaps theirs. However, the approach and the decompositions are different.

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