VISCOSITY METHODS OF APPROXIMATION FOR A COMMON SOLUTION OF A FINITE FAMILY OF ACCRETIVE OPERATORS

JUN-MIN CHEN, LI-JUAN ZHANG, AND TIE-GANG FAN

Abstract. In this paper, we try to extend the viscosity approximation technique to find a particular common zero of a finite family of accretive mappings in a Banach space which is strictly convex reflexive and has a weakly sequentially continuous duality mapping. The explicit viscosity approximation scheme is proposed and its strong convergence to a solution of a variational inequality is proved.

1. Introduction

Let $E$ be a Banach space with a dual space of $E^*$, $C$ a nonempty closed convex subset of $E$, and $T : C \to C$ a mapping. Recall that $T$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A point $x \in C$ is a fixed point of $T$ proved $Tx = x$. Denote by $\text{Fix}(T)$ the set of fixed points of $T$. $f : C \to C$ is a contraction on $C$ if there exists a constant $\beta \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \beta \|x - y\|$, $\forall x, y \in C$. The normalized duality mapping $J$ from $E$ to $2^E^*$ is given by

$$J(x) = \{g \in E^*: \langle x, g \rangle = \|x\|^2 = \|g\|^2\}, x \in E$$

where $E^*$ denotes the dual space of $E$ and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

Recall that an operator $A$ with $D(A)$ and $R(A)$ in $E$ is said to be accretive, if for each $x_i \in D(A)$ and $y_i \in A(x_i)$ ($i = 1, 2$), there is a $j \in J(x_2 - x_1)$ such that

$$\langle y_2 - y_1, j \rangle \geq 0.$$

An accretive operator $A$ is $m$-accretive if $R(I + \lambda A) = E$ for all $\lambda > 0$. Denote by $N(A)$ the zero set of $A$: i.e.,

$$N(A) := A^{-1}(0) = \{x \in D(A) : 0 \in Ax\}.$$
If $A$ is accretive, then we can define, for each $r > 0$, a nonexpansive single-valued mapping $J_r: R(I+rA) \to D(A)$ by $J_r := (I+rA)^{-1}$, which is called the resolvent of $A$. We also know that for an accretive operator $A$, $N(A) = \text{Fix}(J_r)$.

Recently, Zegeye and Shahzad [13] have proved the strong convergence theorem for a finite family of accretive operators, let $l \geq 1$ be a positive integer, and define the set $\Lambda = 1, 2, \ldots, l$. We also can see [6], J. S. Jung also has proved the strong convergence of an iterative method for finding common zeros of a finite family of accretive operators.

**Theorem 1.1.** ([13]) Let $E$ be a strictly convex and real reflexive Banach space $E$ which has a uniformly Gâteaux differentiable norm, and $A$ a nonempty closed convex subset of $E$. Let $A_i : i \in \Lambda : A \to E$ be a finite family of $m$-accretive operators with $\cap_{i=1}^\Lambda A_i \neq \emptyset$. Assume that every nonempty closed bounded convex subset of $E$ has the fixed point property for nonexpansive mappings. For any given $a$, $x_0 \in C$, let $\{x_n\}$ be generated by the algorithm

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)S_r x_n, \quad n \geq 0,$$

where $S_r = a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \cdots + a_l J_{A_l}$ with $J_{A_i} = (I + A_i)^{-1}$, for $i = 0, 1, 2, \ldots, l$, $a_i \in (0, 1)$, $\sum_{i=0}^l a_i = 1$, and $\{\alpha_n\}$ a real sequence satisfying the conditions (C1) $\lim_{n \to \infty} \alpha_n = 0$; (C2) $\sum_{n=0}^\infty \alpha_n = +\infty$ and (C3) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < +\infty$ or (C3)* $\lim_{n \to \infty} \frac{|\alpha_{n+1} - \alpha_n|}{\alpha_{n+1}} = 0$. Then the sequence $\{x_n\}$ converges strongly to a common zero of $\{A_i : i \in \Lambda\}$.

And in [5], L. Hu, L. Liu generalized and extended the result of Zegeye and Shahzad [13], they proved the following theorem:

**Theorem 1.2.** ([5]) Let $E$ be a strictly convex and real reflexive Banach space $E$ which has a uniformly Gâteaux differentiable norm, and $C$ a nonempty closed convex subset of $E$. Let $\{A_i : i \in \Lambda\} : C \to E$ be a finite family of accretive operators satisfying the following range conditions:

$$cl(D(A_i)) \subseteq C \subset \bigcap_{r > 0} R(I + rA_i), \quad i = 1, 2, \ldots, l.$$

Assume that $\cap_{i=1}^\Lambda N(A_i) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $(0, 1)$ and $\{r_n\}$ is a sequence in $(0, +\infty)$, satisfying conditions:

(i) (C1)$\lim_{n \to \infty} \alpha_n = 0$; (C2)$\sum_{n=0}^\infty \alpha_n = +\infty$;

(ii) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$;

(iii) $\lim_{n \to \infty} r_n = r, r \in R^+$.

For any $u \in C$, $x_0 \in C$, the sequence $\{x_n\}$ is given by

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S_{r_n} x_n, \quad n \geq 0,$$

where $S_{r_n} = a_0 I + a_1 J_{r_n}^1 + a_2 J_{r_n}^2 + \cdots + a_l J_{r_n}^l$ with $J_{r_n}^i = (I + r_n A_i)^{-1}$, for $i = 0, 1, 2, \ldots, l$, $a_i \in (0, 1)$, $\sum_{i=0}^l a_i = 1$. Then the sequence $\{x_n\}$ converges strongly to a common zero of $\{A_i : i \in \Lambda\}$.
The viscosity iterative has been studied by many researchers (see, [7], [8], [3], [12]). In 2000, Moudafi [7] introduced viscosity approximation method and proved that if $E$ is a real Hilbert space, for given $x_0 \in C$, the sequence $\{x_n\}$ generated by the algorithm

$$x_{n+1} := \alpha_n f(x_n) + (1 - \alpha_n)T x_n, \quad n \geq 0,$$

(3)

where $f : C \to C$ is a contraction mapping with constant $\beta \in (0,1)$ and $\alpha_n \subseteq (0,1)$ satisfies certain conditions, converges strongly to a fixed point of $T$ in $C$ which is the unique solution to the following variational inequality:

$$\langle (I - f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \text{Fix}(T).$$


In 2006, Paul-Emile Maingé [8] considered the general iterative method

$$x_{n+1} = \alpha_n T_n x_n + (1 - \alpha_n) J_{r_n} A x_n,$$

(4)

for calculating a particular zero of $A$, an $m$-accretive operator in a Banach space $E$, $T_n$ being a sequence of nonexpansive self-mappings in $E$. Under suitable conditions on the parameters and $E$, they stated strong and weak convergence results of $\{x_n\}$.

Motivated and inspired by above works, in this paper, we introduce and study the following iterative algorithm in strictly convex reflexive Banach spaces $E$ with a weakly sequentially continuous duality mapping from $E$ to $E^*$ for given $x_0 \in C$, let the sequence $\{x_n\}$ be defined by

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S r_n x_n, \quad \forall n \geq 0,$$

(5)

where $S r_n = a_0 I + a_1 J_{r_n}^1 + a_2 J_{r_n}^2 + \cdots + a_l J_{r_n}^l$ with $J_{r_n}^i = (I + r_n A_i)^{-1}$ for $i = 1, 2, \cdots, l$, $a_i \in (0,1)$, $\sum_{i=0}^l a_i = 1$ and $\{r_n\} \subset (0, +\infty)$. $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$. The present results improve and extend many known results in the literature.

2. Preliminaries

Recall that a gauge function $\phi : R^+ \to R^+$ such that $\phi(0) = 0$ and $\lim_{t \to \infty} \phi(t) = \infty$. The duality mapping $J_\phi : E \to E^*$ associated with a gauge function $\phi$ is defined by $J_\phi(x) = \{u^* \in E^* : \langle x, u^* \rangle = \|x\|\|u^*\|, \|u^*\| = \phi(\|x\|), \forall x \in E\}$. In the particular case $\phi(t) = t$, the duality map $J = J_\phi$ is called the normal duality map. We note that $J_\phi(x) = \frac{\phi(\|x\|)}{\|x\|}J(x)$, for $x \neq 0$. It is known that if $E$ is smooth then $J_\phi$ is single valued and norm-to-weak* continuous (see[2]).

Following Browder [1], we say that a Banach space $E$ has the weak continuous duality mapping if there exists a gauge function $\phi$ for which the duality map $J_\phi$ is single valued and weak to weak* sequentially continuous (i.e., if
is a sequence in $E$ weakly convergent to a point $x$, then the sequence \( \{J_\phi(x_n)\} \) converges weakly* to \( J_\phi(x) \). If Banach space $E$ admits weakly sequentially continuous duality mapping \( J \), then by ([4] Lemma 1), we get that duality mapping \( J \) is single-valued. It is well known \( l^p(1 < p < \infty) \) spaces have a weakly continuous duality mapping \( J_\phi \) with a gauge function \( \phi(t) = t^{p-1} \).

Setting \( \Phi(t) = \int_0^t \phi(x)dx, \quad t \geq 0, \) one can see that \( \Phi(t) \) is a convex function and \( J_\phi = \partial \Phi(\|x\|) \), for \( x \in E \), where \( \partial \) denotes the subdifferential in the sense of convex analysis.

Recall that a Banach space $E$ is said to be smooth if and only if the duality mapping \( J \) is single-valued. A Banach space $E$ is called strictly convex if for $a_i \in (0, 1), i \in \Lambda$, such that $\sum_{i=1}^l a_i = 1$, we have $\|a_1 x_1 + a_2 x_2 + \cdots + a_l x_l\| < 1$ for $x_i \in U, i \in \Lambda$ and $x_i \neq x_j$ for some $i \neq j$. For in a strictly convex Banach space we have that if $\|x_1\| = \|x_2\| = \cdots = \|x_l\| = \|a_1 x_1 + a_2 x_2 + \cdots + a_l x_l\|,$ for $x_i \in E, a_i \in (0, 1), i \in \Lambda$ and $\sum_{i=1}^l a_i = 1$, then $x_1 = x_2 = \cdots = x_l$.

Let $C$ a nonempty closed convex subset of $E$ and $Q$ a mapping of $E$ onto $C$. Then $Q$ is said to be sunny if $Q(Q(x) + t(x - Q(x))) = Q(x)$ for all $x \in E$ and $t \geq 0$. A mapping $Q$ of $E$ into $E$ is said to be a retraction if $Q^2 = Q$. If a mapping $Q$ is a retraction, then $Q(z) = z$ for every $z \in R(Q)$, where $R(Q)$ is the range of $Q$. A subset $C$ of $E$ is said to be a sunny nonexpansive retract of $E$ if there exists a sunny nonexpansive retraction of $E$ onto $C$ and it is said to be a nonexpansive retract of $E$ if there exists a nonexpansive retraction of $E$ onto $C$. If $E = H$, the metric projection $P_C$ is a sunny nonexpansive retraction from $H$ to any closed convex subset of $H$.

**Lemma 2.1.** (see [10]) Let $E$ be a smooth Banach space and $C$ a nonempty subset of $E$. Let $Q : E \to C$ be a retraction and $J$ the normalized duality mapping on $E$. Then the following are equivalent:

(i) $Q$ is sunny nonexpansive;

(ii) $\langle x - Q(x), J(y - Q(x)) \rangle \leq 0$, for all $x \in E$ and $y \in K$.

We note that Lemma 2.1 still holds if the normalized duality map $J$ is replaced with the general duality map $J_\phi$, where $\phi$ is a gauge function.

**Lemma 2.2.** (see [11]) Let \( \{s_n\} \) be a sequence of nonnegative real numbers satisfying the following relation:

\[
s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n \mu_n, \quad \forall n \geq 0,
\]

where (i) $0 < \alpha_n < 1$, (ii) $\sum_{n=1}^\infty \alpha_n = \infty$. Suppose, either $\sigma_n = o(\alpha_n)$, or $\sum_{n=1}^\infty \sigma_n < \infty$, where $\sigma_n = \alpha_n \mu_n$ or (iii) $\limsup_{n \to \infty} \mu_n \leq 0$. Then $s_n \to 0$ as $n \to \infty$.

**Lemma 2.3.** (see [2]) Let $E$ be a real Banach space. Then for all $x, y \in E$ we get that

\[
\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, j_\phi(x + y) \rangle, \quad j_\phi \in J_\phi.
\]  

(6)
Lemma 2.4. (The Resolvent Identity) For \( \lambda > 0 \) and \( \mu > 0 \) and \( x \in E \),
\[
J_{\lambda}x = J_{\mu}(\frac{\mu}{\lambda}x + (1 - \frac{\mu}{\lambda})J_{\lambda}x).
\]

Lemma 2.5. (see [9]) Let \( \{x_n\} \) and \( \{y_n\} \) be bounded sequences in a Banach space \( E \) such that
\[
x_{n+1} = \beta_n x_n + (1 - \beta_n) y_n, \quad \forall n \geq 0,
\]
where \( \{\beta_n\} \) is a sequence in \((0, 1)\) such that \( 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1 \). Assume
\[
\limsup_{n \to \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.
\]
Then \( \lim_{n \to \infty} \|y_n - x_n\| = 0 \).

Lemma 2.6. ([13]) Let \( C \) be a nonempty closed convex subset of a strictly convex Banach space \( E \). Let \( \{A_i : 1 < i < l\} : C \to E \) be a finite family of accretive operators such that \( \bigcap_{i=1}^l N(A_i) \neq \emptyset \), satisfying the range conditions:
\[
closure(D(A_i)) \subseteq C \subset \bigcap_{r>0} R(I + rA_i), \quad i = 1, 2, \ldots, l.
\]
Let \( a_0, a_1, \ldots, a_l \) be real numbers in \((0, 1)\) such that \( \sum_{i=0}^l a_i = 1 \) and \( S_r = a_0 I + a_1 J^1_r + a_2 J^2_r + \cdots + a_l J^l_r \), where \( J^i_r = (I + rA_i)^{-1} \) and \( r > 0 \). Then \( S_r \) is nonexpansive and \( \text{Fix}(S_r) = \bigcap_{i=1}^l N(A_i) \).

Lemma 2.7. (Demiclosedness Principle) If \( K \) is a closed convex subset of a real space \( E \) satisfying Opial’s condition and \( T \) is a nonexpansive mapping, then \( x_n \rightharpoonup x \) and \( (I - T)x_n \to y \) implies that \( (I - T)x = y \).

3. Main results

Throughout this section, we assume:
(i) \( E \) is a strictly convex reflexive Banach space with a weakly sequentially continuous duality mapping \( J_\phi \) for some gauge \( \phi \). \( C \) is a nonempty closed convex subset of \( E \) which is also a sunny nonexpansive retract of \( E \).
(ii) The real sequence \( \{\alpha_n\} \) satisfies the two conditions: \((C1)\lim_{n \to \infty} \alpha_n = 0\), and \((C2)\sum_{n=0}^{\infty} \alpha_n = +\infty\).

Theorem 3.1. Let \( \{A_i : 1 < i < l\} : C \to E \) be a finite family of accretive operators satisfying the following range conditions:
\[
closure(D(A_i)) \subseteq C \subset \bigcap_{r>0} R(I + rA_i), \quad i = 1, 2, \ldots, l.
\]
Assume that \( \bigcap_{i=1}^l N(A_i) \neq \emptyset \). Let \( \{\alpha_n\} \), \( \{\beta_n\} \) and \( \{\gamma_n\} \) are three sequences in \((0, 1)\) and \( \{r_n\} \) is a sequence in \((0, +\infty)\), satisfying conditions
where $y$ is a contraction with constant $\beta$, and $S_{r_n} = a_0 I + a_1 J_{r_1}^1 + a_2 J_{r_2}^2 + \cdots + a_i J_{r_i}^i$ with $J_{r_i}^i = (I + r_i A_i)^{-1}$, for $i = 0, 1, 2, \cdots, l$, $a_i \in (0, 1)$, $\sum_{i=0}^l a_i = 1$. Then the sequence $\{x_n\}$ converges strongly to $x^* = Q(f(x^*))$, which is a common zero of $\{A_i : i \in \Lambda\}$. Moreover, $x^*$ is the solution of the variational inequality:

\[
\langle (I - f)x^*, J(x^* - x) \rangle \leq 0, \quad \forall x \in \bigcap_{i=1}^l N(A_i).
\]  

(2)

Proof. By Lemma 2.6, this implies that $F := \text{Fix}(S_{r_n}) = \bigcap_{i=1}^l N(A_i) \neq \emptyset$. Take $p \in F$, we obtain

\[
\|x_{n+1} - p\| = \|\alpha_n (f(x_n) - p) + \beta_n (x_n - p) + \gamma_n (S_{r_n} - p)\|
\leq \alpha_n \|f(x_n) - p\| + (1 - \alpha_n) \|x_n - p\|
\leq \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + (1 - \alpha_n) \|x_n - p\|
\leq [1 - (1 - \beta) \alpha_n] \|x_n - p\| + \alpha_n \|f(p) - p\|.
\]

By induction, we obtain for all $n \geq 0$,

\[
\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{1}{1 - \beta} \|f(p) - p\|\}.
\]

Therefore, the sequences $\{x_n\}$, $\{f(x_n)\}$ and $\{S_{r_n} x_n\}$ are bounded. Rewrite the iterative process (7) as follow:

\[
x_{n+1} = \beta_n x_n + (1 - \beta_n) \frac{\alpha_n f(x_n) + \gamma_n S_{r_n} x_n}{1 - \beta_n}
= \beta_n x_n + (1 - \beta_n) y_n,
\]

where $y_n = \frac{\alpha_n}{1 - \beta_n} f(x_n) + \frac{\gamma_n}{1 - \beta_n} S_{r_n} x_n$. We get that $\{y_n\}$ is also bounded. After some manipulation this yields

\[
y_{n+1} - y_n = \frac{\alpha_{n+1}}{1 - \beta_{n+1}} f(x_{n+1}) - \frac{\alpha_n}{1 - \beta_n} f(x_n)
+ \left(1 - \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\right) (S_{r_{n+1}} x_{n+1} - S_{r_n} x_n)
+ \left(\frac{\alpha_n}{1 - \beta_n} - \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\right) S_{r_n} x_n.
\]
By the resolvent identity, it follows that
\[
\|J_{r_{n+1}}^i x_{n+1} - J_{r_n}^i x_n\| = \|J_{r_n}^i \left(\frac{r_n}{r_{n+1}} x_{n+1} + (1 - \frac{r_n}{r_{n+1}})J_{r_{n+1}}^i x_{n+1}\right) - J_{r_n}^i x_n\|
\leq \left\| \frac{r_n}{r_{n+1}} (x_{n+1} - x_n) + (1 - \frac{r_n}{r_{n+1}})(J_{r_{n+1}}^i x_{n+1} - J_{r_n}^i x_n) \right\|
\leq \frac{r_n}{r_{n+1}} \|x_{n+1} - x_n\| + \left| 1 - \frac{r_n}{r_{n+1}} \right| M,
\]
where \(M = \sup_{n \geq 1} \{x_n - J_{r_{n+1}}^i x_n\}\). Since \(S_{r_n} = a_0 I + \sum_{i=1}^l a_i J_{r_n}^i\), we have
\[
\|S_{r_{n+1}} x_{n+1} - S_{r_n} x_n\| = \|a_0 (x_{n+1} - x_n) + \sum_{i=1}^l a_i (J_{r_{n+1}}^i x_{n+1} - J_{r_n}^i x_n)\|
\leq a_0 \|x_{n+1} - x_n\| + \sum_{i=1}^l a_i \|J_{r_{n+1}}^i x_{n+1} - J_{r_n}^i x_n\|
\leq \left[ \frac{r_n}{r_{n+1}} + a_0 (1 - \frac{r_n}{r_{n+1}}) \right] \|x_{n+1} - x_n\|
+ (1 - a_0) \left| 1 - \frac{r_n}{r_{n+1}} \right| M.
\]
It follows that
\[
\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|
\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \|f(x_{n+1}) - f(x_n)\| + (1 - \frac{\alpha_{n+1}}{1 - \beta_{n+1}}) \|S_{r_{n+1}} x_{n+1} - S_{r_n} x_n\|
+ \left| \frac{\alpha_n}{1 - \beta_n} - \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \right| \|S_{r_n} x_n\| - \|x_{n+1} - x_n\|
\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \|f(x_{n+1}) - f(x_n)\| + \left| \frac{\alpha_n}{1 - \beta_n} - \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \right| \|f(x_n)\|
+ (1 - \frac{\alpha_{n+1}}{1 - \beta_{n+1}})(\frac{r_n}{r_{n+1}} + a_0 (1 - \frac{r_n}{r_{n+1}})) \|x_{n+1} - x_n\|
+ (1 - \frac{\alpha_{n+1}}{1 - \beta_{n+1}})(1 - a_0) \left| 1 - \frac{r_n}{r_{n+1}} \right| M
+ \left| \frac{\alpha_n}{1 - \beta_n} - \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \right| \|S_{r_n} x_n\| - \|x_{n+1} - x_n\|
= \left\{ (1 - \frac{\alpha_{n+1}}{\beta_{n+1}}) \left[ \frac{r_n}{r_{n+1}} + a_0 (1 - \frac{r_n}{r_{n+1}}) \right] + \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \beta - 1 \right\} \|x_{n+1} - x_n\|
+ (1 - \frac{\alpha_{n+1}}{\beta_{n+1}})(1 - a_0) \left| 1 - \frac{r_n}{r_{n+1}} \right| M
+ \left| \frac{\alpha_n}{1 - \beta_n} - \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \right| \|S_{r_n} x_n\| + f(\|x_n\|),
\]
from \(\{x_n\}\), \(\{f(x_n)\}\) and \(\{S_{r_n} x_n\}\) are bounded, \(\lim_{n \to \infty} r_n = r\), \(\lim_{n \to \infty} a_n = 0\), we have
\[
\limsup_{n \to \infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.
\]
Consequently, by Lemma 2.5, we obtain \( \lim_{n \to \infty} \|y_n - x_n\| = 0. \)

\[
\|x_{n+1} - \frac{\beta_n x_n + \gamma_n S_{r_n} x_n}{1 - \alpha_n}\| = \|\alpha_n f(x_n) + (1 - \alpha_n) \beta_n x_n + \gamma_n S_{r_n} x_n\|
\leq \alpha_n \|f(x_n) - \frac{\beta_n x_n + \gamma_n S_{r_n} x_n}{1 - \alpha_n}\| \\
\to 0 \quad (n \to \infty),
\]

and

\[
\|x_{n+1} - x_n\| = \|\alpha_n f(x_n) + \gamma_n S_{r_n} x_n - x_n\|
\leq (1 - \beta_n) \|y_n - x_n\| \\
\to 0.
\]

Obviously,

\[
\|x_n - \frac{\beta_n x_n + \gamma_n S_{r_n} x_n}{1 - \alpha_n}\| = \frac{\gamma_n}{1 - \alpha_n} \|x_n - S_{r_n} x_n\|.
\]

By the conditions \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \lim sup_{n \to \infty} \beta_n < 1 \), it follows that \( \lim inf_{n \to \infty} \gamma_n > 0 \). Therefore, we obtain

\[
\|x_n - S_{r_n} x_n\| \leq \frac{1 - \alpha_n}{\gamma_n} \|x_n - x_{n+1}\| \left( \left\|x_{n+1} - \frac{\beta_n x_n + \gamma_n S_{r_n} x_n}{1 - \alpha_n}\right\| \right) \to 0 \quad (n \to \infty).
\]

By the resolvent identity and \( S_{r_n} = a_n I + \sum_{i=1}^l J^i_{r_n} \), this implies that

\[
\|S_{r_n} x_n - S_r x_n\| = \| \sum_{i=1}^l a_i (J^i_{r_n} x_n - J_r x_n) \|
\leq \sum_{i=1}^l a_i \| J^i_{r_n} \left( \frac{r}{r_n} x_n + \left( 1 - \frac{r}{r_n} \right) J^i_{r_n} x_n \right) - J_r x_n \|
\leq \sum_{i=1}^l a_i \| J^i_{r_n} \left( \frac{r}{r_n} x_n + \left( 1 - \frac{r}{r_n} \right) J^i_{r_n} x_n - x_n \right) \|
\leq \sum_{i=1}^l a_i \| 1 - \frac{r}{r_n} \| \| x_n - J^i_{r_n} x_n \| \to 0 \quad (n \to \infty).
\]

Hence, we have

\[
\|x_n - S_r x_n\| \leq \|x_n - S_{r_n} x_n\| + \|S_{r_n} x_n - S_r x_n\| \to 0 \quad (n \to \infty).
\]

Next we shall show that

\[
\lim sup_{n \to \infty} \langle x^* - f(x^*), J_\phi(x^* - x_{n+1}) \rangle \leq 0.
\]
Since $E$ is reflexive and $\{x_n\}$ is bounded, we may assume $x_{n_k} \to \omega$ such that
c$$
\lim_{n \to \infty} \sup (x^* - f(x^*), J_\phi(x^* - x_{n_k})) = \lim_{k \to \infty} \sup (x^* - f(x^*), J_\phi(x^* - x_{n_k})).
$$

From the Dimclosedness Principle, we have $\omega \in F$. On the other hand, from
the standard characterization of retraction onto $\omega$

$$
\lim_{n \to \infty} \sup (x^* - f(x^*), J_\phi(x^* - x_{n_k})) = \lim_{k \to \infty} \sup (x^* - f(x^*), J_\phi(x^* - x_{n_k}))
$$

$$
= (x^* - f(x^*), J_\phi(x^* - \omega) \leq 0.
$$

From Lemma 2.3, we get that

c$$
\Phi(||x_{n+1} - x^*||)
= \Phi(||\alpha_n(f(x_n) - f(x^*)) + \beta_n(x_n - x^*) + \gamma_n(S_{r_n}x_n - x^*)
+ \alpha_n(f(x^*) - x^*))||)
\leq \Phi(||\alpha_n(f(x_n) - f(x^*)) + \beta_n(x_n - x^*) + \gamma_n(S_{r_n}x_n - x^*))||)
+ \alpha_n(f(x^*) - x^*, J_\phi(x_{n+1} - x^*))
\leq \Phi((\alpha_n\beta + \beta_n + \gamma_n)||x_n - x^*||) + \alpha_n(f(x^*) - x^*, J_\phi(x_{n+1} - x^*))
\leq (1 - \alpha_n(1 - \beta))\Phi(||x_n - x^*||) + \alpha_n(f(x^*) - x^*, J_\phi(x_{n+1} - x^*)).
$$

By the Lemma 2.2, we have $x_n \to x^*$ as $n \to \infty$. Moreover, $x^*$ satisfying
condition (*) follows from the property of $Q$. To show that it is unique, let
$y^* \in F$ be another solution of the variational inequality (*). Then adding
$\langle f(x^*) - x^*, J_\phi(y^* - x^*) \rangle \leq 0$ and $\langle f(y^*) - y^*, J_\phi(x^* - y^*) \rangle \leq 0$, we have
$(1 - \beta)\phi(||x^* - y^*||)||x^* - y^*|| \leq 0$. This implies that $x^* = y^*$.

**Theorem 3.2.** Let $\{A_i : i \in \Lambda\} : C \to E$ be a finite family of accretive
operators satisfying the following range conditions:

$$
\text{cl}(D(A_i)) \subseteq C \subseteq \bigcap_{r > 0} R(I + rA_i), \quad i = 1, 2, \cdots, l.
$$

And assume that $\bigcap_{i=1}^l N(A_i) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be three sequences in
$(0, 1)$ and $r > 0$ a real number satisfying $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$. For any $x_0 \in C$, the sequence $\{x_n\}$ is given by

$$
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_r x_n, \quad n \geq 0,
$$

where $S_r = a_0I + a_1J^1_r + a_2J^2_r + \cdots + a_l J^l_r$, with $J^i_r = (I + rA_i)^{-1}$ for $0 < a_i < 1,$
i $= 0, 1, 2, \cdots, l, \sum_{i=0}^l a_i = 1$. Then the sequence $\{x_n\}$ converges strongly to $x^*$, which is a common zero of $\{A_i : i \in \Lambda\}$ and the unique solution of the variational inequality

$$
\langle (I - f)x^*, J(x^* - x) \rangle \leq 0, \quad \forall x \in \bigcap_{i=1}^l N(A_i).
$$
As direct consequences of Theorem 3.1 and Theorem 3.2, we obtain the two corollaries below:

**Corollary 3.3.** Let \( \{A_i : i \in \Lambda\} : C \to E \) be a finite family of \( m \)-accretive operators. Assume that \( \bigcap_{i=1}^{\infty} N(A_i) \neq \emptyset \). Let \( \{\alpha_n\} \), \( \{\beta_n\} \), \( \{\gamma_n\} \) be three sequences in \((0,1)\) and \( r > 0 \) a real number satisfying the condition \( 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1 \). For any \( x_0 \in C \), the sequence \( \{x_n\} \) is given by

\[
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n Sx_n, \quad n \geq 0,
\]

where \( S = a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \cdots + a_l J_{A_l} \), with \( J_{A_i} = (I + A_i)^{-1} \) for \( 0 < a_i < 1 \), \( i = 0, 1, 2, \ldots, l \), \( \sum_{i=0}^{l} a_i = 1 \). Then the sequence \( \{x_n\} \) converges strongly to \( x^* \), which is a common zero of \( \{A_i : i \in \Lambda\} \) and the unique solution of the variational inequality

\[
\langle (I - f)x^*, J(x^* - x) \rangle \leq 0, \quad \forall x \in \bigcap_{i=1}^{l} N(A_i).
\]

**Corollary 3.4.** Let \( A : C \to E \) be an \( m \)-accretive operator such that \( N(A) \neq \emptyset \). Let \( \{\alpha_n\} \), \( \{\beta_n\} \), \( \{\gamma_n\} \) be three sequences in \((0,1)\) and \( r_n \in (0, +\infty) \), satisfying the conditions \( 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1 \), and \( \lim_{n \to \infty} \frac{\gamma_n}{r_n} = 1 \). For any \( x_0 \in C \), the sequence \( \{x_n\} \) is given by

\[
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n J_{r_n}x_n, \quad n \geq 0,
\]

where \( J_{r_n} = (I + r_n A)^{-1} \). Then the sequence \( \{x_n\} \) converges strongly to \( x^* \), which is a zero of \( A \) and the unique solution of the variational inequality

\[
\langle (I - f)x^*, J(x^* - x) \rangle \leq 0, \quad \forall x \in N(A).
\]

**Theorem 3.5.** Let \( \{A_i : 1 < i < l\} : C \to E \) be a finite family of accretive operators satisfying the following range conditions:

\[
ci(\text{R}(A_i)) \subset C \subset \bigcap_{r>0} \text{R}(I + rA_i), \quad i = 1, 2, \ldots, l.
\]

Assume that \( \bigcap_{i=1}^{l} N(A_i) \neq \emptyset \). Let \( \{\alpha_n\} \), \( \{\beta_n\} \) and \( \{\gamma_n\} \) are three sequences in \((0,1)\) and \( \{r_n\} \) is a sequence in \((0, +\infty)\), satisfying conditions \( 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1 \) and \( \lim_{n \to \infty} r_n = r, \ r \in (0, +\infty) \). For any \( x_0 \in C \), the sequence \( \{x_n\} \) is given by

\[
x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) (\lambda x_n + (1 - \lambda) S_{r_n}x_n), \quad \forall n \geq 0,
\]

where \( f : C \to C \) is a contraction with constant \( \beta \), and \( S_r = a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \cdots + a_l J_{A_l} \), with \( J_{A_i} = (I + A_i)^{-1} \) for \( i = 0, 1, 2, \ldots, l \), \( a_i \in (0, 1) \), \( \sum_{i=0}^{l} a_i = 1 \). Then the sequence \( \{x_n\} \) converges strongly to \( x^* \), which is a common zero of \( \{A_i : i \in \Lambda\} \) and the solution of the variational inequality:

\[
\langle (I - f)x^*, J(x^* - x) \rangle \leq 0, \quad \forall x \in \bigcap_{i=1}^{l} N(A_i). \quad (*)
\]
Proof. Taking $\beta_n = (1 - \alpha_n)\lambda$, $\forall n \in \mathbb{N}$, we have
\[
\lim_{n \to \infty} \beta_n = \lim_{n \to \infty} (1 - \alpha_n)\lambda = \lambda \in (0, 1).
\]
By theorem 3.1, we obtain the conclusion. \qed

References


Jun-Min Chen  
College of Mathematics and Computer  
Hebei University, Baoding 071002, P. R. China  
E-mail address: chenjum01@163.com

Li-Juan Zhang  
College of Mathematics and Computer  
Hebei University, Baoding 071002, P. R. China  
E-mail address: zhanglj@hbu.edu.cn

Tie-Gang Fan  
College of Mathematics and Computer  
Hebei University, Baoding 071002, P. R. China  
E-mail address: ftg@hbu.edu.cn