FINDING THE SKEW-SYMMETRIC SOLVENT TO A QUADRATIC MATRIX EQUATION

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Abstract. In this paper we consider the quadratic matrix equation which can be defined by

\[ Q(X) = AX^2 + BX + C = 0, \]

where \( X \) is a \( n \times n \) unknown real matrix; \( A, B \) and \( C \) are \( n \times n \) given matrices with real elements. Newton’s method is considered to find the skew-symmetric solvent of the nonlinear matrix equations \( Q(X) \). We also show that the method converges the skew-symmetric solvent even if the Fréchet derivative is singular. Finally, we give some numerical examples.

1. Introduction

It is well-known that the main application of quadratic matrix equation

\[ Q(X) = AX^2 + BX + C, \quad A, B, C \in \mathbb{R}^{n \times n}, \tag{1} \]

arises in the quadratic eigenvalue problem

\[ Q(\lambda)x = (\lambda^2 A + \lambda B + C)x = 0. \tag{2} \]

When \( A = A^T, \quad B = -B^T, \quad C = C^T \) in the quadratic eigenvalue problem (2), it has a Hamiltonian eigenstructure, that is, the eigenvalues are symmetric with respect to both axes [9, 11]. Motivation for finding skew-symmetric solvent of the quadratic matrix equation (1) comes from the quadratic eigenvalue problem (2), because any skew-symmetric matrix has a pair of purely imaginary eigenvalues. For solving a skew-symmetric eigenvalue problem [10] we suggest an algorithm and convergent theory for finding the skew-symmetric solvent to the equation (1).

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2. Newton’s methods for $Q(X)$

If we define $E$ as the solution of the linear equation $Q(X) + Q(X)(E) = 0$, where $Q(X)(E)$ is the Fréchet derivative of $Q$ at $X$ in the direction $E$, then Newton’s method for the quadratic matrix equations (1) with the given starting matrix $X_0$ can be written in the iteration form

$$
\begin{align*}
Q_{X_k}(E_k) &= -Q(X_k), \\
X_{k+1} &= X_k + E_k,
\end{align*}
$$

where $k = 0, 1, \ldots$. Thus, each step of Newton’s method requires the finding of solution $E$ of the linear equation

$$Q(X) = -Q(X). \tag{3}$$

The next theorem gives the conditions for the uniqueness of solution of the matrix equation (3).

**Theorem 2.1.** ([8]) The Fréchet derivative $Q_X$ is nonsingular iff

i) the pair $(AX + B, -A)$ is regular (that is, $\det((AX + B) + \lambda A)$ is not identically zero in $\lambda$),

ii) $\lambda(AX + B, -A) \cap \lambda(X) = \emptyset$.

If $A$ is nonsingular, the condition i) holds. Now we give some sufficient conditions for nonsingularity of $Q_X$ at the $Q(X)$ solvent $X$.

**Lemma 2.2.** ([8, Lem. 3.1]) If $A$ is nonsingular then $Q_X$ is nonsingular at

i) a dominant or minimal solvent $S$,

ii) all solvents $S$ if the eigenvalues of $Q(\lambda) = \lambda^2 A + \lambda B + C (\lambda \in \mathbb{C})$ are distinct.

To solve (3) we can apply several method for solving the generalized Sylvester equation described by Chu [1], Epton [5], Gardiner, Laub, Amato and Moler [6] and Golub, Nash and Van Loan [7]. Here, we describe a Schur algorithm for solving equation (3) which is proposed by Davis [2, 3]. First, we compute the Schur decomposition of $X \in \mathbb{C}^{n \times n}$,

$$W^*XW = T,$$

where $W$ is unitary and $T$ is upper triangular. Then, compute the generalized Schur decomposition of the matrices $AX + B$ and $A$,

$$M^*(AX + B)N = H, \quad M^*AN = J,$$

where $M$ and $N$ are unitary, $H$ and $J$ are upper triangular. Equating the $k$th columns and rearranging leads to

$$(H + t_{kk}J) y_k = g_k - \sum_{i=1}^{k-1} t_{ik}J y_i, \quad Y = [y_1 \; y_2 \; \cdots \; y_n].$$

By solving these upper triangular systems in the order of $k = 1, \cdots, n$, $Y$ can be obtained by column at a time.
3. Skew-symmetric solvents of the quadratic matrix equation $Q(X)$

Now, we present an algorithm to find skew-symmetric solution of the $q$-th Newton iteration (3).

**Algorithm 1.** The matrices $A, B, C \in \mathbb{R}^{n \times n}$ and skew-symmetric matrix $X_q \in \mathbb{R}^{n \times n}$ are given. Iteration is started at skew-symmetric matrix $E_{q0} \in \mathbb{R}^{n \times n}$.

$k = 0$;

- $R_0 = -Q(X_q) - (AX_q + B)E_{q_0} - AE_{q_0}X_q$
- $Y_0 = (AX_q + B)^T R_0 + A^T R_0 (X_q)^T$
- $Q_0 = \frac{1}{2} (Y_0 - Y_0^T)$
- $\gamma_0 = \frac{\|R_0\|^2}{\|Q_0\|^2}$

**while** $R_k \neq 0$

- $\gamma_k = \frac{\|R_k\|^2}{\|Q_k\|^2}$
- $E_{qk+1} = E_{qk} + \gamma_k Q_k$
- $R_{k+1} = R_k - \gamma_k [(AX_q + B)Q_k + AQ_k X_q]$
- $Y_{k+1} = (AX_q + B)^T R_{k+1} + A^T R_{k+1} (X_q)^T$
- $\delta_k = \frac{\|Q_k\|^2}{\|Q_k\|^2}$
- $\gamma_{k+1} = \frac{1}{2} (Y_{k+1} - Y_{k+1}^T) + \delta_k Q_k$

**end.**

Note that, the matrices $E_{q_k}$ and $Q_k$ in Algorithm 1 are all skew-symmetric.

From Algorithm 1, we directly obtain the following basic lemmas.

**Lemma 3.1.** Assume that the $q$-th Newton iteration (3) is consistent. The sequences \{ $R_k$ \} and \{ $Q_k$ \} are generated by Algorithm 1, and the integer number $l \geq 0$ such that $\|R_k\| \neq 0$ for all $k = 0, 1, \cdots, l$. Then, we have

$$\text{tr} (R^T R_j) = 0 \quad \text{and} \quad \text{tr} (Q^T Q_j) = 0 \quad \text{for} \quad k > j = 0, 1, \cdots, l. \quad (4)$$

**Proof.** We prove this theorem by principle induction.

Step 1. When $l = 1$, from Algorithm 1 we obtain

$$\text{tr} (R^T R_0) = \text{tr} \left\{ [R_0 - \gamma_0 (AX_q + B)Q_0 - \gamma_0 AQ_0X_q]^T R_0 \right\}$$

$$= \text{tr} (R_0^T R_0) - \gamma_0 \text{tr} \left\{ [(AX_q + B)Q_0 + AQ_0X_q]^T R_0 \right\}$$

$$= \|R_0\|^2 - \gamma_0 \text{tr} \left\{ Q_0^T [AX_q + B]^T R_0 + A^T R_0 (X_q)^T \right\}$$

$$= \|R_0\|^2 - \gamma_0 \text{tr} (Q_0^T Y_0)$$

$$= \|R_0\|^2 - \gamma_0 \text{tr} \left( Q_0^T \frac{Y_0 - Y_0^T}{2} \right)$$

$$= \|R_0\|^2 - \gamma_0 \|Q_0\|^2$$

$$= 0,$$
and
\[
\text{tr} \left( Q_1^T Q_0 \right) = \text{tr} \left[ \left( \frac{Y_1 - Y_1^T}{2} + \delta_0 Q_0 \right)^T Q_0 \right] = \text{tr} \left( Y_1^T Q_0 \right) + \delta_0 \|Q_0\|^2 = \text{tr} \left( Q_0^T Y_1 \right) + \text{tr} \left( Y_1 Q_0 \right) = -\text{tr} \left( Y_1 Q_0 \right) + \text{tr} \left( Y_1 Q_0 \right) = 0.
\]
Suppose that the result (4) holds for \( l = s \). Then, when \( l = s + 1 \)
\[
\text{tr} \left( R_{s+1}^T R_s \right) = \text{tr} \left\{ \left[ R_s - \gamma_s (AX_q + B) Q_s - \gamma_s AQ_s X_q \right]^T R_s \right\} = \text{tr} \left( R_s^T R_s \right) - \gamma_s \text{tr} \left\{ \left[ (AX_q + B) Q_s + AQ_s X_q \right]^T R_s \right\} = \|R_s\|^2 - \gamma_s \text{tr} \left\{ Q_s^T \left[ (AX_q + B)^T R_s + A^T R_s (X_q)^T \right] \right\} = \|R_s\|^2 - \gamma_s \text{tr} \left( Q_s^T Y_q \right) = \|R_s\|^2 - \gamma_s \|Q_s\|^2 - \gamma_s \delta_{s-1} \text{tr} \left( Q_s^T Q_{s-1} \right) = 0,
\]
and
\[
\text{tr} \left( Q_{s+1}^T Q_s \right) = \text{tr} \left[ \left( \frac{Y_{s+1} - Y_{s+1}^T}{2} \right)^T Q_s \right] + \delta_s \text{tr} \left( Q_s^T Q_s \right) = \text{tr} \left( Y_{s+1}^T Q_s \right) + \delta_s \|Q_s\|^2 = \text{tr} \left\{ \left[ (AX_q + B)^T R_{s+1} + A^T R_{s+1} (X_q)^T \right]^T Q_s \right\} + \delta_s \|Q_s\|^2 = \text{tr} \left\{ R_{s+1}^T [AX_q + B] Q_s + AQ_s X_q \right\} + \delta_s \|Q_s\|^2 = \text{tr} \left\{ R_{s+1}^T \frac{1}{\gamma_s} (R_s - R_{s+1}) \right\} + \delta_s \|Q_s\|^2 = -\frac{1}{\gamma_s} \text{tr} \left( R_{s+1}^T R_{s+1} \right) + \delta_s \|Q_s\|^2 = 0.
\]
Step 2. Assume that \( \text{tr} \left( R_j^T R_j \right) = 0 \), \( \text{tr} \left( Q_j^T Q_j \right) = 0 \) for all \( j = 0, 1, \ldots, s - 1 \).
We show that \( \text{tr} \left( R_{s+1}^T R_j \right) = 0 \) and \( \text{tr} \left( Q_{s+1}^T Q_j \right) = 0 \) for \( j = 0, 1, \ldots, s - 1 \).
From Algorithm 1 and accompanying assumptions, we have
\[
\text{tr} \left( R_{s+1}^T R_j \right) = \text{tr} \left( R_j^T R_j \right) - \gamma_s \text{tr} \left\{ \left[ (AX_q + B) Q_s + AQ_s X_q \right]^T R_j \right\}
\]
\[ \begin{align*}
&= -\gamma_s \text{tr} \left\{ Q_s^T \left[ (AX_q + B)^T R_j + A^T R_j (X_q)^T \right] \right\} \\
&= -\gamma_s \text{tr} \left( Q_s^T Y_j \right) \\
&= -\gamma_s \text{tr} \left( Q_s^T \frac{Y_j - Y_j^T}{2} \right) \\
&= -\gamma_s \text{tr} \left[ Q_s^T (Q_j - \delta_{j-1} Q_{j-1}) \right] \\
&= -\gamma_s \text{tr} \left( Q_s^T Q_j \right) + \gamma_s \delta_{j-1} \text{tr} \left( Q_s^T Q_{j-1} \right) \\
&= 0,
\end{align*} \]

and

\[ \begin{align*}
\text{tr} \left( Q_{s+1}^T Q_j \right) &= \text{tr} \left( \frac{Y_{s+1} - Y_{s+1}^T}{2} \right)^T Q_j + \delta_s \text{tr} \left( Q_s^T Q_j \right) \\
&= \text{tr} \left( Y_{s+1}^T Q_j \right) \\
&= \text{tr} \left\{ \left[ (AX_q + B)^T R_{s+1} + A^T R_{s+1} (X_q)^T \right]^T Q_j \right\} \\
&= \text{tr} \left\{ R_{s+1}^T [(AX_q + B) Q_j + AQ_j X_q] \right\} \\
&= \frac{1}{\gamma_j} \text{tr} \left[ R_{s+1}^T (R_j - R_{j+1}) \right] \\
&= 0.
\end{align*} \]

Thus, the result (4) holds for \( l = s + 1 \). Therefore, from Step 1 and Step 2 we complete the proof. \( \Box \)

**Remark 1.** If there exists a positive number \( l \) such that \( R_k \neq 0 \) for all \( k = 0, 1, \ldots, l \), then the sequence \( \{R_k\} \) which is generated by Algorithm 1 is orthogonal set.

**Lemma 3.2.** Let \( E_q \) be a skew-symmetric solution of the \( q \)-th Newton iteration (3), then

\[ \text{tr} \left[ Q_k^T (E_q - E_{q_k}) \right] = \|R_k\|^2, \quad \text{for} \quad k = 0, 1, \ldots. \]

**Proof.** We prove the statement (5) by principle induction. When \( k = 0 \), from Algorithm 1 we have

\[ \begin{align*}
\text{tr} \left[ Q_0^T (E_q - E_{q_0}) \right] &= \text{tr} \left[ \frac{Y_0 - Y_0^T}{2} \right]^T (E_q - E_{q_0}) \\
&= \text{tr} \left[ Y_0^T (E_q - E_{q_0}) \right] \\
&= \text{tr} \left\{ \left[ (AX_q + B)^T R_0 + A^T R_0 (X_q)^T \right]^T (E_q - E_{q_0}) \right\} \\
&= \text{tr} \left\{ R_0^T [(AX_q + B) (E_q - E_{q_0}) + A (E_q - E_{q_0}) X_q] \right\}
\end{align*} \]
From this fact, we obtain

\[ \| R_0 \| = 2. \]

Assume that the statement (5) holds for \( k = l \), i.e., \( \text{tr} [Q_l^T (E_q - Q_l)] = \| R_l \| \).

Therefore, we can easily check that

\[ \text{tr} [Q_l^T (E_q - E_{q+1})] = \text{tr} [Q_l^T (E_q - E_{q+1})] - \gamma_l \text{tr} (Q_l^T Q_l) = 0. \]

From this fact, we obtain

\[
\begin{align*}
\text{tr} [Q_{l+1}^T (E_q - E_{q+1})] &= \text{tr} \left\{ \left[ \frac{Y_{l+1} - Y_{l+1}^T}{2} + \delta_l Q_l \right]^T (E_q - E_{q+1}) \right\} \\
&= \text{tr} \left\{ Q_{l+1}^T [Q_{l+1}^T (E_q - E_{q+1})] \right\} + \delta_l \text{tr} [Q_l^T (E_q - E_{q+1})] \\
&= \text{tr} \left\{ (AX_q + B)^T R_{l+1} + A^T R_{l+1} (X_q)^T \right\} (E_q - E_{q+1}) \\
&= \text{tr} \left\{ R_{l+1}^T [AX_q + B] (E_q - E_{q+1}) + A (E_q - E_{q+1}) X_q \right\} \\
&= \text{tr} \left\{ R_{l+1}^T [-Q(X_q) - (AX_q + B) E_{q+1} - A E_{q+1} X_q] \right\} \\
&= \| R_{l+1} \|^2,
\end{align*}
\]

which completes the proof. \( \square \)

Remark 2. Lemma 3.2 implies that, the \( q \)-th Newton iteration (3) has a skew-symmetric solution if \( R_k \neq 0 \) leads to \( P_k \neq 0 \) for some integer number \( k \). However, if \( P_k \neq 0 \) and \( R_k = 0 \), then the equation (3) is inconsistent.

**Theorem 3.3.** If the \( q \)-th Newton iteration (3) has a skew-symmetric solution, then for a skew-symmetric starting matrix \( E_{q_0} \), a skew-symmetric solution can be obtained, at most, in \( n^2 \) steps.

Proof. Let \( R_k \neq 0 \) for all \( k = 0, 1, \ldots, n^2 - 1 \). Then from Lemma 3.1, the set \( \{R_0, R_1, \ldots, R_{n^2-1} \} \) is an orthogonal basis of the matrix space \( \mathbb{R}^{n \times n} \). Since, the \( q \)-th Newton iteration (3) has a skew-symmetric solution, \( Q_k \neq 0 \) for \( k \) by Lemma 3.2. Therefore, we can evaluate \( E_{q_{2}} \) and \( R_{n^2} \) from Algorithm 1, and \( \text{tr} (R_k^T R_k) = 0 \) for \( k = 0, 1, \ldots, n^2 - 1 \) by Lemma 3.1. However, \( \text{tr} (R_k^T R_k) = 0 \) holds only when \( R_k = 0 \), which implies that \( E_{q_{2}} \) is a solution of the \( q \)-th Newton iteration. By Algorithm 1, it is natural that \( E_{q_{2}} \) is a skew-symmetric matrix. \( \square \)

From Newton’s method and Theorem 3.3, we obtained the following convergence theory.

**Theorem 3.4.** Assume that the quadratic matrix equation (1) has a skew-symmetric solvent and each Newton iteration is consistent for a skew-symmetric
starting matrix $X_0$. The sequence $\{X_k\}$ is generated by Newton’s method with $X_0$ such that

$$\lim_{k \to \infty} X_k = S,$$

and the matrix $S$ is the solvent of $Q(X)$, then $S$ is a skew-symmetric matrix.

**Proof.** Let $E_0, E_1, \ldots, E_k$ be skew-symmetric solution of first, second, $\ldots$, $k$th Newton iteration, respectively. Then, from Newton’s method we can obtain $(k + 1)$th approximation matrix

$$X_{k+1} = X_0 + E_0 + \cdots + E_k,$$

which is also skew-symmetric. Since, the Newton sequence $\{X_k\}$ converges to a solvent $S$, so, it is a skew-symmetric solvent. $\square$

### 4. Numerical experiments

The relative residual $\rho Q(X_k)$ and $\rho P(X_k)$, stop condition $\|R_k\|$ are same as in Section 4.3. We first consider the quadratic matrix equation

$$Q_1(X) \equiv X^2 + \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} X + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = 0 \quad (6)$$

which is dealt by Dennis, Traub and Weber [4]. It has an infinite number of solvents which have a form:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} i & 0 \\ -1 - i & 0 \end{bmatrix}, \begin{bmatrix} -i & 0 \\ -1 + i & 1 \end{bmatrix}, \begin{bmatrix} -z - 1 - i & i(z - 1) \\ iz - 1 & z \end{bmatrix},$$

$$\begin{bmatrix} -z + 1 + i & -i(z - 1) \\ -zi - 1 & z \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 + i & i \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 - i & -i \\ -1 & 0 \end{bmatrix}. \quad (7)$$

where $i = \sqrt{-1}$ and $z \in \mathbb{C}$. There are three skew-symmetric solvents in (6), that is,

$$\begin{bmatrix} 1 + \frac{i}{2} & -\frac{1}{2} - \frac{i}{2} \\ \frac{1}{2} + \frac{i}{2} & \frac{1}{2} - \frac{i}{2} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -\frac{3}{2} - \frac{i}{2} & \frac{1}{2} - \frac{i}{2} \\ -\frac{3}{2} + \frac{i}{2} & \frac{1}{2} + \frac{i}{2} \end{bmatrix}.$$  

Since our researches are progressed in real matrix spaces, we examine a skew-symmetric solvent $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. First, we select the skew-symmetric starting matrix $X_0 = \begin{bmatrix} 0 & 1.001 \\ -1 & 0 \end{bmatrix}$. It is sufficiently close to $S$, since a scalar number $\|S - X_0\| \approx 4.4721e-005$ can be sufficiently small. Sure enough we expected, the skew-symmetric solvent $S$ can be obtained using Newton’s method with Algorithm 1 with the starting matrix $X_0$. The convergence result is displayed in Table 1.
Next, we consider when the Fréchet derivative is singular. Let the quadratic matrix equation be
\[ Q_2(X) \equiv \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} X^2 + \begin{bmatrix} 0 & -4 \\ 0 & -4 \end{bmatrix} X + \begin{bmatrix} 5 & -25 \\ 5 & -25 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{8} \]
Starting Newton’s method with Algorithm 1 at the matrix \[ \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}, \]
then we can be obtained a skew-symmetric solvent \[ \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix}. \]
Figure 1 shows our Newton’s method with the starting matrix converges to a solvent. Therefore, we can know without difficulty this starting matrix enough close to the solvent.

![Figure 1. The convergence result for problem (8) with skew-symmetric matrices.](image)

In this paper, we introduced an iterative method for solving Newton steps (3) and (3) over skew-symmetric. Then we incorporated the method into Newton’s
method to find the skew-symmetric solvent. Our algorithm can be worked even if the Fréchet derivative is singular.

References


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