EXTENDING THE APPLICATION OF THE SHADOWING LEMMA FOR OPERATORS WITH CHAOTIC BEHAVIOUR

IOANNIS K. ARGYROS

Abstract. We use a weaker version of the celebrated Newton–Kantorovich theorem [3] reported by us in [1] to find solutions of discrete dynamical systems involving operators with chaotic behavior. Our results are obtained by extending the application of the shadowing lemma [4], and are given under the same computational cost as before [4]–[6].

1. Introduction

It is well known that complicated behaviour of dynamical systems can easily be detected via numerical experiments. However, it is very difficult to prove mathematically in general that a given system behaves chaotically.

Several authors have worked on various aspects of this problem, see, e.g., [4]–[6], and the references therein. In particular the shadowing lemma [4, p. 1684] proved via the celebrated Newton–Kantorovich theorem [3] was used in [4] to present a computer-assisted method that allows us to prove that a discrete dynamical system admits the shift operator as a subsystem. Motivated by this work and using a weaker version of the Newton–Kantorovich theorem reported by us in [1], [2] (see Theorem 2.1 that follows) we show that it is possible to weaken the shadowing Lemma on on which the work in [4] is based. In particular we show that under weaker hypotheses and the same computational cost a larger upper bound on the crucial norm of operator \( M^{-1} \) (see (7)) is found and the information on location of the shadowing orbit is more precise. Other advantages have already been reported in [1]. Clearly this approach widens the applicability of the shadowing lemma.

2. The shadowing lemma

We need the definitions: Let \( D \subseteq \mathbb{R}^k \) be an open subset of \( \mathbb{R}^k \) (\( k \) a natural number), and let \( f : D \to D \) be an injective operator. Then the pair \((D, f)\) is a discrete dynamical system. Denote by \( S = l^\infty(\mathbb{Z}, \mathbb{R}^k) \) the space of \( \mathbb{R}^k \).
valued bounded sequences \( x = \{x_n\} \) with norm \( \|x\| = \sup_{n \in \mathbb{Z}} |x_n| \). Here we use the Euclidean norm in \( \mathbb{R}^k \) and denote it by \( |\cdot| \), omitting the index 2. A \( \delta_0 \)-pseudo-orbit is a sequence \( y = \{y_n\} \in D^\mathbb{Z} \) with \( |y_{n+1} - f(y_n)| \leq \delta_0 \) \( (n \in \mathbb{Z}) \).

A \( r \)-shadowing orbit \( x = \{x_n\} \) of a \( \delta_0 \)-pseudo-orbit \( y \) is an orbit of \((D, f)\) with \( |y_n - x_n| \leq 2 \) \( (n \in \mathbb{Z}) \).

We need the following semilocal convergence theorem for Newton method [1, page 132, Case 3 for \( \delta = \delta_0 \)].

**Theorem 2.1.** Let \( F : D \subseteq X \to Y \) be a Fréchet differentiable operator. Assume there exist \( x_0 \in D \), positive constant \( \eta, \beta, L_0 \) and \( L \) such that:

\[
\|F'(x_0)^{-1} - 1\| \leq \beta, \tag{1}
\]

\[
\|F'(x_0)^{-1} F(x_0)\| \leq \eta, \tag{2}
\]

\[
\|F'(x) - F'(y)\| \leq L \|x - y\|, \text{ for all } x, y \in D, \tag{3}
\]

\[
\|F'(x) - F'(x_0)\| \leq L_0 \|x - x_0\|, \text{ for all } x \in D, \tag{4}
\]

\[
h_A = \beta L_1 \eta \leq 1, \tag{5}
\]

and

\[
\tilde{U}(x_0, s^*) = \{x \in X : \|x - x_0\| \leq s^*\} \subseteq D,
\]

where

\[
s^* = \lim_{n \to \infty} s_n,
\]

\[
s_0 = 0, s_1 = \eta, s_{n+2} = s_{n+1} + \frac{L(s_{n+1} - s_n)}{2(1 - L_0 s_{n+1})} \quad (n \geq 0),
\]

\[
L_1 = \frac{1}{4} (L + 4 L_0 + \sqrt{L^2 + 8 L_0 L}).
\]

Then, sequence \( \{y_n\} \) \( (n \geq 0) \) generated by Newton’s method

\[
y_{n+1} = y_n - F'(y_n)^{-1} F(y_n) \quad (n \geq 0)
\]

is well defined, remains in \( \tilde{U}(x_0, s^*) \) for all \( n \geq 0 \) and converges to a unique solution \( y^* \in \tilde{U}(x_0, s^*) \), so that estimates

\[
\|y_{n+1} - y_n\| \leq s_{n+1} - s_n
\]

and

\[
\|y_n - y^*\| \leq s^* - s_n \leq 2\eta - s_n
\]

hold for all \( n \geq 0 \).

Moreover \( y^* \) is the unique solution of equation \( F(y) = 0 \) in \( U(x_0, R) \) provided that

\[
L_0 (s^* + R) \leq 2
\]

and

\[
U(x_0, R) \subseteq D.
\]
The advantages of Theorem 2.1 over the Newton-Kantorovich theorem [3] have been explained in detail in [1], [2].

From now on we set $X = Y = \mathbb{R}^k$.

Sufficient conditions for a $\delta_0$-pseudo-orbit $y$ to admit a unique $r$-shadowing orbit are given in the following main result.

**Theorem 2.2.** (Weak version of the shadowing lemma) Let $D \subseteq \mathbb{R}^k$ be open, $f \in C^1_{\text{Lip}}(D,D)$ be injective, $y = \{y_n\} \in D^\mathbb{Z}$ be a given sequence, $\{A_n\}$ be a bounded sequence of $k \times k$ matrices and let $\delta_0, \delta, \ell_0, \ell$ be positive constants. Assume that for the operator $M : S \to S$ with $\{Mz\}_n = z_{n+1} - Az_n$ (6)
is invertible and

$$\|M^{-1}\| \leq a = \frac{1}{\delta + \sqrt{\ell_1 \delta_0}}, \quad (7)$$

where

$$\ell_1 = \frac{1}{4} (\ell + 4 \ell_0 + \sqrt{\ell^2 + 8 \ell_0 \ell}).$$

Then, the numbers $t^*, R$ given by

$$t^* = \lim_{n \to \infty} t_n \quad \quad (8)$$

and

$$R = \frac{2}{\ell_0} - t^* \quad \quad (9)$$
satisfy $0 < t^* \leq R$, where sequence $\{t_n\}$ is given by

$$t_0 = 0, t_1 = \eta, t_{n+2} = t_{n+1} + \frac{\ell (t_{n+1} - t_n)^2}{2(1 - \ell_0 t_{n+1})} \quad \quad (n \geq 0) \quad \quad (10)$$

and

$$\eta = \frac{\delta_0}{\|M^{-1}\| - \delta}. \quad \quad (11)$$

Let $r \in [t^*, R]$. Moreover, assume that

$$\bigcup_{n \in \mathbb{Z}} U(y_n, r) \subseteq D \quad \quad (12)$$

and for every $n \in \mathbb{Z}$

$$|y_{n+1} - f(y_n)| \leq \delta_0, \quad \quad (13)$$

$$|A_n - Df(y_n)| \leq \delta, \quad \quad (14)$$

$$|F'(u) - F'(0)| \leq \ell_0 |u| \quad \quad (15)$$

and

$$|F'(u) - F'(v)| \leq \ell |u - v|, \quad \quad (16)$$

for all $u, v \in U(y_n, r)$. 

Then there is a unique $t^*$-shadowing orbit $x^* = \{x_n\}$ of $y$. Moreover, there is no orbit $\bar{x}$ other than $x^*$ such that
\[\|\bar{x} - y\| \leq r.\]  

Proof. We shall solve the difference equation
\[x_{n+1} = f(x_n) \quad (n \geq 0)\]  
provided that $x_n$ is close to $y_n$. Setting
\[x_n = y_n + z_n\]  
and
\[g_n(z_n) = f(z_n + y_n) - A_n z_n - y_{n+1}\]  
we can have
\[z_{n+1} = A_n z_n + g_n(z_n).\]  
Define $D_0 = \{z = \{z_n\} : \|z\| \leq 2\}$ and nonlinear operator $G : D_0 \to S$, by
\[(G(z))_n = g_n(z_n).\]  
Operator $G$ can naturally be extended to a neighborhood of $D_0$. Equation (21) can be rewritten as
\[F(x) = Mx - G(x) = 0,\]  
where $F$ is an operator from $D_0$ into $S$.

We will show the existence and uniqueness of a solution $x^* = \{x_n\}$ ($n \geq 0$) of equation (23) with $\|x^*\| \leq r$ using Theorem 2.1. Clearly we need to express $\eta, L_0, L$ and $\beta$ in terms of $\|M^{-1}\|, \delta_0, \delta, \ell_0$ and $\ell$.

(i) $\|F'(0)^{-1}F(0)\| \leq \eta$.

Using (13), (14) and (20) we get $\|F(0)\| \leq \delta_0$ and $\|G'(0)\| \leq \delta$, since $[G'(0)(w)]_n = (F'(y_n) - A_n)w_n$.

By (7) and the Banach lemma on invertible operators [3] we get $F'(0)^{-1}$ exists and
\[\|F'(0)^{-1}\| \leq \left(\frac{1}{\|M^{-1}\|} - \delta\right)^{-1}.\]  
That is, $\eta$ can be given by (11).

(ii) $\|F'(0)^{-1}\| \leq \beta$.

By (24) we can set
\[\beta = \left(\frac{1}{\|M^{-1}\|} - \delta\right)^{-1}.\]  

(iii) $\|F'(u) - F'(v)\| \leq L \|u - v\|$.

We can have using (16)
\[|(F'(u) - F'(v))(w)_n| = |(F'(y_n + u_n) - F'(y_n + v_n))w_n| \leq \ell |u_n - v_n| |w_n| .\]  

Hence we can set $L = \ell$. 

(iv) \( \|F'(u) - F'(0)\| \leq L_0 \|u\| \).

By (17) we get
\[
|(F'(u) - F'(0))(w)_n| = |(F'(y_n + u_n) - F'(y_n + 0))w_n| 
\leq \ell_0 |u_n| |w_n|.
\]
That is, we can take \( L_0 = \ell_0 \).

Crucial condition (5) is satisfied by (7) and with the above choices of \( \eta, \beta, L \) and \( L_0 \).

Therefore the claims of Theorem 2.2 follow immediately from the conclusions of Theorem 2.1.

That completes the proof of the theorem. \( \Box \)

**Remark 1.** In general
\[
\ell_0 \leq \ell
\]
holds and \( \ell / \ell_0 \) can be arbitrarily large [1]. If \( \ell_0 = \ell \), Theorem 2.2 reduces to Theorem 1 in [4, p. 1684]. Otherwise our Theorem 2.2 improves Theorem 1 in [4]. Indeed, the upper bound in [4, p. 1684] is given by
\[
\|M^{-1}\| \leq b = \frac{1}{\delta + \sqrt{2\delta_0}}.
\]
By comparing (7) with (29) we deduce
\[
b < a
\]
(if \( \ell_0 < \ell \)).

That is, we have justified the claims made in the introduction.

**References**


Ioannis K. Argyros
Cameron University, Department of Mathematics Sciences, Lawton, OK 73505, USA

E-mail address: iargyros@cameron.edu