GENERALIZED PRIME IDEALS
IN NON-ASSOCIATIVE NEAR-RINGS I

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Abstract. In this paper, the concept of $\ast$-prime ideals in non-associative near-rings is introduced and then will be studied. For this purpose, first we introduce the notions of $\ast$-operation, $\ast$-prime ideal and $\ast$-system in a near-ring. Next, we will define the $\ast$-sequence, $\ast$-strongly nilpotent and $\ast$-prime radical of near-rings, and then obtain some characterizations of $\ast$-prime ideal and $\ast$-prime radical $r_s(I)$ of an ideal $I$ of near-ring $N$.

1. Introduction

A near-ring $N$ is an algebraic system $(N, +, \cdot)$ with two binary operations, say $+$ and $\cdot$ such that $(N, +)$ is a group (not necessarily abelian) with neutral element $0$, $(N, \cdot)$ is a semigroup and $a(b + c) = ab + ac$ for all $a, b, c \in N$.

In this near-ring, if $(N, \cdot)$ is not a semigroup, then $N$ is a non-associative near-ring. If $N$ has a unity $1$, then $N$ is called unitary. An element $d$ in $N$ is called distributive if $(a + b)d = ad + bd$ for all $a$ and $b$ in $N$. A near-ring $N$ is called distributive if every element in $N$ is distributive.

An ideal of $N$ is a subset $I$ of $N$ such that (i) $(I, +)$ is a normal subgroup of $(N, +)$, (ii) $a(I + b) - ab \subset I$ for all $a, b \in N$, (iii) $(I + a)b - ab \subset I$ for all $a, b \in N$. If $I$ satisfies (i) and (ii) then it is called a left ideal of $N$. If $I$ satisfies (i) and (ii) then it is called a right ideal of $N$.

On the other hand, an $N$-subgroup of $N$ is any subset $H$ of $N$ such that (i) $(H, +)$ is a normal subgroup of $(N, +)$, (ii) $NH \subset H$ and (iii) $HN \subset H$. If $H$ satisfies (i) and (ii) then it is called a left $N$-subgroup of $N$. If $H$ satisfies (i) and (ii) then it is called a right $N$-subgroup of $N$. In case, $(H, +)$ is normal in above, we say that normal $N$-subgroup, normal left $N$-subgroup and normal right $N$-subgroup instead of $N$-subgroup, left $N$-subgroup and right $N$-subgroup, respectively.

Note that normal $N$-subgroups of $N$ are not equivalent to ideals of $N$.

We consider the following notations: Given a near-ring $N$,

$$N_0 = \{a \in N \mid 0a = 0\}$$

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which is called the zero symmetric part of $N$,

$$N_c = \{ a \in N \mid 0a = a \} = \{ a \in N \mid ra = a, \text{ for all } r \in N \}$$

which is called the constant part of $N$.

We note that $N_0$ and $N_c$ are subnear-rings of $N$. A near-ring $N$ with the extra axiom $0a = 0$ for all $a \in N$, that is, $N = N_0$ is said to be zero symmetric, also, in case $N = N_c$, $N$ is called a constant near-ring. From the Pierce decomposition theorem, we get the important fact:

$$N = N_0 \oplus N_c$$

as additive groups. So every element $a \in N$ has a unique representation of the form $a = b + c$, where $b \in N_0$ and $c \in N_c$.

Throughout this paper, by a near-ring, we mean a zero-symmetric non-associative near-ring. For basic definitions and results on near-rings, one may refer Pilz [5].

Let $(G, +)$ be a group (not necessarily abelian). In the set

$$M(G) = \{ f : G \to G \}$$

of all the self maps of $G$, if we define the sum $f + g$ of any two mappings $f, g$ in $M(G)$ by the rule $x(f + g) = xf + xg$ for all $x \in G$ and the product $f \cdot g$ by the rule $x(f \cdot g) = (xf)g$ for all $x \in G$, then $(M(G), +, \cdot)$ becomes a near-ring. It is called the self map near-ring of the group $G$. Also, if we define the set

$$M_0(G) = \{ f \in M(G) \mid 0f = 0 \},$$

then $(M_0(G), +, \cdot)$ is a zero symmetric near-ring.

2. Results on $*$-prime ideals and $*$-prime radicals

Groeneveld and Potgieter [1] generalized the notion of prime ideals in associative near-rings and introduced the concept of $f$-prime ideals in associative near-rings. The notion of $f$-prime ideals in associative near rings actually extends the notion of $f$-prime ideals in associative rings due to Murata et al. [2]. Myung [3] introduced the notion of $*$-prime ideals in non-associative rings. Corresponding to $*$-prime ideals in non-associative rings, we can introduce in this paper the $*$-prime ideals in non-associative near-rings. For this purpose, first we define the notions of $*$-system and $*$-prime ideal in a near-ring and prove that complement of a $*$-system is a $*$-prime ideal.

In this section, we define $*$-operation for the purpose of $*$-prime ideals, and obtain some characterizations of $*$-prime ideal and $*$-prime radical.

The concept of $*$-operation for rings was introduced by Myung [3], [4]. We can extend this concept to near-rings as following:

**Definition 1.** Let $F(N)$ be the set of all ideals in $N$. A $*$-operation is a mapping from $F(N) \times F(N)$ into the family of additive subgroups of $N$ satisfying the following conditions.

(i) for $A, B, C, D$ in $F(N)$, if $A \subseteq B$ and $C \subseteq D$, then $A * C \subseteq B * D$. 

Hereafter, by a near-ring we mean a near-ring $N$ in which a $*$-operation is defined.

Now, we may obtain the following examples of $*$-operations in $N$.

**Example 1.** Let $N$ be a near-ring. Define $*$ on $F(N) \times F(N)$ by $A \ast B$ is a normal subgroup generated by \{ab|a \in A, b \in B\}. Then this $*$-operation satisfy the conditions stated in the above Definition 1. For, the conditions (i) and (ii) are trivially true. If $A, B, C \in F(N)$, then $(A + C)(B + C) \subseteq AB + C$. Thus the set of all generators of $(A + C)*(B + C)$ are of the form $ab + c$ for $a \in A, b \in B$ and $c \in C$. Clearly $A \ast B + C$ is a normal subgroup of $(N, +)$ and it contains all the elements of $AB + C$. Thus $(A + C)*(B + C) \subseteq A \ast B + C$. Hence for any near-ring $N$, always $*$-operation exists.

**Definition 2.** A proper ideal $I$ in a near-ring is said to be $*$-prime if $A \ast B \subseteq I$ implies either $A \subseteq I$ or $B \subseteq I$ for $A, B \in F(N)$.

**Definition 3.** A non-empty subset $M$ of $N$ is said to be $*$-system if $A \cap M \neq \emptyset$ and $B \cap M \neq \emptyset$ implies $A \ast B \cap M \neq \emptyset$ for $A, B \in F(N)$.

In the following, we give some examples of $*$-prime ideals in $N$.

**Example 2.** Consider the near-ring $(N, +)$ defined on Dihedral group $(D_8, +)$ according to the scheme $(0,9,0,9,1,3,1,3)$ (p. 415 [5]). This near-ring is non-associative, since $(a+b)((2a+b)(3a+b)) = a+b$ and $((a+b)(2a+b))(3a+b) = 3a+b$. One can check that the proper ideals of the above near-ring are $I_1 = \{0, 2a\}$ and $I_2 = \{0, a, 2a, 3a\}$. This follows from the fact that the above near-ring is distributive and $I_1$ and $I_2$ are the only normal subgroups which are closed under left and right multiplications by elements of $N$. Define $*$ on $F(N) \times F(N)$ as in Example 1. For this $*$-operation, it is easy to observe that $I_2$ is $*$-prime and $I_1$ is not a $*$-prime ideal in $N$.

Now, we can obtain some equivalent conditions of $*$-prime ideals in $N$.

**Proposition 2.1.** Let $I$ be a proper ideal in a near-ring $N$. Then the following are equivalent:

(i) If $A \ast B \subseteq I$ for $A, B \in F(N)$, then either $A \subseteq I$ or $B \subseteq I$.

(ii) If $A \cap C(I) \neq \emptyset$ and $B \cap C(I) \neq \emptyset$, then $(A \ast B) \cap C(I) \neq \emptyset$ for $A, B \in F(N)$. Here $C(I)$ denotes complement of $I$.

(iii) If $a$ and $b$ are in $C(I)$, then $(< a > * < b >) \cap C(I) \neq \emptyset$, where $< x >$ denotes the ideal generated by $x \in N$.

**Proof.** (i) $\Rightarrow$ (ii). Assume the condition (i). If $A \cap C(I) \neq \emptyset$ and $B \cap C(I) \neq \emptyset$, then there exist $a$ in $A$ and $b$ in $B$ such that $a \in C(I)$ and $b \in C(I)$, that is, $a \notin I$ and $b \notin I$. These facts imply that $A \not\subseteq I$ and $B \not\subseteq I$. From the condition (i), we see that $A \ast B \not\subseteq I$, that is, there exists $c \in (A \ast B)$ such that $c \notin I$, equivalently, there exists $c \in (A \ast B)$ such that $c \in C(I)$. Hence, $(A \ast B) \cap C(I) \neq \emptyset$ for $A, B \in F(N)$.
Remark 1. By the above Proposition 2.1, an ideal $I$ is an $*$-prime ideal if and only if $C(I)$ is a $*$-system. Thus in Example 2, the set $M = \{ b, a + b, 2a + b \}$ is a $*$-system.

Definition 4. A sequence $a_0, a_1, \ldots, a_n, \ldots$ of elements in $N$ is said to be a $*$-sequence if $a_n \in < a_{n-1} >$ for all $n \geq 1$.

Lemma 2.2. Every $*$-sequence is a $*$-system in $N$.

Proof. Let $S = \{ a_0, a_1, \ldots, a_n, \ldots \}$ be a $*$-sequence in $N$. If $A \cap S \neq \emptyset$ and $B \cap S \neq \emptyset$, then there exist elements $a_k$ and $a_\ell$ in $S$ such that $a_k \in A$ and $a_\ell \in B$. If $k \geq \ell$, then $a_{k+1} \in < a_k >$ and $< a_k > \subseteq < a_\ell > \subseteq A \cap B$ and so $(A \ast B) \cap S \neq \emptyset$. Thus $S$ is a $*$-system in $N$. 

Definition 5. An element $a \in N$ is said to be $*$-strongly nilpotent if every $*$-sequence $a_0, a_1, \ldots, a_n, \ldots$ with $a_0 = a$ vanishes. That is, there exists an integer $k \geq 1$ such that $a_s = 0$ for all $s \geq k$.

Definition 6. If $I$ is a proper ideal of $N$, then the $*$-prime radical $r_S(I)$ of $I$ is the set of all elements $x \in N$ such that every $*$-system that contains $x$ contains an element of $I$.

Proposition 2.3. For an ideal $I$ of a near-ring $N$, $r_S(I)$ is the intersection of all $*$-prime ideals in $N$ containing $I$.

Proof. Let $x \in r_S(I)$. Suppose $x \not\in \cap P_i$, where $P_i$ is a $*$-prime ideal containing $I$. By assumption there exists a $*$-prime ideal $P$ such that $x \not\in P$ and $I \subseteq P$. Since $P$ is a $*$-prime ideal, $C(P)$ is a $*$-system containing $x$ and $C(P) \cap I = \emptyset$. This is a contradiction. Hence $r_S(I) \subseteq \cap P_i$.

Conversely, if $x \in \cap P_i$ and $x \not\in r_S(I)$, then there exists a $*$-system $M$ such that $x \in M$ and $M \cap I = \emptyset$. This implies that $C(M) = P$ is a $*$-prime ideal and $x \not\in P$, a contradiction. Thus $\cap P_i \subseteq r_S(I)$.

Proposition 2.4. Let $N$ be a near-ring. Then $r_S(N) = \{ n \in N/n \text{ is } *-\text{strongly nilpotent} \}$.

Proof. Let $x \in r_S(N)$. If $x$ is not $*$-strongly nilpotent, then there exists a $*$-sequence $S = \{ a_0, a_1, \ldots, a_n, \ldots \}$ with $a_0 = x$ and $a_n \neq 0$ for all $n \geq 1$. By Lemma 2.2, $S$ is a $*$-system. Again by Proposition 2.1, $C(S)$ is a $*$-prime ideal and note that $x \not\in C(S)$. Thus $x \not\in r_S(N)$, a contradiction.

Conversely let $x$ be a $*$-strongly nilpotent. If $x \not\in r_S(N)$, then there exists a $*$-prime ideal $P$ such that $x \not\in P$. By Proposition 2.1, $C(P)$ is a $*$-system and $x \in C(P)$. Since $a_0 = x < x > \cap C(P)$, by the definition of $*$-system we get $(< a_0 > \ast < a_0 >) \cap C(P) \neq \emptyset$. Let $a_1 \in (< a_0 > \ast < a_0 >) \cap C(P)$. Since $< a_1 > \cap C(P) \neq \emptyset$ we get an element $a_2 \in (< a_1 > \ast < a_1 >) \cap C(P)$. Continuing in this way we get a $*$-sequence $S = \{ a_0, a_1, \ldots \}$ with $a_0 = x$. Note that $S \subseteq C(P)$. By the assumption, $x$ is $*$-strongly nilpotent, there exists
an integer $k \geq 1$ such that $a_s = 0$ for all $s \geq k$. Thus $a_k = 0 \in P$ and so $P \cap C(P) \neq \emptyset$, a contradiction. Thus $x \in r_S(N)$.

References


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