ON \( L \)-FUZZY \( \omega \)-BASICALLY DISCONNECTED SPACES

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Abstract. In this paper \( L \)-fuzzy \( \omega \)-closed and \( L \)-fuzzy \( \omega \)-open sets are introduced. Also a new class of \( L \)-fuzzy topological space called \( L \)-fuzzy \( \omega \)-basically disconnected space is introduced. Several characterizations and some interesting properties are also given.

1. Introduction

The fuzzy concept has invaded almost all branches of Mathematics since the introduction of the concept by Zadeh[14]. Fuzzy sets have applications in many fields such as information [10] and control [11]. The theory of fuzzy topological spaces was introduced and developed by Chang [3] and since then various important notions in classical topology have been extended to fuzzy topological spaces. Rodabaugh [7] discussed normality and the \( L \)-fuzzy unit interval. He [8] also studied fuzzy addition in the \( L \)-fuzzy real line. Hoeche [6] studied the characterizations of \( L \)-topologies by \( L \)-valued neighbourhoods. An \( L \)-fuzzy normal spaces and Tietze extension theorem were discussed by Tomash Kubiak [13]. The concept of \( \omega \)-open set was studied in [9]. The purpose of this paper is to introduce \( L \)-fuzzy \( \omega \)-closed, \( L \)-fuzzy \( \omega \)-open sets and a new class of \( L \)-fuzzy topological spaces called \( L \)-fuzzy \( \omega \)-basically disconnected space. In this connection several characterizations and some interesting properties are also given.

2. Preliminaries

Definition 2.1. ([1]) Let \((X, T)\) be a fuzzy topological space and \( \lambda \) be a fuzzy set in \((X, T)\). \( \lambda \) is called a fuzzy \( G_\delta \)-set if \( \lambda = \bigcap_{i=1}^{\infty} \lambda_i \) where each \( \lambda_i \in T, i \in I \).
Definition 2.2. ([1]) Let $(X, T)$ be a fuzzy topological space and $\lambda$ be a fuzzy set in $(X, T)$. $\lambda$ is called a fuzzy $F_\sigma$-set if $\lambda = \bigvee_{i=1}^{\infty} \lambda_i$ where each $\lambda_i \in T$, $i \in I$.

Definition 2.3. ([2]) Throughout this paper $(L, \leq, ', \wedge, \vee)$ stands for an infinitely distributive lattice with an order reversing involution. Such a lattice being complete has a least element 0 and a greatest element 1. Let $X$ be a non-empty set. An $L$-fuzzy set in $X$ is an element of the set $L^X$ of all functions from $X$ to $L$.

Definition 2.4. The $L$-fuzzy real line $R(L)$ is the set of all monotone decreasing elements $\lambda$ in $L^R$ satisfying $\vee\{\lambda(t)/t \in R\} = 1$ and $\wedge\{\lambda(t)/t \in R\} = 0$, after the identification of $\lambda, \mu \in L^R$ if $\lambda(t-) = \mu(t-)$ and $\lambda(t+) = \mu(t+)$ for all $t \in R$ where $\lambda(t-) = \wedge\{\lambda(s)/s < t\}$ and $\lambda(t+) = \vee\{\lambda(s)/s > t\}$. The natural $L$-fuzzy topology on $R(L)$ is generated from the subbases $\{L_t, R_t/t \in R\}$, where $L_t(\lambda) = \lambda(t-)$ and $R_t(\lambda) = \lambda(t+)$. The $L$-fuzzy unit interval $I(L)$ is a subset of $R(L)$ such that $[\lambda] \in I(L)$ if $\lambda(t) = 1$ for $t < 0$ and $\lambda(t) = 0$ for $t > 1$. It is equipped with the subspace $L$-fuzzy topology.

Definition 2.5. ([13]) If $A \in L^X$ is crisp, then $(A, T_A)$ is an $L$-fuzzy topological space called a crisp subspace of $(X, T)$, where $T_A = \{U/A|U \in T\}$ is called the subspace $L$-fuzzy topology.

Definition 2.6. ([9]) A subset of a topological space $(X, T)$ is called $\omega$-closed in $(X, T)$ if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open in $(X, T)$. A subset $A$ is called $\omega$-open in $(X, T)$ if its complement, $A^C$ is $\omega$-closed.

Definition 2.7. ([12]) Let $(X, T)$ be any fuzzy topological space. $(X, T)$ is called fuzzy basically disconnected if the closure of every fuzzy open $F_\sigma$ set is fuzzy open.

3. Characterizations and properties of $L$-fuzzy $\omega$-basically disconnected spaces

In this section a new class of set called $L$-fuzzy $\omega$-closed set and thereby a new class of space called $L$-fuzzy $\omega$-basically disconnected space is introduced. Some interesting properties and characterizations are also discussed.

Definition 3.1. Let $(X, T)$ be any $L$-fuzzy topological space and $\lambda$ be any $L$-fuzzy set in $(X, T)$. $\lambda$ is called

(a) an $L$-fuzzy $G_\delta$ set if $\lambda = \bigwedge_{i=1}^{\infty} \lambda_i$ where each $\lambda_i$ is $L$-fuzzy open.

(b) an $L$-fuzzy $F_\sigma$ set if $\lambda = \bigvee_{i=1}^{\infty} \lambda_i$ where each $(1 - \lambda_i)$ is $L$-fuzzy open.
Definition 3.2. Let $\lambda$ be any $L$-fuzzy set in the $L$-fuzzy topological space $(X,T)$. Then we define
\[
L - \, \text{int}(\lambda) = \mathcal{V}\{\mu/\mu \leq \lambda \text{ and } \mu \text{ is } L - \text{fuzzy open}\},
\]
\[
L - \, \text{cl}(\lambda) = \mathcal{W}\{\mu/\mu \geq \lambda \text{ and } \mu \text{ is } L - \text{fuzzy closed}\}.
\]

Definition 3.3. Let $\lambda$ be any $L$-fuzzy set in the $L$-fuzzy topological space $(X,T)$. $\lambda$ is called $L$-fuzzy semi-open if $\lambda \leq L - \text{cl}(L - \text{int}(\lambda))$.

Definition 3.4. An $L$-fuzzy set $\lambda$ of an $L$-fuzzy topological space $(X,T)$ is called $L$-fuzzy $\omega$-closed in $(X,T)$ if $L - \text{cl}(\lambda) \leq \mu$ whenever $\lambda \leq \mu$ and $\mu$ is $L$-fuzzy semi-open in $(X,T)$. The complement of $L$-fuzzy $\omega$-closed set is $L$-fuzzy $\omega$-open.

Note 3.1. (a) Let $(X,T)$ be an $L$-fuzzy topological space. An $L$-fuzzy set $\lambda$ in $(X,T)$ which is both $L$-fuzzy $\omega$-open and $L$-fuzzy $F_{\sigma}$ is denoted by $L$-fuzzy $\omega$-open $F_{\sigma}$.

(b) Let $(X,T)$ be an $L$-fuzzy topological space. An $L$-fuzzy set $\lambda$ in $(X,T)$ which is both $L$-fuzzy $\omega$-closed and $L$-fuzzy $G_{\delta}$ is denoted by $L$-fuzzy $\omega$-closed $G_{\delta}$.

Notation 1. An $L$-fuzzy set $\lambda$ which is both $L$-fuzzy $\omega$-open $F_{\sigma}$ and $L$-fuzzy $\omega$-closed $G_{\delta}$ is denoted by $L$-fuzzy $\omega$-$\text{COGF}$.

Definition 3.5. Let $(X,T)$ be an $L$-fuzzy topological space. For any $L$-fuzzy set $\lambda$ in $(X,T)$, $L$-fuzzy $\omega^*$-closure of $\lambda$ (briefly, $L\omega^*$-$\text{cl}(\lambda)$ is defined as $L\omega^*$-$\text{cl}(\lambda) = \mathcal{W}\{\mu/\mu \geq \lambda \text{ and } \mu \text{ is } L$-fuzzy $\omega$-closed $G_{\delta}\}$.

Definition 3.6. Let $(X,T)$ be an $L$-fuzzy topological space. For any $L$-fuzzy set $\lambda$ in $(X,T)$, $L$-fuzzy $\omega^*$-interior of $\lambda$ (briefly, $L\omega^*$-$\text{int}(\lambda)$) is defined as $L\omega^*$-$\text{int}(\lambda) = \mathcal{V}\{\mu/\mu \leq \lambda \text{ and } \mu \text{ is } L$-fuzzy $\omega$-open $F_{\sigma}\}$.

Remark 3.1. Let $(X,T)$ be an $L$-fuzzy topological space. For any $L$-fuzzy set $\lambda$ in $(X,T)$

(a) $1 - L\omega^*$-$\text{int}(\lambda) = L\omega^*$-$\text{cl}(1 - \lambda)$,

(b) $1 - L\omega^*$-$\text{cl}(\lambda) = L\omega^*$-$\text{int}(1 - \lambda)$.

Definition 3.7. Let $(X,T)$ and $(Y,S)$ be any two $L$-fuzzy topological spaces. A mapping $f : (X,T) \to (Y,S)$ is called $L$-fuzzy $\omega^*$-continuous if $f^{-1}(\lambda)$ is $L$-fuzzy $\omega$-closed $G_{\delta}$ in $(X,T)$ for every $L$-fuzzy closed and $L$-fuzzy $G_{\delta}$ set $\lambda$ in $(Y,S)$.

Definition 3.8. Let $(X,T)$ and $(Y,S)$ be any two $L$-fuzzy topological spaces. A mapping $f : (X,T) \to (Y,S)$ is called $L$-fuzzy $\omega^*$-irresolute if the inverse image of every $L$-fuzzy $\omega$-open $F_{\sigma}$ set in $(Y,S)$ is $L$-fuzzy $\omega$-open $F_{\sigma}$ in $(X,T)$.

Definition 3.9. Let $(X,T)$ and $(Y,S)$ be any two $L$-fuzzy topological spaces. A mapping $f : (X,T) \to (Y,S)$ is said to be $L$-fuzzy $\omega^*$-open if the image of every $L$-fuzzy $\omega$-open $F_{\sigma}$ set in $(X,T)$ is $L$-fuzzy $\omega$-open $F_{\sigma}$ in $(Y,S)$. 

Proposition 3.1. Let \((X,T)\) and \((Y,S)\) be any two \(L\)-fuzzy topological spaces. Then \(f: (X,T) \to (Y,S)\) is \(L\)-fuzzy \(\omega^*\)-irresolute iff 
\[f(\omega^*\text{-cl}(\lambda)) \leq \omega^*\text{-cl}(f(\lambda)),\]
for every \(L\)-fuzzy set \(\lambda\) in \((Y,S)\).

Proposition 3.2. Let \((X,T)\) and \((Y,S)\) be any two \(L\)-fuzzy topological spaces and let \(f: (X,T) \to (Y,S)\) be an \(L\)-fuzzy \(\omega^*\)-open surjective function. Then 
\[f^{-1}(\omega^*\text{-cl}(\lambda)) \leq \omega^*\text{-cl}(f^{-1}(\lambda)),\]
for each \(L\)-fuzzy set \(\lambda\) in \((Y,S)\).

Definition 3.10. Let \((X,T)\) be any \(L\)-fuzzy topological space. \((X,T)\) is called \(L\)-fuzzy \(\omega\)-basically disconnected if the \(L\)-fuzzy \(\omega\)-closure of every \(L\)-fuzzy \(\omega\)-open \(F_\sigma\) set is \(L\)-fuzzy \(\omega\)-open \(F_\sigma\).

Proposition 3.3. For an \(L\)-fuzzy topological space \((X,T)\) the following statements are equivalent:

(a) \((X,T)\) is an \(L\)-fuzzy \(\omega\)-basically disconnected space,
(b) For each \(L\)-fuzzy \(\omega\)-closed \(G_\delta\) set \(\lambda\), \(\omega^*\text{-int}(\lambda)\) is \(L\)-fuzzy \(\omega\)-closed \(G_\delta\),
(c) For each \(L\)-fuzzy \(\omega\)-open \(F_\sigma\) set \(\lambda\), \(\omega^*\text{-cl}(\lambda) + \omega^*\text{-cl}(1 - \omega^*\text{-cl}(\lambda)) = 1,
(d) For every pair of \(L\)-fuzzy \(\omega\)-open \(F_\sigma\) sets \(\lambda\) and \(\mu\) such that \(\omega^*\text{-cl}(\lambda) + \omega^*\text{-cl}(\mu) = 1\), we have \(\omega^*\text{-cl}(\lambda) + \omega^*\text{-cl}(\mu) = 1\).

Proof. (a)\(\Rightarrow\)(b) Let \(\lambda\) be any \(L\)-fuzzy \(\omega\)-closed \(G_\delta\) set. Then \(1 - \lambda\) is \(L\)-fuzzy \(\omega\)-open \(F_\sigma\). Now \(\omega^*\text{-cl}(1 - \lambda) = 1 - \omega^*\text{-int}(\lambda)\). By (a), \(\omega^*\text{-cl}(1 - \lambda)\) is \(L\)-fuzzy \(\omega\)-open, which implies that \(\omega^*\text{-int}(\lambda)\) is \(L\)-fuzzy \(\omega\)-closed \(G_\delta\).

(b)\(\Rightarrow\)(c) Let \(\lambda\) be any \(L\)-fuzzy \(\omega\)-open \(F_\sigma\) set. Then
\[\omega^*\text{-cl}(\lambda) + \omega^*\text{-cl}(1 - \omega^*\text{-cl}(\lambda)) = \omega^*\text{-cl}(\lambda) + \omega^*\text{-cl}(\omega^*\text{-int}(1 - \lambda)).\]  
(3.1)
Since \(\lambda\) is \(L\)-fuzzy \(\omega\)-open \(F_\sigma\), \(1 - \lambda\) is \(L\)-fuzzy \(\omega\)-closed \(G_\delta\). Hence by (b), \(\omega^*\text{-int}(1 - \lambda)\) is \(L\)-fuzzy \(\omega\)-closed \(G_\delta\). Therefore by 3.1,
\[\omega^* - cl(\lambda) + \omega^* - cl(1 - \omega^* - cl(\lambda)) = \omega^* - cl(\lambda) + \omega^* - int(1 - \lambda)
= \omega^* - cl(\lambda) + 1 - \omega^* - cl(\lambda)
= 1.\]
Therefore, \(\omega^*\text{-cl}(\lambda) + \omega^*\text{-cl}(1 - \omega^*\text{-cl}(\lambda)) = 1.\)

(c)\(\Rightarrow\)(d) Let \(\lambda\) and \(\mu\) be \(L\)-fuzzy \(\omega\)-open \(F_\sigma\) sets such that
\[\omega^* - cl(\lambda) + \mu = 1.\]  
(3.2)
Then by (c),
\[1 = \omega^* - cl(\lambda) + \omega^* - cl(1 - \omega^* - cl(\lambda)) = \omega^* - cl(\lambda) + \omega^* - cl(\mu).\]
Therefore, \(\omega^*\text{-cl}(\lambda) + \omega^*\text{-cl}(\mu) = 1.\)

(d)\(\Rightarrow\)(a) Let \(\lambda\) be any \(L\)-fuzzy \(\omega\)-open \(F_\sigma\) set. Put \(\mu = 1 - \omega^*\text{-cl}(\lambda)\). Then \(\omega^*\text{-cl}(\lambda) + \mu = 1\). Therefore by (d), \(\omega^*\text{-cl}(\lambda) + \omega^*\text{-cl}(\mu) = 1\). This implies \(\omega^*\text{-cl}(\lambda)\) is \(L\)-fuzzy \(\omega\)-open \(F_\sigma\) and so \((X,T)\) is \(L\)-fuzzy \(\omega\)-basically disconnected. \(\square\)
Proposition 3.4. Let \((X, \mathcal{T})\) be any \(L\)-fuzzy \(\omega\)-basically disconnected space and \((Y, \mathcal{S})\) be any \(L\)-fuzzy topological space. Let \(f : (X, \mathcal{T}) \to (Y, \mathcal{S})\) be \(L\)-fuzzy \(\omega^*\)-irresolute, \(L\)-fuzzy \(\omega^*\)-open and surjective function. Then \((Y, \mathcal{S})\) is \(L\)-fuzzy \(\omega\)-basically disconnected.

Proof. The proof follows from the concepts of \(L\)-fuzzy \(\omega^*\)-irresolute, \(L\)-fuzzy \(\omega^*\)-open maps and by the Propositions 3.1 and 3.2. \(\square\)

Definition 3.11. Let \(\{(X_\alpha, \mathcal{T}_\alpha) / \alpha \in \Delta \}\) be a family of disjoint \(L\)-fuzzy topological spaces. Let \(X = \bigcup_{\alpha \in \Delta} X_\alpha\). Define \(T = \{ \lambda \in L^X / \lambda/X_\alpha \text{ is } L\)-fuzzy \(\omega\)-open \}

\(F_\sigma\) in \((X_\alpha, \mathcal{T}_\alpha)\). Then \((X, T)\) is an \(L\)-fuzzy topological space called the \(L\)-fuzzy topological sum of \(\{(X_\alpha, \mathcal{T}_\alpha) / \alpha \in \Delta \}\).

Proposition 3.5. Let \(\{(X_\alpha, \mathcal{T}_\alpha) / \alpha \in \Delta \}\) be a family of disjoint \(L\)-fuzzy \(\omega\)-basically disconnected spaces and let \((X, T)\) be their \(L\)-fuzzy topological sum. Then \((X, T)\) is \(L\)-fuzzy \(\omega\)-basically disconnected.

Proof. Let \(\lambda\) be an \(L\)-fuzzy \(\omega\)-open \(F_\sigma\) set in \((X, T)\). Then \(\lambda/X_\alpha\) is \(L\)-fuzzy \(\omega\)-open \(F_\sigma\) in \((X_\alpha, \mathcal{T}_\alpha)\). Since \((X_\alpha, \mathcal{T}_\alpha)\) is \(L\)-fuzzy \(\omega\)-basically disconnected, \(L^\omega^*\)-\(cl_{X_\alpha}(\lambda/X_\alpha)\) is \(L\)-fuzzy \(\omega\)-open \(F_\sigma\) in \((X_\alpha, \mathcal{T}_\alpha)\). Now \(L^\omega^*\)-\(cl_X(\lambda/X_\alpha) = L^\omega^*\)-\(cl_{X_\alpha}(\lambda/X_\alpha)\), which implies that \(L^\omega^*\)-\(cl_X(\lambda)\) is \(L\)-fuzzy \(\omega\)-open \(F_\sigma\) in \((X, T)\). Therefore \((X, T)\) is \(L\)-fuzzy \(\omega\)-basically disconnected. \(\square\)

Definition 3.12. Let \((X, \mathcal{T})\) be an \(L\)-fuzzy topological space. A mapping \(f : X \to R(L)\) is called lower (resp. upper) \(L\)-fuzzy \(\omega^*\)-continuous if \(f^{-1}(R_t)\) (resp. \(f^{-1}(L_t)\)) is \(L\)-fuzzy \(\omega\)-open \(F_\sigma\) (resp. \(L\)-fuzzy \(\omega\)-open \(F_\sigma/L\)-fuzzy \(\omega\)-closed \(G_\delta\)), for each \(t \in R\).

Proposition 3.6. Let \((X, \mathcal{T})\) be an \(L\)-fuzzy topological space. Then \((X, T)\) is \(L\)-fuzzy \(\omega\)-basically disconnected iff for all \(L\)-fuzzy \(\omega\)-open \(F_\sigma\) set \(\lambda\) and an \(L\)-fuzzy \(\omega\)-closed \(G_\delta\) set \(\mu\) such that \(\lambda \leq \mu\), \(L^\omega^*\)-\(cl(\lambda) \leq L^\omega^*\)-\(int(\mu)\).

Proof. Let \(\lambda\) be \(L\)-fuzzy \(\omega\)-open \(F_\sigma\) and \(\mu\) be \(L\)-fuzzy \(\omega\)-closed \(G_\delta\) with \(\lambda \leq \mu\). Then by (b) of Proposition 3.3, \(L^\omega^*-\text{int}(\mu)\) is \(L\)-fuzzy \(\omega\)-closed \(G_\delta\). Also since \(\lambda\) is \(L\)-fuzzy \(\omega\)-open \(F_\sigma\), \(L^\omega^*-\text{cl}(\lambda) \leq L^\omega^*\)-\(int(\mu)\). Conversely let \(\mu\) be any \(L\)-fuzzy \(\omega\)-closed \(G_\delta\) set. Then \(L^\omega^*-\text{int}(\mu)\) is \(L\)-fuzzy \(\omega\)-open \(F_\sigma\) in \((X, T)\) and \(L^\omega^*-\text{int}(\mu) \leq \mu\). Therefore by assumption, \(L^\omega^*-\text{cl}(L^\omega^*-\text{int}(\mu)) \leq L^\omega^*-\text{int}(\mu)\). This implies that \(L^\omega^*-\text{int}(\mu)\) is \(L\)-fuzzy \(\omega\)-closed \(G_\delta\). Hence by (b) of Proposition 3.3, it follows that \((X, T)\) is \(L\)-fuzzy \(\omega\)-basically disconnected. \(\square\)

Remark 3.2. Let \((X, \mathcal{T})\) be an \(L\)-fuzzy \(\omega\)-basically disconnected space. Let \(\{\lambda_i, 1 - \mu_i / i \in N\}\) be a collection such that \(\lambda_i\)'s, are \(L\)-fuzzy \(\omega\)-open \(F_\sigma\) and \(\mu_i\)'s are \(L\)-fuzzy \(\omega\)-closed \(G_\delta\) and let \(\lambda, \mu\) are \(L\)-fuzzy \(\omega\)-COGF. If \(\lambda_i \leq \lambda \leq \mu_j\) and \(\lambda_i \leq \mu \leq \mu_j\) for all \(i, j \in N\), then there exists an \(L\)-fuzzy \(\omega\)-COGF set \(\gamma\) such that \(L^\omega^*-\text{cl}(\lambda_i) \leq \gamma \leq L^\omega^*-\text{int}(\mu_j)\), for all \(i, j \in N\).
Proposition 3.7. Let \((X, T)\) be an \(L\)-fuzzy \(\omega\)-basically disconnected space. Let \(\{\lambda_q\}_{q \in Q}\) and \(\{\mu_r\}_{r \in Q}\) be monotone increasing collections of \(L\)-fuzzy \(\omega\)-open \(F_\sigma\) sets and \(L\)-fuzzy \(\omega\)-closed \(G_\delta\) sets of \((X, T)\) and suppose that \(\lambda_{q_1} \leq \mu_{q_2}\) whenever \(q_1 < q_2\) \((Q\) is the set of all rational numbers). Then there exists a monotone increasing collection \(\{\gamma_r\}_{r \in Q}\) of \(L\)-fuzzy \(\omega\)-COGF sets of \((X, T)\) such that \(L\omega^*\text{-cl}(\lambda_{q_1}) \leq \gamma_{q_2}\) and \(\gamma_{q_1} \leq L\omega^*\text{-int}(\mu_{q_2})\) whenever \(q_1 < q_2\).

Proposition 3.8. Let \((X, T)\) be any \(L\)-fuzzy topological space; let \(\lambda \in L^X\) and let \(f : X \to R(L)\) be such that

\[
\begin{align*}
\lambda(x), & \quad \text{if } 0 \leq t \leq 1 \\
0, & \quad \text{if } t > 0.
\end{align*}
\]

for all \(x \in X\). Then \(f\) is lower (resp. upper) \(L\)-fuzzy \(\omega^*\)-continuous iff \(\lambda\) is \(L\)-fuzzy \(\omega\)-open \(F_\sigma\) (resp. \(L\)-fuzzy \(\omega\)-open \(F_\sigma\)/\(L\)-fuzzy \(\omega\)-closed \(G_\delta\)).

Remark 3.2, Proposition 3.7 and Proposition 3.8 can be established by the concepts of \(L\)-fuzzy \(\omega\)-COGF set, \(L\)-fuzzy \(\omega^*\)-interior, \(L\)-fuzzy \(\omega^*\)-closure and the lemmas given in [13] with some slight suitable modifications.

Definition 3.13. The characteristic function of \(\lambda \in L^X\) is the map \(\chi_\lambda : X \to I(L)\) defined by \(\chi_\lambda(x) = (\lambda(x)), x \in X\).

Proposition 3.9. Let \((X, T)\) be an \(L\)-fuzzy topological space and let \(\lambda \in L^X\). Then \(\chi_\lambda\) is lower (resp. upper) \(L\)-fuzzy \(\omega^*\)-continuous iff \(\lambda\) is \(L\)-fuzzy \(\omega\)-open \(F_\sigma\) (resp. \(L\)-fuzzy \(\omega\)-open \(F_\sigma\)/\(L\)-fuzzy \(\omega\)-closed \(G_\delta\)).

Proof. The proof follows from Proposition 3.8.

Definition 3.14. Let \((X, T)\) and \((Y, S)\) be any two \(L\)-fuzzy topological spaces. A mapping \(f : (X, T) \to (Y, S)\) is called strong \(F_\sigma\) \(L\)-fuzzy \(\omega^*\)-continuous if \(f^{-1}(\lambda)\) is \(L\)-fuzzy \(\omega\)-COGF set of \((X, T)\), for every \(L\)-fuzzy \(\omega\)-open \(F_\sigma\) set \(\lambda\) of \((Y, S)\).

Proposition 3.10. Let \((X, T)\) be an \(L\)-fuzzy topological space. Then the following statements are equivalent:

(a) \((X, T)\) is an \(L\)-fuzzy \(\omega\)-basically disconnected space.
(b) If \(g, h : X \to R(L)\) where \(g\) is lower \(L\)-fuzzy \(\omega^*\)-continuous, \(h\) is upper \(L\)-fuzzy \(\omega^*\)-continuous, then there exists \(f \in C_{F_\sigma}\) \(L\omega(X)\) such that \(g \leq f \leq h\). \([C_{F_\sigma}\) \(L\omega(X)\) = collection of all strong \(F_\sigma\) \(L\)-fuzzy \(\omega^*\)-continuous function on \(X\) with values in \(R(L)\)].
(c) If \(\lambda\) is \(L\)-fuzzy \(\omega\)-closed \(G_\delta\) and \(\mu\) is \(L\)-fuzzy \(\omega\)-open \(F_\sigma\) sets such that \(\mu \leq \lambda\), then there exists a strong \(F_\sigma\) \(L\)-fuzzy \(\omega^*\)-continuous function \(f : X \to I(L)\) such that \(\mu \leq (1 - \lambda_1)f \leq R_0f \leq \lambda\).

Proof. (a) \(\Rightarrow\) (b) can be established by the concept of \(L\)-fuzzy \(\omega\)-COGF set and the theorem 3.7 of Kubiak [13] with some slight suitable modifications.
(b)⇒ (c) Suppose λ is L-fuzzy ω-closed $G_δ$ and μ is L-fuzzy ω-open $F_σ$ such that μ ≤ λ. Then $χ_μ \leq χ_λ$ where $χ_μ$ and $χ_λ$ are lower and upper L-fuzzy $ω^*$-continuous respectively. Hence by (b), there exists a strong $F_δ$ L-fuzzy $ω^*$-continuous function $f : X \rightarrow R(L)$ such that, $χ_μ \leq f \leq χ_λ$. Clearly $f(x) \in I(L)$, for all $x \in X$ and $μ = (1 - L_1)χ_μ \leq (1 - L_1)f \leq R_0f \leq R_0χ_λ = λ$. Therefore $μ \leq (1L_1)f \leq R_0f \leq λ$.

(c)⇒ (a) $(1 - L_1)f$ and $R_0f$ are L-fuzzy $ω$-COGF sets. By Proposition 3.6, $(X, T)$ is an L-fuzzy ω-basically disconnected space.

**Proposition 3.11.** Let $(X, T)$ be an L-fuzzy ω-basically disconnected space and let $A \subseteq X$ be such that $χ_A$ is L-fuzzy ω*-open. Let $f : (A, T/A) \rightarrow I(L)$ be strong $F_σ$ L-fuzzy ω*-continuous. Then $f$ has a strong $F_σ$ L-fuzzy ω*-continuous extension over $(X, T)$.

**Proof.** Let $g, h : X \rightarrow I(L)$ be such that $g = f = h$ on $A$ and $g(x) = (0)$, $h(x) = (1)$ if $x \notin A$. We now have

$$R_tg = \begin{cases} μ_t \wedge χ_A, & \text{if } t \geq 0 \\ 1, & \text{if } t < 0 \end{cases}$$

where $μ_t$ is L-fuzzy ω-open $F_σ$ and is such that $μ_t/A = R_tf$ and

$$L_th = \begin{cases} λ_t \wedge χ_A, & \text{if } t \leq 1 \\ 1, & \text{if } t > 1 \end{cases}$$

where $λ_t$ is L-fuzzy ω-open $F_σ/L$-fuzzy ω*-closed $G_δ$ and is such that $λ_t/A = L_tf$. Thus $g$ is lower L-fuzzy ω*-continuous $h$ is upper L-fuzzy ω*-continuous and $g \leq h$. By Proposition 3.10, there is a strong $F_σ$ L-fuzzy ω*-continuous function $F : X \rightarrow I(L)$ such that $g \leq F \leq h$. Hence $F \equiv f$ on $A$.

**Acknowledgement**

The authors express their sincere thanks to the referee for his valuable comments regarding the improvement of the paper.

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