ON PROJECTIVELY FLAT FINSLER SPACE WITH AN APPROXIMATE INFINITE SERIES \((\alpha, \beta)\)-METRIC

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Abstract. We introduced a Finsler space \(F^n\) with an approximate infinite series \((\alpha, \beta)\)-metric \(L(\alpha, \beta) = \beta \sum_{k=0}^{r} \left(\frac{\alpha}{\beta}\right)^k\), where \(\alpha < \beta\) and investigated it with respect to Berwald space ([12]) and Douglas space ([13]). The present paper is devoted to finding the condition that is projectively flat on a Finsler space \(F^n\) with an approximate infinite series \((\alpha, \beta)\)-metric above.

1. Introduction

A Finsler metric function \(L\) in a differentiable manifold \(M^n\) is called an \((\alpha, \beta)\)-metric, if \(L\) is a positively homogeneous function of degree one of a Riemannian metric \(\alpha = (a_{ij}y^iy^j)^{1/2}\) and a non-vanishing 1-form \(\beta = b_iy^i\) on \(M^n\). An infinite series \((\alpha, \beta)\)-metric \(L(\alpha, \beta) = \beta^2/(\beta - \alpha)\) is expressed as an infinite series form, where \(\alpha < \beta\). We introduced an approximate infinite series \((\alpha, \beta)\)-metric \(L(\alpha, \beta) = \beta \sum_{k=0}^{r} \left(\frac{\alpha}{\beta}\right)^k\) as the \(r\)-th finite series \((\alpha, \beta)\)-metric form and investigated it in [12] and [13].

A change \(L \longrightarrow \overline{L}\) of a Finsler metric on a same underlying manifold \(M^n\) is called projective, if any geodesic in \((M^n, L)\) remains to be a geodesic in \((M^n, \overline{L})\) and vice versa. A Finsler space is called projective flat if it is projective to a locally Minkowski space. The condition for a Finsler space with \((\alpha, \beta)\)-metric to be projectively flat was studied by M. Matsumoto [7]. Aikou, Hashiguchi and Yamauchi [2] give interesting results on the projective flatness of Matsumoto space.

The purpose of the present paper is to find condition that is projectively flat on a Finsler space with an approximate infinite series \((\alpha, \beta)\)-metric.
2. Preliminaries

In a Finsler space \((M^n, L)\), the metric

\[
L(\alpha, \beta) = \beta \left\{ \sum_{k=0}^{r} \left( \frac{\alpha}{\beta} \right)^k \right\}
\]  

(2.1)

is called an approximate infinite series \((\alpha, \beta)\)-metric. The infinite series \((\alpha, \beta)\)-metric is expressed as

\[
\lim_{r \to \infty} \beta \left\{ \sum_{k=0}^{r} \left( \frac{\alpha}{\beta} \right)^k \right\} = \frac{\beta^2}{\beta - \alpha}
\]

for \(\alpha < \beta\) in (2.1). If \(r = 0\), then \(L = \beta\) is a non-vanishing 1-form. If \(r = 1\), then \(L = \alpha + \beta\) is a Randers metric. The condition for a Randers space to be projectively flat was given by Hashiguchi-Ichijō [4], and M. Matsumoto [7]. Therefore in this paper, we suppose that \(r > 1\).

Let \(\gamma^i_{jk}\) be the Christoffel symbols with respect to \(\alpha\) and denote by \((;\)\) the covariant differentiation with respect to \(\gamma^i_{jk}\). From the differential 1-form \(\beta(x,y) = \beta_i(x) y^i\) we define

\[
2r_{ij} = b_{ij} + b_{ji}, \quad 2s_{ij} = b_{ij} - b_{ji} = (\partial_j b_i - \partial_i b_j),
\]

\[
s^i_j = a^{ir} s_{rj}, \quad b^i = a^{ir} b_r, \quad b^2 = a^{sr} b_r b_s.
\]

We shall denote the homogeneous polynomials in \((y^i)\) of degree \(r\) by \(hp_r\) for brevity and the subscription 0 means contraction by \(y^i\), for instance, \(\mu_0 = \mu_i y^i\). In the following we denote \(L_\alpha = \partial_\alpha L, \quad L_\beta = \partial_\beta L, \quad L_{\alpha\alpha} = \partial_\alpha \partial_\alpha L\).

Now the following Matsumoto’s theorem [7] is well-known.

**Theorem 2.1.** A Finsler space \((M^n, L)\) with an \((\alpha, \beta)\)-metric \(L(\alpha, \beta)\) is projectively flat if and only if for any point of space \(M^n\) there exist local coordinate neighborhoods containing the point such that \(\gamma^i_{jk}\) satisfies:

\[
(\gamma^i_{00} - 2\alpha \beta s_0)/\alpha^2 + (\alpha L_\beta/L_\alpha) s_0 + (L_{\alpha\alpha}/L_{\alpha})(C + \alpha r_{00}/2\beta)(\alpha^2 b^2/\beta - y^i) = 0,
\]

(2.2)

where \(C\) is given by

\[
C + (\alpha^2 L_\beta/L_\alpha) s_0 + (\alpha L_{\alpha\alpha}/\beta^2 L_\alpha)(\alpha^2 b^2 - \beta^2)(C + \alpha r_{00}/2\beta) = 0.
\]

(2.3)

The equation (2.3) is rewritten in the form

\[
(C + \alpha r_{00}/2\beta)(1 + (\alpha L_{\alpha\alpha}/\beta^2 L_\alpha)(\alpha^2 b^2 - \beta^2)) - (\alpha/2\beta) r_{00} - (2\alpha L_\beta/L_\alpha) s_0 = 0,
\]

(2.4)

that is,

\[
C + \alpha r_{00}/2\beta = \frac{\alpha \beta (r_{00} L_\alpha - 2\alpha L_\beta s_0)}{2(\beta^2 L_\alpha + \alpha L_{\alpha\alpha}(\alpha^2 b^2 - \beta^2))}.
\]
Therefore (2.2) leads us to
\[
\{ L_\alpha (\alpha^2 \gamma_0^i 0 - \gamma_000 y^i) + 2\alpha^3 L_\beta s^i_0 \} \{ \beta^2 L_\alpha + \alpha L_\alpha \alpha (\alpha^2 b^2 - \beta^2) \}
+ \alpha^3 L_\alpha (r_{00} L_\alpha - 2\alpha L_\beta s_0) (\alpha^2 b^2 - \beta y^i) = 0.
\]

(2.5)

We shall state the following lemma for later:

**Lemma 2.2.** ([3]) If \( \alpha^2 \equiv 0 \pmod{\beta} \), that is, \( a_{ij}(x) y^i y^j \) contains \( b_i(x) y^i \) as a factor, then the dimension is equal to two and \( b^2 \) vanishes. In this case we have \( \delta = d_i(x) y^i \) satisfying \( \alpha^2 = \beta \delta \) and \( d_i b^i = 2 \).

### 3. Projectively flat space

In the present section, we find the condition that a Finsler space \( F^n \) with the \( r \)-th approximate infinite series \((\alpha, \beta)\)-metric (2.1) be projectively flat. In the \( n \)-dimensional Finsler space \( F^n \) with the approximate infinite series \((\alpha, \beta)\)-metric (2.1), we have
\[
L_\alpha = \sum_{k=0}^{r} k \left( \frac{\alpha}{\beta} \right)^{k-1}, \quad L_\beta = -\sum_{k=0}^{r} (k-1) \left( \frac{\alpha}{\beta} \right)^{k},
\]
\[
L_{\alpha \alpha} = \frac{1}{\beta} \sum_{k=0}^{r} k(k-1) \left( \frac{\alpha}{\beta} \right)^{k-2}.
\]

(3.1)

Here, by means of (2.5) and (3.1) we have
\[
\begin{align*}
&\left\{ \sum_{k=0}^{r} k \left( \frac{\alpha}{\beta} \right)^{k-1} (\alpha^2 \gamma_0^i 0 - \gamma_000 y^i) - 2\alpha^3 s^i_0 \sum_{k=0}^{r} (k-1) \left( \frac{\alpha}{\beta} \right)^{k} \right\} \\
&\quad \times \left\{ \beta^2 \sum_{k=0}^{r} k \left( \frac{\alpha}{\beta} \right)^{k-1} + (\alpha^2 b^2 - \beta^2) \sum_{k=0}^{r} k(k-1) \left( \frac{\alpha}{\beta} \right)^{k-1} \right\} \\
&\quad + \alpha^2 \sum_{k=0}^{r} k(k-1) \left( \frac{\alpha}{\beta} \right)^{k-1} \left\{ r_{00} \sum_{k=0}^{r} k \left( \frac{\alpha}{\beta} \right)^{k-1} \right\} \\
&\quad + 2\alpha s_0 \sum_{k=0}^{r} (k-1) \left( \frac{\alpha}{\beta} \right)^{k} \times (\alpha^2 b^2 - \beta y^i) = 0.
\end{align*}
\]

(3.2)

We shall divide our consideration in two cases of which \( r \) is even or odd.

(1) Case of \( r = 2h \), where \( h \) is a positive integer.
When \( r = 2h \), we have

\[
\sum_{k=0}^{r} k \left( \frac{\alpha}{\beta} \right)^k = \frac{\beta}{\beta^{2h}} \sum_{k=0}^{2h} k \alpha^{k-1} \beta^{2h-k},
\]

\[
\sum_{k=0}^{r} (k-1) \left( \frac{\alpha}{\beta} \right)^k = \frac{1}{\beta^{2h}} \sum_{k=0}^{2h} (k-1) \alpha^k \beta^{2h-k},
\]

(3.3)

\[
\sum_{k=0}^{r} k(k-1) \left( \frac{\alpha}{\beta} \right)^{k-1} = \frac{1}{\beta^{2h-1}} \sum_{k=0}^{2h} k(k-1) \alpha^{k-1} \beta^{2h-k}.
\]

Separating the rational and irrational parts in \( y_i \) with respect to (3.3), we obtain

\[
\begin{align*}
2h \sum_{k=0}^{h-1} k \alpha^{k-1} \beta^{2h-k} & = \sum_{k=0}^{h-1} (2k+1) \alpha^{2k} \beta^{2h-2k-1} + \alpha \sum_{k=1}^{h} 2k \alpha^{2k-2} \beta^{2h-2k} \\
& = M + \alpha K,
\end{align*}
\]

(3.4)

\[
\begin{align*}
2h \sum_{k=0}^{h-1} (k-1) \alpha^k \beta^{2h-k} & = \sum_{k=0}^{h} (2k-1) \alpha^{2k} \beta^{2h-2k} \\
& + \alpha^3 \sum_{k=1}^{h-1} 2k \alpha^{2k-2} \beta^{2h-2k-1} \\
& = L + \alpha^3 N,
\end{align*}
\]

\[
\begin{align*}
2h \sum_{k=0}^{h-1} k(k-1) \alpha^{k-1} \beta^{2h-k} & = \alpha^2 \sum_{k=1}^{h-1} (2k+1)2k \alpha^{2k-2} \beta^{2h-2k-1} \\
& + \alpha \sum_{k=1}^{h} 2k(2k-1) \alpha^{2k-2} \beta^{2h-2k} \\
& = \alpha^2 Q + \alpha P,
\end{align*}
\]

where

\[
K = \sum_{k=1}^{h} 2k \alpha^{2k-2} \beta^{2h-2k}, \quad L = \sum_{k=0}^{h} (2k-1) \alpha^{2k} \beta^{2h-2k},
\]

\[
M = \sum_{k=0}^{h-1} (2k+1) \alpha^{2k} \beta^{2h-2k-1}, \quad N = \sum_{k=1}^{h-1} 2k \alpha^{2k-2} \beta^{2h-2k-1},
\]

\[
P = \sum_{k=1}^{h} 2k(2k-1) \alpha^{2k-2} \beta^{2h-2k}, \quad Q = \sum_{k=1}^{h-1} (2k+1)2k \alpha^{2k-2} \beta^{2h-2k-1}.
\]
Substituting (3.3) and (3.4) into (3.2), we have
\[
(a^2 \gamma_0^i - \gamma_0^0 y^i) \beta [\beta^2 (M^2 + 2aKM + \alpha^2 K^2) + \alpha (a^2 b^2 - \beta^2) (MP + \alpha (KP + MQ) + \alpha^2 KQ)] - 2a^3 s^i_0 [\beta^2 (LM + \alpha KL + \alpha^3 MN + \alpha^4 KN) + \alpha (a^2 b^2 - \beta^2) (LP + \alpha LQ + \alpha^3 NP + \alpha^4 NQ) + (a^2 b^2 - \beta y^i) \alpha^2 (MP + \alpha (KP + MQ) + \alpha^2 KQ)] + 2a^2 s_0 (LP + \alpha LQ + \alpha^3 NP + \alpha^4 NQ) = 0.
\]

The above is rewritten in the form
\[
A + \alpha B = 0,
\]
where
\[
A = (a^2 \gamma_0^i - \gamma_0^0 y^i) \{ \beta^2 (M^2 + \alpha^2 K^2) + \beta \alpha^2 (a^2 b^2 - \beta^2) (MQ + KP) \}
- 2a^4 s^i_0 [\beta^2 (KL + \alpha^2 MN) + (a^2 b^2 - \beta^2) (LP + \alpha^4 NQ)]
+ \alpha^2 (a^2 b^2 - \beta y^i) \{ \beta \alpha^2 r_{00} (MQ + KP) + 2a^2 s_0 (LP + \alpha^4 NQ) \},
\]
\[
B = (a^2 \gamma_0^i - \gamma_0^0 y^i) \{ 2\beta^3 K M + \beta (a^2 b^2 - \beta^2) (MP + \alpha^2 KQ) \}
- 2a^2 s^i_0 [\beta^2 (LM + \alpha^4 KN) + \alpha^2 (a^2 b^2 - \beta^2) (LQ + \alpha^2 NP)]
+ \alpha^2 (a^2 b^2 - \beta y^i) \{ \beta \alpha^2 r_{00} (MP + \alpha^2 KQ) + 2a^2 s_0 (LQ + \alpha^2 NP) \}.
\]

Since A, B are rational parts and \( \alpha \) is an irrational part in \( y^i \), we have \( A = 0 \) and \( B = 0 \), that is,
\[
(a^2 \gamma_0^i - \gamma_0^0 y^i) \{ \beta^2 (M^2 + \alpha^2 K^2) + \beta \alpha^2 (a^2 b^2 - \beta^2) (MQ + KP) \}
- 2a^4 s^i_0 [\beta^2 (KL + \alpha^2 MN) + (a^2 b^2 - \beta^2) (LP + \alpha^4 NQ)]
+ \alpha^2 (a^2 b^2 - \beta y^i) \{ \beta \alpha^2 r_{00} (MQ + KP) + 2a^2 s_0 (LP + \alpha^4 NQ) \} = 0, \tag{3.5}
\]
\[
(a^2 \gamma_0^i - \gamma_0^0 y^i) \{ 2\beta^3 K M + \beta (a^2 b^2 - \beta^2) (MP + \alpha^2 KQ) \}
- 2a^2 s^i_0 [\beta^2 (LM + \alpha^4 KN) + \alpha^2 (a^2 b^2 - \beta^2) (LQ + \alpha^2 NP)]
+ \alpha^2 (a^2 b^2 - \beta y^i) \{ \beta \alpha^2 r_{00} (MP + \alpha^2 KQ) + 2a^2 s_0 (LQ + \alpha^2 NP) \} = 0. \tag{3.6}
\]

Eliminating \( (a^2 \gamma_0^i - \gamma_0^0 y^i) \) from (3.5) and (3.6), we have
\[
2a^4 s^i_0 [\beta^2 (KL + \alpha^2 MN) + (a^2 b^2 - \beta^2) (LP + \alpha^4 NQ)]
- \beta^2 (M^2 + a^2 K^2) + \alpha^2 (a^2 b^2 - \beta^2) (MQ + KP)
\times \{ \beta^2 (M^2 + \alpha^4 KN) + a^2 (a^2 b^2 - \beta^2) (LQ + \alpha^2 NP) \}
- (a^2 b^2 - \beta y^i) \{ [2\beta^2 K M + (a^2 b^2 - \beta^2) (MP + \alpha^2 KQ)]
\times \{ \beta \alpha^2 r_{00} (MQ + KP) + 2a^2 s_0 (LP + \alpha^4 NQ) \}
- (a^2 b^2 - \beta y^i) \{ [2\beta^2 K M + (a^2 b^2 - \beta^2) (MP + \alpha^2 KQ)]
\times \{ \beta r_{00} (MP + \alpha^2 KQ) + 2a^2 s_0 (LQ + \alpha^2 NP) \} = 0. \tag{3.7}
\]
Transvecting (3.7) by \( b \), we get

\[
2s_0 [\alpha^2(2\beta^2KM + (\alpha^2b^2 - \beta^2)(MP + \alpha^2KQ))(KL + \alpha^2MN) \\
- [\beta^2(M^2 + \alpha^2K^2) + \alpha^2(\alpha^2b^2 - \beta^2)(MQ + KP)](LM + \alpha^4KN)]
- \beta r_{00}(\alpha^2b^2 - \beta^2)(2\alpha^2KM(MQ + KP) - (M^2 + \alpha^2K^2)(MP + \alpha^2KQ)) = 0.
\]

Thus the term of (3.8) which seemingly does not contain \( \alpha^2 \) is \( 2(\beta s_0 - r_{00})\beta^{8h - 2} \). Therefore there exists \( hp(8h - 2) : V_{sh-2} \) such that

\[
2(\beta s_0 - r_{00})\beta^{8h - 2} = \alpha^2 V_{sh-2}.
\]

We suppose that \( \alpha^2 \not\equiv 0 \pmod{\beta} \) due to Lemma 2.2. From (3.9) there exists a function \( k = k(x) \) satisfying \( V_{sh-2} = k\beta^{8h - 2} \), which leads to

\[
2(\beta s_0 - r_{00}) = k\alpha^2.
\]

Substituting (3.10) into (3.8), we have

\[
k(x)[\alpha^2(2\beta^2KM + (\alpha^2b^2 - \beta^2)(MP + \alpha^2KQ))(KL + \alpha^2MN) \\
- [\beta^2(M^2 + \alpha^2K^2) + \alpha^2(\alpha^2b^2 - \beta^2)(MQ + KP)](LM + \alpha^4KN)]
+ r_{00} \left[ 2[2\beta^2KM + (\alpha^2b^2 - \beta^2)(MP + \alpha^2KQ)](KL + \alpha^2MN) \\
- \beta^2(K^2LM + \alpha^2KM^2N + \alpha^4K^3N) - (\alpha^2b^2 - \beta^2)(MQ + KP) \\
(LM + \alpha^4KN) - 2\beta^2(\alpha^2b^2 - \beta^2)KM(MQ + KP) + \beta^2b^2[M^4P \\
+ \alpha^2\{KM(MQ + KP) + \alpha^2K^3Q\} - \beta^4\{KM(MQ + KP) + \alpha^2K^3Q\}] - \beta^4\{KM(MQ + KP) + \alpha^2K^3Q\} - \beta^2M^3(2L_1 + \beta^2P_1) \right] = 0,
\]

where

\[
L_1 = \sum_{k=1}^{h}(2k - 1)\alpha^{2k-2}\beta^{2h-2k},
\]

\[
P_1 = \sum_{k=2}^{h}2k(2k - 1)\alpha^{2k-4}\beta^{2h-2k}.
\]

Here the term of (3.11) which seemingly does not contain \( \alpha^2 \) is \( \beta^{8h-3}\{k\beta^2 + 2(b^2 - 7)r_{00}\} \). Thus there exists \( hp(8h - 3) : V_{sh-3} \) such that

\[
\beta^{8h-3}\{k\beta^2 + 2(b^2 - 7)r_{00}\} = \alpha^2 V_{sh-3}.
\]

From (3.12) there exists a function \( h = h(x) \) satisfying \( V_{sh-3} = h(x)\beta^{8h-3} \), and hence

\[
k(x)\beta^2 + 2(b^2 - 7)r_{00} = h(x)\alpha^2.
\]
Since $\alpha^2 \not\equiv 0 \pmod{\beta}$, we obtain $k(x) = 0$, which leads to
\[ r_{00} = \frac{h(x)}{2(b^2 - \gamma)} \alpha^2, \]  
(3.14)
where we assume $b^2 \not\equiv 0$.

Substituting $k(x) = 0$ and (3.14) into (3.10), we have
\[ \beta s_0 = \frac{h(x)}{2(b^2 - \gamma)} \alpha^2, \]
which leads to $s_0 = 0$ by virtue of $h(x) = 0$, and hence $r_{00} = 0$ from (3.14), that is, $s_j = 0$ and $r_{ij} = 0$.

Substituting $s_0 = 0$ and $r_{00} = 0$ into (3.7), we have
\[
\begin{align*}
&\frac{h(x)}{2(b^2 - \gamma)} \alpha^2
\times \{\beta^2(KL + \alpha^2 MN) + (\alpha^2b^2 - \beta^2)(LP + \alpha^4 NQ)\}
\times \{\beta^2(M^2 + \alpha^2 K^2) + \alpha^2(\alpha^2b^2 - \beta^2)(MQ + KP)\}
\times \{\beta^2(LM + \alpha^4 KN) + \alpha^2(\alpha^2b^2 - \beta^2)(LQ + \alpha^2 NP)\} = 0.
\end{align*}
\]
Hence the term of (3.15) which seemingly does not contain $\alpha^2$ is $\beta^{8h+1}s^i_0$. Then there exists $hp(8h) : V_{8h}$ such that
\[
\beta^{8h+1}s^i_0 = \alpha^2 V_{8h}.
\]
(3.16)

From $\alpha^2 \not\equiv 0 \pmod{\beta}$, there exists from (3.16) a function $\rho = \rho(x)$ satisfying $V_{8h} = \rho(x)\beta^{8h}$, and hence
\[ \beta s^i_0 = \rho(x) \alpha^2, \]
which leads to $s^i_0 = 0$ by virtue of $\rho(x) = 0$, that is, $s_{ij} = 0$.

Consequently we have $r_{ij} = 0$ and $s_{ij} = 0$, that is, $b_{ij} = 0$ is obtained.

Next, substituting $s_0 = 0$, $r_{00} = 0$ and $s^i_0 = 0$ into (3.5) we have
\[
(\alpha^2\gamma_{00} - \gamma_{000}y^i)\{\beta^2(M^2 + \alpha^2 K^2) + \alpha^2(\alpha^2b^2 - \beta^2)(MQ + KP)\} = 0.
\]
(3.17)
Thus the term of (3.17) which seemingly does not contain $\alpha^2$ is $-\gamma_{000}y^i\beta^{4h}$. Therefore there exists $hp(1) : \mu_0 = \mu_0(x)y^i$ such that
\[
\gamma_{000} = \mu_0 \alpha^2.
\]
(3.18)
Substituting (3.18) into (3.17), we have
\[
\gamma_{000} = \mu_0 \alpha^2.
\]
(3.18)

Substituting (3.18) into (3.17), we have
\[
(\gamma_{00} - \mu_0 y^i)D = 0,
\]
where
\[
D = \beta^2(M^2 + \alpha^2 K^2) + \alpha^2(\alpha^2b^2 - \beta^2)(MQ + KP).
\]
(3.19)
From (3.19) if $D = 0$, then the term of $D = 0$ which seemingly does not contain $\alpha^2$ is $\beta^{4h}$. In this case, there exists $hp(4h - 2) : V_{4h-2}$ such that
\[
\beta^{4h} = \alpha^2 V_{4h-2}.
\]
Hence we have $V_{4h-2} = 0$, which leads to a contradiction, that is, $D \not= 0$. Therefore we obtain $\gamma_{00} = \mu_0 y^i$, that is,
\[
2\gamma_{j}^i k = \mu_j \delta_k^i + \mu_k \delta_j^i,
\]
(3.20)
which shows that the associated Riemannian space is projectively flat.

Conversely it is easy to see that (3.2) is a consequence of \( b_{i,j} = 0 \) and (3.20).

(2) Case of \( r = 2h + 1 \), where \( h \) is a positive integer.

When \( r = 2h + 1 \), we have

\[
\sum_{k=0}^{r} k \left( \frac{\alpha}{\beta} \right)^{k-1} = \frac{1}{\beta^{2h}} \sum_{k=0}^{2h+1} k \alpha^{k-1} \beta^{2h-k} + 1,
\]

\[
\sum_{k=0}^{r} (k - 1) \left( \frac{\alpha}{\beta} \right)^{k} = \frac{1}{\beta^{2h}} \sum_{k=0}^{2h+1} (k - 1) \alpha^{k} \beta^{2h-k} + 1,
\]

\[
\sum_{k=0}^{r} k(k - 1) \left( \frac{\alpha}{\beta} \right)^{k-1} = \frac{1}{\beta^{2h+1}} \sum_{k=0}^{2h+1} k(k - 1) \alpha^{k-1} \beta^{2h-k} + 1.
\] (3.21)

Separating the rational and irrational parts in \( y_i \) with respect to (3.21), we have

\[
\sum_{k=0}^{2h+1} k \alpha^{k-1} \beta^{2h-k} + 1 = \sum_{k=0}^{h} (2k + 1) \alpha^{2k} \beta^{2h-k} + 1 + \alpha \sum_{k=1}^{h} 2k \alpha^{2k-2} \beta^{2h-k} + 1
\]

\[= O + \alpha \beta K,\]

\[
\sum_{k=0}^{2h+1} (k - 1) \alpha^{k} \beta^{2h-k} + 1 = \sum_{k=0}^{h} (2k - 1) \alpha^{2k} \beta^{2h-k} + 1 + \alpha \sum_{k=1}^{h} 2k \alpha^{2k-2} \beta^{2h-k} + 1
\]

\[= \beta L + \alpha^3 K,\]

\[
\sum_{k=0}^{2h+1} k(k - 1) \alpha^{k-1} \beta^{2h-k} + 1 = \sum_{k=1}^{h} (2k + 1) 2k \alpha^{2k-2} \beta^{2h-k} + 1 + \alpha \left( \sum_{k=1}^{h} 2k(2k - 1) \alpha^{2k-2} \beta^{2h-k} + 1 \right)
\]

\[= \alpha^2 R + \alpha \beta P,\]

where

\[O = \sum_{k=0}^{h} (2k + 1) \alpha^{2k} \beta^{2h-k} + 1,\]

\[R = \sum_{k=1}^{h} (2k + 1) 2k \alpha^{2k-2} \beta^{2h-k} + 1.\]
Substituting (3.21) and (3.22) into (3.2), we have
\[
\{ \beta (\alpha^2 \gamma_0' - \gamma_000y') (O + \alpha \beta K) - 2\alpha^3 s_0 (BL + \alpha^3 K) \} \\
\times \{ \beta^2 (O + \alpha \beta K) + \alpha (\alpha^2 b^2 - \beta^2) (\alpha R + \beta P) \} \\
+ \alpha^3 (\alpha R + \beta P) \{ \beta r_{00} (O + \alpha \beta K) + 2\alpha s_0 (\beta L + \alpha^3 K) \} \\
\times (\alpha^2 b^i - \beta y^i) = 0.
\]
(3.23)

Separating the rational and irrational parts in \( y^i \), we obtain
\[
A' + \alpha B' = 0,
\]
that is, \( A' = 0 \) and \( B' = 0 \) because \( \alpha \) is an irrational part in \( y^i \), where
\[
A' = \beta (\alpha^2 \gamma_0' - \gamma_000y') \{ \beta^2 (O + \alpha^2 \beta^2 K^2) + \alpha^2 (\alpha^2 b^2 - \beta^2) (OR + \beta^2 KP) \} \\
- 2\alpha^4 s_0 \{ \beta^2 (\beta^2 L K + \alpha^2 KO) + (\alpha^2 b^2 - \beta^2) (\beta^2 LP + \alpha^4 KR) \} \\
+ \alpha^4 \{ \beta r_{00} (OR + \beta^2 KP) + 2s_0 (\alpha^4 KR + \beta^2 LP) \} (\alpha^2 b^i - \beta y^i) = 0,
\]
\[
B' = \beta (\alpha^2 \gamma_0' - \gamma_000y') \{ 2\beta^2 KO + (\alpha^2 b^2 - \beta^2) (OP + \alpha^2 KR) \} \\
- 2\alpha^2 s_0 \{ \beta^2 (LO + \alpha^4 K^2) + \alpha^2 (\alpha^2 b^2 - \beta^2) (LR + \alpha^2 KP) \} \\
+ \alpha^2 \{ \beta r_{00} (\alpha^2 KR + OP) + 2\alpha^2 s_0 (LR + \alpha^2 KP) \} (\alpha^2 b^i - \beta y^i) = 0.
\]
From (3.24) we have \(- \gamma_000y^i \beta^{4h+3} = \alpha^2 W_{4h+5} \), where \( W_{4h+5} \) is a \( hp(4h+5) \). Therefore there exists \( hp(1) : v_0 \) satisfying
\[
\gamma_000 = v_0 \alpha^2.
\]
(3.26)

Next, eliminating \((\alpha^2 \gamma_0' - \gamma_000y')\) from (3.24) and (3.25), we have
\[
2s_0 \{ \alpha^2 (\alpha^2 b^2 - \beta^2) (OP + \alpha^2 KR) \} \\
\times \{ 2\beta^2 KO + (\alpha^2 b^2 - \beta^2) (OR + \beta^2 KP) \} \\
\times \{ \beta^2 (O + \alpha^4 K^2) + \alpha^2 (\alpha^2 b^2 - \beta^2) (LR + \alpha^2 KP) \} \\
\times \{ \beta^2 (O + \alpha^2 \beta^2 K^2) + \alpha^2 (\alpha^2 b^2 - \beta^2) (OR + \beta^2 KP) \} \\
\times \{ \beta^2 (O + \alpha^2 \beta^2 K^2) + \alpha^2 (\alpha^2 b^2 - \beta^2) (OR + \beta^2 KP) \} = 0.
\]
(3.27)

Transvecting (3.27) by \( b_i \), we have
\[
2s_0 \{ \alpha^2 (\beta^2 L K + \alpha^2 KO) \} (2\beta^2 KO + (\alpha^2 b^2 - \beta^2) (OP + \alpha^2 KR) \\
- (LO + \alpha^4 K^2) \{ \beta^2 (O + \alpha^2 \beta^2 K^2) + \alpha^2 (\alpha^2 b^2 - \beta^2) (OR + \beta^2 KP) \} \\
- \beta (\alpha^2 b^2 - \beta^2) r_{00} \{ 2\alpha^2 KO (OR + \beta^2 KP) - (\alpha^2 KR + OP) (O^2 \\
+ \alpha^2 \beta^2 K^2) \} = 0.
\]
(3.28)
The terms of (3.28) which does not contain $\alpha^2$ are found in $2\beta^{h+1}(\beta s_0 - r_{00})$. Thus there exists $hp(8h + 1) : W_{sh+1}$ such that

$$2\beta^{h+1}(\beta s_0 - r_{00}) = \alpha^2 W_{sh+1}. \quad (3.29)$$

We suppose that $\alpha^2 \not\equiv 0 \pmod{\beta}$ owing to Lemma 2.2. Therefore there exists from (3.29) a function $f = f(x)$ satisfying $W_{sh+1} = f\beta^{h+1}$, which leads to

$$2(\beta s_0 - r_{00}) = f(x)\alpha^2. \quad (3.30)$$

Substituting (3.30) into (3.28), we obtain

$$f(x)\alpha^2 \left[ \alpha^2(\beta^2LK + \alpha^2KO)(2\beta^2KO + (\alpha^2b^2 - \beta^2)(OP + \alpha^2KR)) \right.\]$$

$$- (LO + \alpha^4K^2)(\beta^2(2\beta^2K^2) + \alpha^2(\alpha^2b^2 - \beta^2)(OR + \beta^2KP)) \]$$

$$+ r_{00}[2\alpha^2(\beta^2LK + \alpha^2KO)(2\beta^2KO + (\alpha^2b^2 - \beta^2)(OP + \alpha^2KR)) \]$$

$$- 2\alpha^2\beta^2(\alpha^2K^2O^2 + \beta^2K^2LO + \alpha^4\beta^2K^4) - 2\alpha^4(\alpha^2b^2 - \beta^2)(LO \quad (3.31)$$

$$+ \alpha^4K^2)(OR + \beta^2KP) - 2\alpha^2\beta^2(\alpha^2b^2 - \beta^2)KO(OR + \beta^2KP) \]$$

$$+ \alpha^2\beta^2b^2O^3P + \alpha^4\beta^2b^2(\beta^2KR + \alpha^2\beta^2K^3R + \beta^2K^2OP) \]$$

$$- \alpha^2\beta^4(KO^2R + \alpha^2\beta^2K^3R + \beta^2K^2OP) - \beta^2O^3(2L + \beta^2P) \]$$

$$= 0.$$
Thus the term of (3.33) which seemingly does not contain $\alpha^2$ is included in the form: $\beta^8 \{ f(x) \beta^2 + 2(b^2 - 7)r_{00} \}$. Therefore there exists $hp(8h) : W_{8h}$ such that
\[
\beta^8 \{ f(x) \beta^2 + 2(b^2 - 7)r_{00} \} = \alpha^2 W_{8h}. \tag{3.34}
\]
In this case, there exists from (3.34) a function $g = g(x)$ satisfying $W_{8h} = g(x)\beta^8$, which takes the follow of form
\[
f(x) \beta^2 + 2(b^2 - 7)r_{00} = g(x)\alpha^2.
\]
From $\alpha^2 \not\equiv 0$ (mod. $\beta$), it follows that $f(x)$ must vanish and hence we have
\[
r_{00} = \frac{g(x)}{2(b^2 - 7)} \alpha^2, \tag{3.35}
\]
where we assume $b^2 \neq 7$. Substituting $f(x) = 0$ and (3.35) into (3.30), we have
\[
\beta s_0 = \frac{g(x)}{2(b^2 - 7)} \alpha^2,
\]
which leads to $s_0 = 0$ and $r_{00} = 0$, that is, $s_j = 0$ and $r_{ij} = 0$. Substituting $s_0 = 0$ and $r_{00} = 0$ into (3.27), we obtain
\[
s^i_0 \{ \alpha^2 \{ \beta^2 (\beta^2 L K + \alpha^2 K O) + (\alpha^2 b^2 - \beta^2)(\beta^2 L P + \alpha^4 K R) \} \\
\{ \beta^2 K O + (\alpha^2 b^2 - \beta^2)(O P + \alpha^2 K R) \} - \{ \beta^2 (L O + \alpha^4 K^2) \\
+ \alpha^2 (\alpha^2 b^2 - \beta^2)(L R + \alpha^2 K P) \} \{ \beta^2 (O^2 + \alpha^2 b^2 K^2) \\
+ \alpha^2 (\alpha^2 b^2 - \beta^2)(O R + \alpha^2 K P) \} \} = 0.
\]
Thus the term of (3.36) which seemingly does not contain $\alpha^2$ is $\beta^{8h+4} s^i_0$. Then there exists $hp(8h + 3) : W_{8h+3}$ such that
\[
s^i_0 \beta^{8h+4} = \alpha^2 W_{8h+3}. \tag{3.37}
\]
From $\alpha^2 \not\equiv 0$ (mod. $\beta$), there exists from (3.37) a function $h = h(x)$ satisfying $W_{8h+3} = h \beta^{8h+3}$, and hence
\[
\beta s^i_0 = h(x)\alpha^2,
\]
which leads to $s^i_0 = 0$, that is, $s_{ij} = 0$ by virtue of $h(x) = 0$.

Consequently we obtain $r_{ij} = 0$ and $s_{ij} = 0$, that is, $b_{ij} = 0$ is obtained. Substituting $s_0 = 0$, $r_{00} = 0$, $s^i_0 = 0$ and (3.26) into (3.23), we have
\[
\gamma^i_0 q^i = \mu \alpha^2,
\]
which leads to
\[
2 \gamma^i_0 = \mu \alpha^2.
\]
which shows that the associated Riemannian space is projectively flat.

Conversely, it is easy to see that (3.2) is a consequence of $b_{ij} = 0$ and (3.38).

Consequently we obtain the same results from both case of $r = 2h$ and case of $r = 2h + 1$.

Hence we have the following
Theorem 3.1. A Finsler space $F^n$ ($n > 2$) with an approximate infinite $(\alpha, \beta)$-metric (2.1) provided $b^2 \neq 7$ is projectively flat if and only if $h_{ij} = 0$ is satisfied, and the associated Riemannian space $(M^n, \alpha)$ is projectively flat if and only if $2\gamma_j^i k = \mu_j \delta_k^i + \mu_k \delta_j^i$ is obtained. Then $F^n$ is a Berwald space.

References


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