COMMON FIXED POINT THEOREMS FOR TWO MAPPINGS IN $\mathcal{M}$-FUZZY METRIC SPACES

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Abstract. In this paper, we prove some common fixed point theorems for two nonlinear mappings in complete $\mathcal{M}$-fuzzy metric spaces. Our main results improved versions of several fixed point theorems in complete fuzzy metric spaces.

1. Introduction and preliminaries

The concept of fuzzy sets was introduced initially by Zadeh [27] in 1965. Since then, to apply this concept in topology and analysis, many authors [9, 17, 19, 24] have expansively developed the theory of fuzzy sets and application. George and Veeramani [8] and Kramosil and Michalek [11] have introduced the concept of fuzzy topological spaces induced by fuzzy metric which have very important applications in quantum particle physics particularly in connections with both string and $E$-infinity theory which were given and studied by El-Naschie [3-6]. Many authors [7, 10, 12, 18, 20, 23] have proved fixed point theorem in fuzzy (probabilistic) metric spaces. Vasuki [25] obtained the fuzzy version of common fixed point theorem which had extra conditions. In fact, Vasuki [25] proved fuzzy common fixed point theorem by a strong definition of a Cauchy sequence (see Note 3.13 and Definition 3.15 of [8], also [23, 26]).

On the other hand, Dhage [1, 2] introduced the notion of generalized metric or $D$-metric spaces and claimed that $D$-metric convergence defines a Hausdorff topology and $D$-metric is sequentially continuous in all the three variables. Many authors have used these claims in proving fixed point theorems in $D$-metric spaces, but, unfortunately, almost all theorems in $D$-metric spaces are not valid (see [13-16, 22]).

Recently, Sedghi et al. [21] introduced $D^*$-metric which is a probable modification of the definition of $D$-metric introduced by Dhage [1, 2] and proved...
some basic properties in $D^*$-metric spaces. Also, using the concept of the $D^*$-metrics, they defined $M$-fuzzy metric space and proved some related fixed point theorems for some nonlinear mappings in complete $M$-fuzzy metric spaces.

In this paper, we prove some common fixed point theorems for two nonlinear mappings in complete $M$-fuzzy metric spaces. Our main results improved versions of several fixed point theorems in complete fuzzy metric spaces.

In what follows $(X, D^*)$ will denote a $D^*$-metric space, $N$ the set of all natural numbers and $R^+$ the set of all positive real numbers.

**Definition 1.1.** ([21]) Let $X$ be a nonempty set. A generalized metric (or $D^*$-metric) on $X$ is a function: $D^*$ : $X^3 \rightarrow R^+$ that satisfies the following conditions: for any $x, y, z, a \in X$,

1. $D^*(x, y, z) \geq 0$,
2. $D^*(x, y, z) = 0$ if and only if $x = y = z$,
3. $D^*(x, y, z) = D^*(p\{x, y, z\})$ (symmetry), where $p$ is a permutation function,
4. $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$.

The pair $(X, D^*)$ is called a generalized metric space (or $D^*$-metric space).

Some immediate examples of such a function are as follows:

(a) $D^*(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$.
(b) $D^*(x, y, z) = d(x, y) + d(y, z) + d(z, x)$, where $d$ is the ordinary metric on $X$.
(c) If $X = R^n$, then we define $D^*(x, y, z) = (||x - y||^p + ||y - z||^p + ||z - x||^p)^{\frac{1}{p}}$ for any $p \in R^+$.
(d) If $X = R^+$, then we define

$$D^*(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\ \max\{x, y, z\}, & \text{otherwise}. \end{cases}$$

In a $D^*$-metric space $(X, D^*)$, we can prove that $D^*(x, y, z) = D^*(y, x, z)$. Let $(X, D^*)$ be a $D^*$-metric space. For any $r > 0$, define the open ball with the center $x$ and radius $r$ as follows:

$$B_{D^*}(x, r) = \{ y \in X : D^*(x, y, y) < r \}.$$

**Example 1.2.** Let $X = R$. Denote $D^*(x, y, z) = |x - y| + |y - z| + |z - x|$ for all $x, y, z \in R$. Thus we have

$$B_{D^*}(1, 2) = \{ y \in R : D^*(1, y, y) < 2 \} = \{ y \in R : |y - 1| + |y - 1| < 2 \} = \{ y \in R : |y - 1| < 1 \} = (0, 2).$$

**Definition 1.3.** ([21]) Let $(X, D^*)$ be a $D^*$-metric space and $A \subset X$. 

(1) If, for any \( x \in A \), there exists \( r > 0 \) such that \( B_{D^*}(x, r) \subset A \), then \( A \) is called an open subset of \( X \).

(2) A subset \( A \) of \( X \) is said to be \( D^* \)-bounded if there exists \( r > 0 \) such that \( D^*(x, y, y) < r \) for all \( x, y \in A \).

(3) A sequence \( \{x_n\} \) in \( X \) is said to be convergent to a point \( x \in X \) if
\[
D^*(x_n, x, x) = D^*(x, x, x_n) \to 0 \quad (n \to \infty).
\]
That is, for any \( \epsilon > 0 \), there exists \( n_0 \in N \) such that
\[
D^*(x, x, x_n) \leq \epsilon, \quad \forall n \geq n_0.
\]
Equivalently, for any \( \epsilon > 0 \), there exists \( n_0 \in N \) such that
\[
D^*(x_n, x_n, x_m) \leq \epsilon, \quad \forall n, m \geq n_0.
\]

(4) A sequence \( \{x_n\} \) in \( X \) is called a Cauchy sequence if, for any \( \epsilon > 0 \), there exits \( n_0 \in N \) such that
\[
D^*(x_n, x_n, x_m) \leq \epsilon, \quad \forall n, m \geq n_0.
\]

(5) A \( D^* \)-metric space \((X, D^*)\) is said to be complete if every Cauchy sequence is convergent.

Let \( \tau \) be the set of all \( A \subset X \) with \( x \in A \) if and only if there exists \( r > 0 \) such that \( B_{D^*}(x, r) \subset A \). Then \( \tau \) is a topology on \( X \) (induced by the \( D^* \)-metric \( D^* \)).

**Definition 1.4.** ([21]) Let \((X, D^*)\) be a \( D^* \)-metric space. \( D^* \) is said to be continuous function on \( X^3 \times (0, \infty) \) if
\[
\lim_{n \to \infty} D^*(x_n, y_n, z_n) = D^*(x, y, z)
\]
whenever a sequence \( \{(x_n, y_n, z_n)\} \) in \( X^3 \) converges to a point \((x, y, z) \in X^3 \), i.e.,
\[
\lim_{n \to \infty} x_n = x, \quad \lim_{n \to \infty} y_n = y, \quad \lim_{n \to \infty} z_n = z.
\]

**Remark 1.5.** ([21]) (1) Let \((X, D^*)\) be a \( D^* \)-metric space. Then \( D^* \) is continuous function on \( X^3 \).

(2) If a sequence \( \{x_n\} \) in \( X \) converges to a point \( x \in X \), then the limit \( x \) is unique.

(3) If a sequence \( \{x_n\} \) in \( X \) is converges to a point \( x \), then \( \{x_n\} \) is a Cauchy sequence in \( X \).

Recently, motivated by the concept of \( D^* \)-metrics, Sedghi et al. [21] introduced the concept of \( M \)-fuzzy metric spaces and their properties and, further, proved some related common fixed theorems for some contractive type mappings in \( M \)-fuzzy metric spaces.

**Definition 1.6.** ([21]) A binary operation \( \ast : [0, 1] \times [0, 1] \to [0, 1] \) is a continuous \( t \)-norm if it satisfies the following conditions:

(1) \( \ast \) is associative and commutative,
Two typical examples of continuous $t$-norm are $a \ast b = ab$ and $a \ast b = \min(a, b)$. 

**Definition 1.7.** ([21]) A 3-tuple $(X, \mathcal{M}, \ast)$ is called an $\mathcal{M}$-fuzzy metric space if $X$ is an arbitrary (non-empty) set, $\ast$ is a continuous $t$-norm and $\mathcal{M}$ is a fuzzy set on $X^3 \times (0, \infty)$ satisfying the following conditions: for all $x, y, z, a \in X$ and $t, s > 0$,

1. $\mathcal{M}(x, y, z, t) > 0$,
2. $\mathcal{M}(x, y, z, t) = 1$ if and only if $x = y = z$,
3. $\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)$ (symmetry), where $p$ is a permutation function,
4. $\mathcal{M}(x, y, a, t) \ast \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t + s),$
5. $\mathcal{M}(x, y, z, t) : X^3 \times (0, \infty) \rightarrow [0, 1]$ is continuous with respect to $t$.

**Remark 1.8.** Let $(X, \mathcal{M}, \ast)$ be an $\mathcal{M}$-fuzzy metric space. Then, for any $t > 0$, $\mathcal{M}(x, x, y, t) = \mathcal{M}(x, y, y, t)$.

Let $(X, \mathcal{M}, \ast)$ be an $\mathcal{M}$-fuzzy metric space. For any $t > 0$, the open ball $B_\mathcal{M}(x, r, t)$ with the center $x \in X$ and radius $0 < r < 1$ is defined by

$$B_\mathcal{M}(x, r, t) = \{y \in X : \mathcal{M}(x, y, y, t) > 1 - r\}.$$ 

A subset $A$ of $X$ is called an open set if, for all $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B_\mathcal{M}(x, r, t) \subseteq A$.

**Definition 1.9.** ([21]) Let $(X, \mathcal{M}, \ast)$ be an $\mathcal{M}$-fuzzy metric space.

1. A sequence $\{x_n\}$ in $X$ is said to be convergent to a point $x \in X$ if $\mathcal{M}(x, x, x_n, t) \rightarrow 1$ as $n \rightarrow \infty$ for any $t > 0$.
2. A sequence $\{x_n\}$ is called a Cauchy sequence if, for any $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mathcal{M}(x_n, x_m, x_m, t) > 1 - \epsilon, \forall n, m \geq n_0$.
3. An $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, \ast)$ is said to be complete if every Cauchy sequence in $X$ is convergent.

**Example 1.10.** Let $X$ be a nonempty set and $D^*$ be the $D^*$-metric on $X$. Denote $a \ast b = ab$ for all $a, b \in [0, 1]$. For any $t \in [0, \infty[$, define

$$\mathcal{M}(x, y, z, t) = \frac{t}{t + D^*(x, y, z)}, \forall x, y, z \in X.$$ 

It is easy to see that $(X, \mathcal{M}, \ast)$ is a $\mathcal{M}$-fuzzy metric space.

**Remark 1.11.** Let $(X, \mathcal{M}, \ast)$ be a fuzzy metric space. If we define $\mathcal{M} : X^3 \times (0, \infty) \rightarrow [0, 1]$ by

$$\mathcal{M}(x, y, z, t) = \mathcal{M}(x, y, t) \ast \mathcal{M}(y, z, t) \ast \mathcal{M}(z, x, t), \forall x, y, z \in X,$$

then $(X, \mathcal{M}, \ast)$ is an $\mathcal{M}$-fuzzy metric space.
Lemma 1.12. ([21]) Let \((X, \mathcal{M}, \ast)\) be an \(\mathcal{M}\)-fuzzy metric space. Then, for all \(x, y, z \in X\) and \(t > 0\), \(\mathcal{M}(x, y, z, t)\) is nondecreasing with respect to \(t\).

Definition 1.13. ([21]) Let \((X, \mathcal{M}, \ast)\) be an \(\mathcal{M}\)-fuzzy metric space. \(\mathcal{M}\) is said to be continuous function on \(X^3 \times (0, \infty)\) if
\[
\lim_{n \to \infty} \mathcal{M}(x_n, y_n, z_n, t_n) = \mathcal{M}(x, y, z, t)
\]
whenever a sequence \(\{(x_n, y_n, z_n, t_n)\}\) in \(X^3 \times (0, \infty)\) converges to a point \((x, y, z, t) \in X^3 \times (0, \infty)\), that is,
\[
\lim_{n \to \infty} x_n = x, \quad \lim_{n \to \infty} y_n = y, \quad \lim_{n \to \infty} z_n = z,
\]
\[
\lim_{n \to \infty} \mathcal{M}(x, y, z, t_n) = \mathcal{M}(x, y, z, t).
\]

Lemma 1.14. ([21]) Let \((X, \mathcal{M}, \ast)\) be an \(\mathcal{M}\)-fuzzy metric space. Then \(\mathcal{M}\) is continuous function on \(X^3 \times (0, \infty)\).

Lemma 1.15. ([21]) Let \((X, \mathcal{M}, \ast)\) be an \(\mathcal{M}\)-fuzzy metric space. If we define
\[
E_{\lambda, \mathcal{M}}(x, y, z) = \inf \{t > 0 : \mathcal{M}(x, y, z, t) > 1 - \lambda\}, \quad \forall \lambda \in (0, 1),
\]
then we have the following:

1. For any \(\mu \in (0, 1)\), there exists \(\lambda \in (0, 1)\) such that
\[
E_{\mu, \mathcal{M}}(x_1, x_1, x_2) \leq E_{\lambda, \mathcal{M}}(x_1, x_1, x_2) + E_{\lambda, \mathcal{M}}(x_2, x_2, x_3) + \cdots + E_{\lambda, \mathcal{M}}(x_n, x_{n-1}, x_n)
\]
for any \(x_1, x_2, \cdots, x_n \in X\).

2. A sequence \(\{x_n\}\) is convergent in an \(\mathcal{M}\)-fuzzy metric space \((X, \mathcal{M}, \ast)\) if and only if \(E_{\lambda, \mathcal{M}}(x_n, x_n, x) \to 0\). Also, the sequence \(\{x_n\}\) is a Cauchy sequence in \(X\) if and only if it is a Cauchy sequence with \(E_{\lambda, \mathcal{M}}\).

Lemma 1.16. ([21]) Let \((X, \mathcal{M}, \ast)\) be an \(\mathcal{M}\)-fuzzy metric space. If there exists \(k > 1\) such that
\[
\mathcal{M}(x_n, x_n, x_{n+1}, t) \geq \mathcal{M}(x_0, x_0, x_{1}, k^n t), \quad \forall n \geq 1,
\]
then \(\{x_n\}\) is a Cauchy sequence in \(X\).

Definition 1.17. ([7]) We say that an \(\mathcal{M}\)-fuzzy metric space \((X, \mathcal{M}, \ast)\) has the property (C) if it satisfies the following condition: For some \(x, y, z \in X\),
\[
\mathcal{M}(x, y, z, t) = C, \quad \forall t > 0, \quad \implies \quad C = 1.
\]
2. The main results

Now, we are ready to give main results in this paper.

**Theorem 2.1.** Let \((X, \mathcal{M}, \ast)\) be a complete \(\mathcal{M}\)-fuzzy metric space and \(S, T\) be two self-mappings of \(X\) satisfying the following conditions:

(i) there exists a constant \(k \in (0, 1)\) such that

\[
\mathcal{M}(Sx, TSx, Ty, kt) \geq \gamma(\mathcal{M}(x, Sx, y, t)), \quad \forall x, y \in X,
\]  

or

\[
\mathcal{M}(Ty, STy, Sx, kt) \geq \gamma(\mathcal{M}(y, Ty, x, t)), \quad \forall x, y \in X,
\]

where \(\gamma : [0, 1] \to [0, 1]\) is a function such that \(\gamma(a) \geq a\) for all \(a \in [0, 1]\),

(ii) \(ST = TS\).

If \((X, \mathcal{M}, \ast)\) have the property \((\mathcal{C})\), then \(S\) and \(T\) have a unique common fixed point in \(X\).

**Proof.** Let \(x_0\) be an arbitrary point in \(X\), define

\[
\begin{aligned}
x_{2n+1} &= Tx_{2n}, \\
x_{2n+2} &= Sx_{2n+1},
\end{aligned}
\]

\((2.3)\)

(1) Let \(d_m(t) = \mathcal{M}(x_m, x_{m+1}, x_{m+1}, t)\) for any \(t > 0\). Then, for any even \(m = 2n \in N\), by (2.1) and (2.3), we have

\[
\begin{aligned}
d_{2n}(kt) &= \mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, kt) \\
&= \mathcal{M}(Sx_{2n-1}, Tx_{2n}, Tx_{2n}, kt) \\
&= \mathcal{M}(Sx_{2n-1}, TSx_{2n-1}, Tx_{2n}, kt) \\
&\geq \gamma(\mathcal{M}(x_{2n-1}, Sx_{2n-1}, x_{2n}, kt)) \\
&\geq \mathcal{M}(x_{2n-1}, x_{2n}, x_{2n}, kt) \\
&= d_{2n-1}(t).
\end{aligned}
\]

Thus \(d_{2n}(kt) \geq d_{2n-1}(t)\) for all even \(m = 2n \in N\) and \(t > 0\).

Similarly, for any odd \(m = 2n + 1 \in N\), we have also

\[
d_{2n+1}(kt) \geq d_{2n}(t).
\]

Hence we have

\[
d_{2n+1}(kt) \geq d_{2n}(t), \quad \forall n \geq 1.
\]

Thus, by (2.4), we have

\[
\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t) \geq \mathcal{M}(x_{n-1}, x_n, x_n, \frac{1}{k} t) \geq \cdots \geq \mathcal{M}(x_0, x_1, x_1, \frac{1}{k^n} t).
\]

Therefore, by Lemma 1.16, \(\{x_n\}\) is a Cauchy sequence in \(X\) and, by the completeness of \(X\), \(\{x_n\}\) converges to a point \(x\) in \(X\) and so

\[
\lim_{n \to \infty} x_{2n+1} = \lim_{n \to \infty} Tx_{2n} = \lim_{n \to \infty} Sx_{2n+1} = x.
\]

\[
\lim_{n \to \infty} x_{2n+2} = x.
\]
Now, we prove that $Tx = x$. Replacing $x, y$ by $x_{2n-1}, x$, respectively, in (i), we obtain

$$
\mathcal{M}(Sx_{2n-1}, TSx_{2n-1}, Tx, kt) \geq \gamma(\mathcal{M}(x_{2n-1}, Sx_{2n-1}, x, t)),
$$

that is,

$$
\mathcal{M}(x_{2n}, x_{2n+1}, Tx, kt) \geq \gamma(\mathcal{M}(x_{2n-1}, x_{2n}, x, t)) \geq \mathcal{M}(x_{2n-1}, x_{2n}, x, t). \tag{2.5}
$$

Letting $n \to \infty$ in (2.5), we have

$$
\mathcal{M}(x, x, Tx, kt) \geq \mathcal{M}(x, x, x, t) = 1,
$$

which implies that $Tx = x$, that is, $x$ is a fixed point of $T$.

Next, we prove that $Sx = x$. Replacing $x, y$ by $x, x_{2n}$, respectively, in (2.1), we obtain

$$
\mathcal{M}(Sx, TSx, Tx_{2n}, kt) \geq \gamma(\mathcal{M}(x, Sx, x_{2n}, t)) \geq \mathcal{M}(x, Sx, x_{2n}, t).
$$

By (ii), since $TS = ST$, we get

$$
\mathcal{M}(Sx, Sx, Tx_{2n}, kt) \geq \gamma(\mathcal{M}(x, Sx, x_{2n}, t)) \geq \mathcal{M}(x, Sx, x_{2n}, t). \tag{2.6}
$$

Letting $n \to \infty$ in (2.6), we have

$$
\mathcal{M}(Sx, Sx, x, kt) \geq \mathcal{M}(x, Sx, x, t)
$$

and hence

$$
\mathcal{M}(x, Sx, x, t) \geq \mathcal{M}(x, Sx, x, \frac{1}{k} t)
\geq \mathcal{M}(x, Sx, x, \frac{1}{k^2} t)
\geq \mathcal{M}(x, Sx, x, \frac{1}{k^n} t).
$$

On the other hand, it follows from Lemma 1.12 that

$$
\mathcal{M}(x, Sx, x, k^n t) \leq \mathcal{M}(x, Sx, x, t).
$$

Hence $\mathcal{M}(x, Sx, x, t) = C$ for all $t > 0$. Since $(X, \mathcal{M}, *)$ has the property $(C)$, it follows that $C = 1$ and so $Sx = x$, that is, $x$ is a fixed point of $S$. Therefore, $x$ is a common fixed point of the self-mappings $S$ and $T$. 
(2) By using (2.2) and (2.3), let 
\[ d_m(t) = M(x_{m+1}, x_m, x_m, t) \]
for any \( t > 0 \). Then, for any even \( m = 2n \in N \), we have
\[ d_{2n}(kt) = M(x_{2n+1}, x_{2n}, x_{2n}, kt) \]
\[ = M(Tx_{2n}, Sx_{2n-1}, Sx_{2n-1}, kt) \]
\[ = M(Tx_{2n}, STx_{2n-2}, Sx_{2n-1}, kt) \]
\[ \geq \gamma(M(x_{2n}, Tx_{2n-2}, x_{2n-1}, t)) \]
\[ \geq M(x_{2n}, Tx_{2n-2}, x_{2n-1}, t) \]
\[ \geq M(x_{2n}, x_{2n-1}, x_{2n-1}, t) \]
\[ = d_{2n-1}(t). \]
Thus \( d_{2n}(kt) \geq d_{2n-1}(t) \) for all even \( m = 2n \in N \) and \( t > 0 \).
Similarly, for any odd \( m = 2n + 1 \in N \), we have also
\[ d_{2n+1}(kt) \geq d_{2n}(t). \]
Hence we have
\[ d_n(kt) \geq d_{n-1}(t), \quad \forall n \geq 1. \]
The remains of the proof are almost same to the case of (2.1).
Now, to prove the uniqueness, let \( x' \) be another common fixed point of \( S \) and \( T \). Then we have
\[ M(x, x', kt) = M(Sx, TSx, Tx', kt) \]
\[ \geq \gamma(M(x, Sx, x', t)) \]
\[ \geq M(x, x', t), \]
which implies that
\[ M(x, x', t) \geq M(x, x', \frac{1}{k} t) \]
\[ \geq M(x, x', \frac{1}{k^2} t) \]
\[ \geq \cdots \]
\[ \geq M(x, x', \frac{1}{k^n} t). \]
On the other hand, it follows from Lemma 2.12 that
\[ M(x, x', t) \leq M(x, x', \frac{1}{k^n} t) \]
and hence \( M(x, x', t) = C \) for all \( t > 0 \). Since \((X, M, *)\) has the property \((C)\), it follows that \( C = 1 \), that is, \( x = x' \). Therefore, \( x \) is a unique common fixed point of \( S \) and \( T \). This completes the proof. \( \square \)

By Theorem 2.1, we have the following:
Corollary 2.2. Let \((X, \mathcal{M}, *)\) be a complete \(\mathcal{M}\)-fuzzy metric space. Let \(T\) be a mapping from \(X\) into itself such that there exists a constant \(k \in (0,1)\) such that
\[
\mathcal{M}(Tx, T^2x, Ty, kt) \geq \mathcal{M}(x, Tx, y, t), \quad \forall x, y \in X.
\]
If \((X, \mathcal{M}, *)\) have the property \((C)\), then \(T\) have a unique fixed point in \(X\).

Proof. By Theorem 2.1, if we set \(\gamma(a) = a\) and \(S = T\), then the conclusion follows. \(\square\)

Corollary 2.3. Let \((X, \mathcal{M}, *)\) be a complete \(\mathcal{M}\)-fuzzy metric space. Let \(T\) be a mapping from \(X\) into itself such that there exists a constant \(k \in (0,1)\) such that
\[
\mathcal{M}(T^n x, T^{2n} x, T^n y, kt) \geq \mathcal{M}(x, T^n x, y, t)
\]
for all \(x, y \in X\) and \(n \geq 2\). If \((X, \mathcal{M}, *)\) has the property \((C)\), then \(T\) have a unique fixed point in \(X\).

Proof. By Corollary 2.2, \(T^n\) have a unique fixed point in \(X\). Thus there exists \(x \in X\) such that \(T^n x = x\). Since
\[
T^{n+1} x = T^n (Tx) = T(T^n x) = Tx,
\]
we have \(Tx = x\). \(\square\)

Next, by using Lemma 1.16 and the property \((C)\), we can prove the main results in this paper.

Theorem 2.4. Let \((X, \mathcal{M}, *)\) be a complete \(\mathcal{M}\)-fuzzy metric space with \(t * t = t\) for all \(t \in [0,1]\). Let \(S\) and \(T\) be mappings from \(X\) into itself such that there exists a constant \(k \in (0,1)\) such that
\[
\mathcal{M}(Sx, Ty, Ty, kt)
\]
\[
\geq a(t)\mathcal{M}(x, Sx, x, t) + b(t)\mathcal{M}(y, Ty, Ty, t)
\]
\[
+ c(t)\mathcal{M}(x, Ty, Ty, at) + d(t)\mathcal{M}(y, Sx, Sx, (2 - \alpha)t)
\]
\[
+ e(t)\mathcal{M}(x, y, y, t)
\]
for all \(x, y \in X\) and \(\alpha \in (0, 2)\), where \(a, b, c, d, e : [0, \infty) \to [0,1]\) are five functions such that
\[
a(t) + b(t) + c(t) + d(t) + e(t) = 1, \quad \forall t \in [0, \infty).
\]
Then \(S\) and \(T\) have a unique common fixed point in \(X\).

Proof. Let \(x_0 \in X\) be an arbitrary point. Then there exist \(x_1, x_2 \in X\) such that \(x_1 = Sx_0\) and \(x_2 = Tx_1\). Inductively, we can construct a sequence \(\{x_n\}\) in \(X\) such that
\[
\begin{cases}
x_{2n+1} = Sx_{2n}, \\
x_{2n+2} = Tx_{2n+1},
\end{cases} \quad \forall n \geq 0.
\]
(2.8)

Now, we show that \(\{x_n\}\) is a Cauchy sequence in \(X\). If we set
\[
d_m(t) = \mathcal{M}(x_m, x_{m+1}, x_{m+1}, t), \quad \forall t > 0,
\]
(2.9)
then we prove that \( \{d_m(t)\} \) is increasing with respect to \( m \in N \). In fact, for any odd \( m = 2n + 1 \in N \), we have

\[
\begin{align*}
    &d_{2n+1}(kt) \\
    &= M(x_{2n+1}, x_{2n+2}, x_{2n+2}, kt) \\
    &= M(Sx_{2n}, Tx_{2n+1}, Tx_{2n+1}, kt) \\
    &\geq a(t)M(x_{2n}, Sx_{2n}, Sx_{2n}, t) + b(t)M(x_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, t) \\
    &\quad + c(t)M(x_{2n}, Tx_{2n+1}, Tx_{2n+1}, at) \\
    &\quad + d(t)M(x_{2n+1}, Sx_{2n}, Sx_{2n}, (2 - \alpha)t) \\
    &\quad + e(t)M(x_{2n}, x_{2n+1}, x_{2n+1}, t) \\
    &= a(t)M(x_{2n}, x_{2n+1}, x_{2n+1}, t) + b(t)M(x_{2n+1}, x_{2n+2}, x_{2n+2}, t) \\
    &\quad + c(t)M(x_{2n}, x_{2n+2}, x_{2n+2}, at) \\
    &\quad + d(t)M(x_{2n+1}, x_{2n+1}, x_{2n+1}, (2 - \alpha)t) \\
    &\quad + e(t)M(x_{2n}, x_{2n+1}, x_{2n+1}, t)
\end{align*}
\]

and so

\[
\begin{align*}
    d_{2n+1}(kt) &\geq a(t)d_{2n}(t) + b(t)d_{2n+1}(t) + c(t)d_{2n}(t) * d_{2n+1}(qt) \\
    &\quad + d(t) + e(t)d_{2n}(t). \\
\end{align*}
\]

The equality in (2.10) is true because, if set \( \alpha = 1 + q \) for any \( q \in (k, 1) \), then

\[
\begin{align*}
    &M(x_{2n}, x_{2n+2}, x_{2n+2}, (1 + q)t) \\
    &= M(x_{2n}, x_{2n}, x_{2n+2}, (1 + q)t) \\
    &\geq M(x_{2n}, x_{2n}, x_{2n+1}, t) * M(x_{2n+1}, x_{2n+2}, x_{2n+2}, qt) \\
    &= d_{2n}(t) * d_{2n+1}(qt). \\
\end{align*}
\]

Now, we claim that

\[
d_{2n+1}(t) \geq d_{2n}(t), \quad \forall n \geq 1.
\]

In fact, if \( d_{2n+1}(t) < d_{2n}(t) \), then since

\[
d_{2n+1}(qt) * d_{2n}(t) \geq d_{2n+1}(qt) * d_{2n+1}(qt) = d_{2n+1}(qt)
\]

in (3.10), we have

\[
\begin{align*}
    d_{2n+1}(kt) &> a(t)d_{2n+1}(qt) + b(t)d_{2n+1}(qt) + c(t)d_{2n+1}(qt) \\
    &\quad + d(t)d_{2n+1}(qt) + e(t)d_{2n+1}(qt) \\
    &= d_{2n+1}(qt)
\end{align*}
\]
and so $d_{2n+1}(kt) > d_{2n+1}(qt)$, which is a contradiction. Hence $d_{2n+1}(t) \geq d_{2n}(t)$ for all $n \in \mathbb{N}$ and $t > 0$. By (2.10), we have

$$
d_{2n+1}(kt) \geq a(t)d_{2n}(qt) + b(t)d_{2n}(qt) + c(t)d_{2n}(qt) + d(t)d_{2n}(qt) + e(t)d_{2n}(qt)
= d_{2n}(qt).
$$

Now, if $m = 2n$, then, by (2.9), we have

$$
d_{2n}(kt) = \mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, kt)
= \mathcal{M}(Sx_{2n-1}, Tx_{2n}, Tx_{2n}, kt)
\geq a(t)\mathcal{M}(x_{2n-1}, Sx_{2n-1}, Sx_{2n-1}, t) + b(t)\mathcal{M}(x_{2n}, Tx_{2n}, Tx_{2n}, t)
+ c(t)\mathcal{M}(x_{2n-1}, Tx_{2n}, Tx_{2n}, \alpha(t))
+ d(t)\mathcal{M}(x_{2n}, x_{2n-1}, Sx_{2n-1}, (2 - \alpha(t))
+ e(t)\mathcal{M}(x_{2n-1}, x_{2n}, x_{2n}, t)
= a(t)\mathcal{M}(x_{2n-1}, x_{2n}, x_{2n}, t) + b(t)\mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t)
+ c(t)\mathcal{M}(x_{2n-1}, x_{2n+1}, x_{2n+1}, \alpha(t)) + d(t)\mathcal{M}(x_{2n}, x_{2n}, x_{2n}, (2 - \alpha(t))
+ e(t)\mathcal{M}(x_{2n-1}, x_{2n}, x_{2n}, t)
$$

and so

$$
d_{2n}(kt) \geq a(t)d_{2n-1}(t) + b(t)d_{2n}(t) + c(t)d_{2n-1}(t) * d_{2n}(qt)
+ d(t) + e(t)d_{2n-1}(t).
\tag{2.11}
$$

The equality in (2.11) is true because, if $\alpha = 1 + q$ for any $q \in (k, 1)$, then

$$
\mathcal{M}(x_{2n-1}, x_{2n+1}, x_{2n+1}, (1 + q)t)
= \mathcal{M}(x_{2n-1}, x_{2n-1}, x_{2n+1}, (1 + q)t)
\geq \mathcal{M}(x_{2n-1}, x_{2n-1}, x_{2n}, t) * \mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, qt)
= d_{2n-1}(t) * d_{2n}(qt).
$$

Now, we also claim that

$$
d_{2n}(t) \geq d_{2n-1}(t), \quad \forall n \geq 1.
$$

In fact, if $d_{2n}(t) < d_{2n-1}(t)$, then, since

$$
d_{2n}(qt) * d_{2n-1}(t) \geq d_{2n}(qt) * d_{2n}(qt) = d_{2n}(qt)
$$

in (3.11), we have

$$
d_{2n}(kt)
> a(t)d_{2n}(qt) + b(t)d_{2n}(qt) + c(t)d_{2n}(qt) + d(t)d_{2n}(qt) + e(t)d_{2n}(qt)
= d_{2n}(qt)
$$
and so $d_{2n}(kt) > d_{2n}(qt)$, which is a contradiction. Hence $d_{2n}(t) \geq d_{2n-1}(t)$ for all $n \in N$ and $t > 0$. By (2.11), we have

$$
d_{2n}(kt) \\
\geq a(t)d_{2n-1}(qt) + b(t)d_{2n-1}(qt) + c(t)d_{2n-1}(qt) + d_{2n-1}(qt) \\
+ e(t)d_{2n-1}(qt) + e(t)d_{2n-1}(qt) \\
= d_{2n-1}(qt)
$$

and so $d_{2n}(kt) \geq d_{2n-1}(qt)$. Thus we have

$$
d_n(kt) \geq d_{n-1}(qt), \quad \forall n \geq 1.
$$

Therefore, it follows that

$$
\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t) \geq \mathcal{M}(x_{n-1}, x_n, x_n, q/t) \geq \cdots \geq \mathcal{M}(x_0, x_1, x_1, (q/k)^n t).
$$

Hence, by Lemma 1.16, $\{x_n\}$ is a Cauchy sequence in $X$ and, by the completeness of $X$, $\{x_n\}$ converges to a point $x \in X$ and

$$
\lim_{n \to \infty} x_{2n+1} = \lim_{n \to \infty} Sx_{2n} = \lim_{n \to \infty} Tx_{2n+1} = \lim_{n \to \infty} x_{2n+2} = x.
$$

Now, we prove that $Sx = x$. In fact, letting $\alpha = 1$, $x = x$ and $y = x_{2n+1}$ in (2.7), respectively, we obtain

$$
\mathcal{M}(Sx, Tx_{2n+1}, Tx_{2n+1}, kt) \\
\geq a(t)\mathcal{M}(x, Sx, Sx, t) + b(t)\mathcal{M}(x_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, t) \\
+ c(t)\mathcal{M}(x, Tx_{2n+1}, Tx_{2n+1}, t) + d(t)\mathcal{M}(x_{2n+1}, Sx, Sx, t) \\
+ e(t)\mathcal{M}(x, x_{2n+1}, Sx_{2n+1}, t).
$$

If $Sx \neq x$, then, letting $n \to \infty$ in (23.12), we have

$$
\mathcal{M}(Sx, x, x, kt) \\
\geq a(t)\mathcal{M}(x, Sx, Sx, t) + b(t)\mathcal{M}(x, x, x, t) \\
+ c(t)\mathcal{M}(x, x, x, t) + d(t)\mathcal{M}(x, Sx, Sx, t) + e(t)\mathcal{M}(x, x, x, t) \\
\geq \mathcal{M}(x, x, Sx, t),
$$

which is a contradiction. Thus it follows that $Sx = x$.

Similarly, we can prove that $Tx = x$. In fact, again, replacing $x$ by $x_{2n}$ and $y$ by $x$ in (2.7), respectively, for $\alpha = 1$, we have

$$
\mathcal{M}(Sx_{2n}, Tx, Tx, kt) \\
\geq a(t)\mathcal{M}(x_{2n}, Sx_{2n}, Sx_{2n}, t) + b(t)\mathcal{M}(x, Tx, Tx, t) \\
+ c(t)\mathcal{M}(x_{2n}, Tx, Tx, t) + d(t)\mathcal{M}(x, Sx_{2n}, Sx_{2n}, t) \\
+ e(t)\mathcal{M}(x_{2n}, x, x, t)
$$

(2.13)
and so, if $Tx \neq x$, letting $n \to \infty$ in (2.13), we have
\[
\mathcal{M}(x, Tx, Tx, kt) \\
\geq a(t)\mathcal{M}(x, x, x, t) + b(t)\mathcal{M}(x, Tx, t)
\]
\[
+ c(t)\mathcal{M}(x, Tx, Tx, t) + d(t)\mathcal{M}(x, x, t) + e(t)\mathcal{M}(x, x, t)
\]
\[
> \mathcal{M}(x, Tx, Tx, t),
\]
which implies that $Tx = x$. Therefore, $Sx = Tx = x$ and $x$ is a common fixed point of the self-mappings $S$ and $T$ of $X$.

The uniqueness of a common fixed point $x$ is easily verified by using the hypothesis. In fact, if $x'$ be another fixed point of $S$ and $T$, then, for $\alpha = 1$, by (2.7), we have
\[
\mathcal{M}(x, x', x', kt) \\
= \mathcal{M}(Sx, Tx', Tx', kt) \\
\geq a(t)\mathcal{M}(x, Sx, Sx, t) + b(t)\mathcal{M}(x', Tx', Tx', t) \\
+ c(t)\mathcal{M}(x, Tx', Tx', t) + d(t)\mathcal{M}(x', Sx, Sx, t) + e(t)\mathcal{M}(x, x', x', t) \\
> \mathcal{M}(x, x', x', t).
\]
and so $x = x'$.

**Example 2.5.** Let $(X, \mathcal{M}, \ast)$ be an $\mathcal{M}$-fuzzy metric space, where $X = [0, 1]$ with $t$-norm defined $a \ast b = \min\{a, b\}$ for all $a, b \in [0, 1]$ and
\[
\mathcal{M}(x, y, z, t) = \frac{t}{t + |x - y| + |y - z| + |x - z|}, \quad \forall t > 0, x, y, z \in X.
\]
Define the self-mappings $T$ and $S$ on $X$ as follows:
\[
Tx = 1, \quad Sx = \begin{cases} 
1 & \text{if } x \text{ is rational}, \\
0 & \text{if } x \text{ is irrational}.
\end{cases}
\]
We can find the functions $a, b, c, d, e : [0, \infty) \to [0, 1]$ such that $a(t) + b(t) + c(t) + d(t) + e(t) = 1$ and the following inequality holds:
\[
\mathcal{M}(Sx, Ty, Ty, kt) \\
\geq a(t)\mathcal{M}(x, Sx, Sx, t) + b(t)\mathcal{M}(y, Ty, Ty, t) \\
+ c(t)\mathcal{M}(x, Ty, Ty, \alpha t) + d(t)\mathcal{M}(y, Sx, Sx, (2 - \alpha)t) \\
+ e(t)\mathcal{M}(x, y, y, t).
\]
It is easy to see that the all the conditions of Theorem 3.4 hold and $1$ is a unique common fixed point of $S$ and $T$.

From Theorem 2.4, we have the following:
Corollary 2.6. Let \((X, \mathcal{M}, \ast)\) be a complete \(\mathcal{M}\)-fuzzy metric space with \(t \ast t = t\) for all \(t \in [0, 1]\). Let \(S\) be a mapping from \(X\) into itself such that there exists \(k \in (0, 1)\) such that
\[
\mathcal{M}(Sx, Sy, Sy, kt) \\
\geq a(t) \mathcal{M}(x, Sx, Sx, t) + b(t) \mathcal{M}(y, Sx, Sy, t) \\
+ c(t) \mathcal{M}(y, Sy, Sy, \alpha t) + d(t) \mathcal{M}(y, Sx, Sx, (2 - \alpha) t) \\
+ e(t) \mathcal{M}(x, y, y, t)
\]
for all \(x, y \in X\) and \(\alpha \in (0, 2)\), where \(a, b, c, d, e : [0, \infty) \rightarrow [0, 1]\) are five functions such that
\[a(t) + b(t) + c(t) + d(t) + e(t) = 1, \; \forall t \in [0, \infty).\]

Then \(S\) have a unique common fixed point in \(X\).

Corollary 2.7. Let \((X, \mathcal{M}, \ast)\) be a complete \(\mathcal{M}\)-fuzzy metric space with \(t \ast t = t\) for all \(t \in [0, 1]\). Let \(S\) be a mapping from \(X\) into itself such that there exists \(k \in (0, 1)\) such that
\[
\mathcal{M}(Sx, y, y, kt) \\
\geq a(t) \mathcal{M}(x, Sx, Sx, t) + b(t) \mathcal{M}(y, y, y, t) \\
+ c(t) \mathcal{M}(y, Sx, Sx, (2 - \alpha) t) + d(t) \mathcal{M}(y, y, y, t)
\]
for all \(x, y \in X\) and \(\alpha \in (0, 2)\), where \(a, b, c, d : [0, \infty) \rightarrow [0, 1]\) are five functions such that
\[a(t) + b(t) + c(t) + d(t) = 1, \; \forall t \in [0, \infty).\]

Then \(S\) have a unique common fixed point in \(X\).

Corollary 2.8. Let \((X, \mathcal{M}, \ast)\) be a complete \(\mathcal{M}\)-fuzzy metric space with \(t \ast t = t\) for all \(t \in [0, 1]\). Let \(S\) and \(T\) be mappings from \(X\) into itself such that there exists \(k \in (0, 1)\) such that
\[
\mathcal{M}(S^n x, T^n y, T^n y, kt) \\
\geq a(t) \mathcal{M}(x, S^n x, S^n x, t) + b(t) \mathcal{M}(y, T^n y, T^n y, t) \\
+ c(t) \mathcal{M}(x, T^n y, T^n y, (2 - \alpha) t) + d(t) \mathcal{M}(y, S^n x, S^n x, (2 - \alpha) t) \\
+ e(t) \mathcal{M}(x, y, y, t)
\]
for all \(x, y \in X\), \(\alpha \in (0, 2)\) and \(n, m \geq 2\), where \(a, b, c, d, e : [0, \infty) \rightarrow [0, 1]\) are five functions such that
\[a(t) + b(t) + c(t) + d(t) + e(t) = 1, \; \forall t \in [0, \infty).\]
If \(S^n T = T S^n\) and \(T^n S = S T^n\), then \(S\) and \(T\) have a unique common fixed point in \(X\).
Proof. By Theorem 2.4, $S^n$ and $T^m$ have a unique common fixed point in $X$. That is, there exists a unique point $z \in X$ such that $S^n(z) = T^m(z) = z$. Since $S(z) = S(S^n(z)) = S^n(S(z))$ and $S(z) = S(T^m(z)) = T^m(S(z))$, that is, $S(z)$ is fixed point $S^n$ and $T^m$ and so $S(z) = z$. Similarly, $T(z) = z$. This completes the proof. □

Corollary 2.9. Let $(X, \mathcal{M}, \ast)$ be a complete $\mathcal{M}$-fuzzy metric space with $t \ast t = t$ for all $t \in [0, 1]$. Let $S$ and $T$ be mappings from $X$ into itself such that there exists $k \in (0, 1)$ such that

$$\mathcal{M}(Sx, Ty, Ty, kt) \geq a(t)\mathcal{M}(x, Sx, Sx, t) + b(t)\mathcal{M}(y, Ty, Ty, t)$$

for all $x, y \in X$ and $\alpha \in (0, 2)$, where $a, b : [0, \infty) \to [0, 1]$ are two functions such that

$$a(t) + b(t) = 1, \quad \forall t \in [0, \infty).$$

Then $S$ and $T$ have a unique common fixed point in $X$.

References


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