ON ITERATIVE APPROXIMATION OF COMMON FIXED POINTS OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS WITH APPLICATIONS

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ABSTRACT. In this paper, the problem of iterative approximation of common fixed points of asymptotically nonexpansive is investigated in the framework of Banach spaces. Weak convergence theorems are established. A necessary and sufficient condition for strong convergence is also discussed. As an application of main results, a variational inequality is investigated.

1. Introduction

Recently, iterative algorithms for computing common fixed points of nonlinear mappings has been considered by many authors ([1]–[6]).

From the method of generating iterative sequences, we can divide iterative algorithms into explicit algorithms and implicit algorithms. Recently, both explicit Mann-type iterative algorithms and implicit Mann-type iterative algorithms have been extensively studied for approximating common fixed points of nonlinear mappings ([7]–[16]).

In this paper, we consider the problem of approximating common fixed points of asymptotically nonexpansive mappings based on a general implicit iterative algorithm which includes an explicit iterative process as a special case. As an application of main results, a variational inequality is investigated in a uniformly convex and \( q \)-uniformly smooth Banach space.

2. Preliminaries

Let \( E \) be a real Banach space and \( E^* \) the dual space of \( E \). Let \( J_q \), where \( q > 1 \), denote the generalized duality mapping from \( E \) into \( 2^{E^*} \) give by

\[
J_q(x) = \{ f^* \in E^*: \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1} \}, \quad \forall x \in E,
\]

Received October 9, 2012; Accepted November 20, 2012.

2000 Mathematics Subject Classification. 47H09, 47J05, 47J2.

Key words and phrases. asymptotically nonexpansive mapping; fixed point; iterative process; nonexpansive mapping; variational inequality.

The first author was supported by Kyungnam University research fund, 2012.

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where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. In particular, $J_2$ is called the normalized duality mapping which is usually denoted by $J$. It is well known (see, for example, [17]) that $J_2(x) = \|x\|^{q-2} J(x)$ if $x \neq 0$.

Let $U_E = \{x \in E : \|x\| = 1\}$. A Banach space $E$ is said to be strictly convex if for all $x, y \in E$ which are linearly independent, $\|x + y\| < \|x\| + \|y\|$. This condition is equivalent to the following:

$$
\|x\| = \|y\| = 1, \quad \text{and} \quad x \neq y \implies \frac{\|x + y\|}{2} < 1.
$$

$E$ is said to be uniformly convex if for any two sequences $\{x_n\}$ and $\{y_n\}$ in $E$ such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n \to \infty} \|x_n + y_n\| = 2$, then $\lim_{n \to \infty} \|x_n - y_n\| = 0$ holds. It is known that a uniformly convex Banach space is reflexive and strictly convex.

A Banach space $E$ is said to be smooth if the limit

$$
\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}
$$

exists for all $x, y \in U_E$. It is said to be uniformly smooth if the limit is attained uniformly for all $x, y \in U_E$.

The modulus of smoothness of $E$ is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$
\rho_E(\tau) = \sup \left\{ \frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| \leq 1, \|y\| \leq \tau \right\}, \quad \forall \tau \geq 0.
$$

The Banach space $E$ is uniformly smooth if and only if $\lim_{\tau \to \infty} \frac{\rho_E(\tau)}{\tau} = 0$.

A Banach space $E$ is said to be $q$-uniformly smooth if there exists a constant $c > 0$ such that $\rho_E(\tau) \leq c\tau^q$. It is shown in [17] that there is no Banach space which is $q$-uniformly smooth with $q > 2$. Hilbert spaces, $L^p$ (or $l^p$) spaces and Sobolev space $W^p_1$, where $p \geq 2$, are $2$-uniformly smooth. Typical examples of both uniformly convex and uniformly smooth Banach spaces are $L^p$, where $p > 1$. More precisely, $L^p$ is $\min\{p, 2\}$-uniformly smooth for every $p > 1$.

$E$ is said to satisfy Opial’s condition (see [18]) if, for each sequence $\{x_n\}$ in $E$, $x_n \rightharpoonup x$, where $\rightharpoonup$ denotes weak convergence, implies that

$$
\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|. \quad \forall y \in E (y \neq x).
$$

Let $C$ be a nonempty subset of $E$ and $T : C \to C$ be a mapping. In this paper, the symbol $F(T)$ stands for the fixed point set of $T$. $T$ is said to be nonexpansive if

$$
\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.
$$

$T$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$
\|T^nx - T^ny\| \leq k_n \|x - y\|. \quad \forall x, y \in C, \forall n \geq 1.
$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [19] as a generalization of the class of nonexpansive mappings. They
proved that if \( C \) is a nonempty, closed, convex, and bounded subset of a real uniformly convex Banach space, then every asymptotically nonexpansive self mapping has a fixed point (see [19]).

In order to prove our main results, we still need the following lemmas.

**Lemma 2.1.** ([20]) Let \( C \) be a nonempty, closed, and convex subset of a uniformly convex Banach space \( E \). Let \( T : C \to C \) be an asymptotically nonexpansive mapping. Then \( I - T \) is demiclosed at zero, that is, \( x_n \to x \) and \( x_n - Tx_n \to 0 \) imply that \( x = Tx \).

**Lemma 2.2.** ([21]) Let \( \{a_n\}, \{b_n\} \) and \( \{c_n\} \) be nonnegative sequences satisfying the following condition:

\[
a_{n+1} \leq (1 + b_n)a_n + c_n, \quad \forall n \geq n_0,
\]

where \( n_0 \) is some nonnegative integer, \( \sum_{n=1}^{\infty} b_n < \infty \) and \( \sum_{n=1}^{\infty} c_n < \infty \). Then the limit \( \lim_{n \to \infty} a_n \) exists.

**Lemma 2.3.** ([15]) Let \( E \) be a uniformly convex Banach space, \( r > 0 \) a positive number and \( B_r(0) \) a closed ball of \( E \) with the center at zero. Then there exists a continuous, strictly increasing and convex function \( g : [0, \infty) \to [0, \infty) \) with \( g(0) = 0 \) such that

\[
\left\| \sum_{s=1}^{m} (\alpha_s x_s) \right\|^2 \leq \sum_{s=1}^{m} (\alpha_s \|x_s\|^2) - \alpha_i \alpha_j g(\|x_i - x_j\|), \quad \forall i, j \in \{1, 2, \ldots, r\},
\]

where \( x_1, x_2, \ldots, x_m \in B_r(0) \) and \( \alpha_1, \alpha_2, \ldots, \alpha_m \in (0, 1) \) such that \( \sum_{i=1}^{m} \alpha_i = 1 \).

**3. Main results**

Let \( C \) be a nonempty, closed and convex subset of a Banach space \( E \). Let \( T : C \to C \) be an asymptotically nonexpansive mapping with the sequence \( \{k_n\} \). For every \( u \in C \) and \( t_n \in (0, 1) \), define a mapping \( T_n : C \to C \) by

\[
T_n x = t_n u + (1 - t_n)T^n x, \quad \forall x \in C, \quad \forall n \geq 1,
\]

If \((1 - t_n)k_n < 1\), for every \( n \geq 1 \), then \( T_n \) is a contraction. Hence, by the Banach contraction principal, there exists a unique fixed point of \( T_n \), for every \( n \geq 1 \).

Let \( x_0 \) be chosen and \( r \geq 1 \) a positive integer. Let \( \{\alpha_n\}, \{\beta_{n,1}\}, \{\beta_{n,2}\}, \ldots, \{\beta_{n,r}\}, \{\gamma_{n,1}\}, \{\gamma_{n,2}\}, \ldots, \{\gamma_{n,r}\} \) be real sequences in \((0, 1)\) such that

\[
\alpha_n + \sum_{m=1}^{r} \beta_{n,m} + \sum_{m=1}^{r} \gamma_{n,m} = 1.
\]

Let \( S_m, T_m : C \to C \) be asymptotically nonexpansive mappings, for every \( m \in \{1, 2, \ldots, r\} \).
Find $x_1$ by solving the following equation

$$x_1 = \alpha_1 x_0 + \sum_{m=1}^{r} \beta_{1,m} S_m x_0 + \sum_{m=1}^{r} \gamma_{1,m} T_m x_1.$$ 

Find $x_2$ by solving the following equation

$$x_2 = \alpha_2 x_1 + \sum_{m=1}^{r} \beta_{2,m} S_m^2 x_1 + \sum_{m=1}^{r} \gamma_{2,m} T_m^2 x_2.$$ 

\ldots

Find $x_n$ by solving the following equation

$$x_n = \alpha_n x_{n-1} + \sum_{m=1}^{r} \beta_{n,m} S_m^n x_{n-1} + \sum_{m=1}^{r} \gamma_{n,m} T_m^n x_n.$$ 

\ldots

In view of the above, we have the following implicit iterative algorithm

$$x_0 \in C, \quad x_n = \alpha_n x_{n-1} + \sum_{m=1}^{r} \beta_{n,m} S_m^n x_{n-1} + \sum_{m=1}^{r} \gamma_{n,m} T_m^n x_n, \quad \forall n \geq 1. \quad (3.1)$$

If $S_m = I$, where $I$ is the identity mapping, for every $m \in \{1, 2, \ldots, r\}$, then (3.1) is reduced the following.

$$x_0 \in C, \quad x_n = (\alpha_n + \sum_{m=1}^{r} \beta_{n,m}) x_{n-1} + \sum_{m=1}^{r} \gamma_{n,m} T_m^n x_n, \quad \forall n \geq 1. \quad (3.2)$$

If $T_m = I$, where $I$ stands for the identity mapping, for every $m \in \{1, 2, \ldots, r\}$, then (3.1) is reduced the following.

$$x_0 \in C, \quad x_n = \frac{\alpha_n}{1 - \sum_{m=1}^{r} \gamma_{n,m}} x_{n-1} + \frac{\sum_{m=1}^{r} \beta_{n,m}}{1 - \sum_{m=1}^{r} \gamma_{n,m}} S_m^n x_{n-1}, \quad \forall n \geq 1. \quad (3.3)$$

Now, we need the following proposition for our main results.

**Proposition 3.1.** Let $C$ be a nonempty, closed and convex subset of a uniformly convex Banach space $E$. Let $S_m, T_m : C \to C$ be asymptotically nonexpansive mappings with the sequence $\{s_{n,m}\}$ and $\{t_{n,m}\}$, for every $m \in \{1, 2, \ldots, r\}$, where $r \geq 1$. Assume that $\mathcal{F} = \cap_{m=1}^{r} F(S_m) \cap \cap_{m=1}^{r} F(T_m)$ is nonempty. Let $t_n = \max\{t_{n,m} : 1 \leq m \leq r\}$ and $s_n = \max\{s_{n,m} : 1 \leq m \leq r\}$. Assume that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, where $k_n = \max\{s_{n,t_n} : 1 \leq m \leq r\}$. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence generated by (3.1), where $\{\alpha_n\}$, $\{\beta_{1,1}\}$, $\{\beta_{1,2}\}$, \ldots, $\{\beta_{n,r}\}$, $\{\gamma_{1,1}\}$, $\{\gamma_{1,2}\}$, \ldots, $\{\gamma_{n,r}\}$ are real number sequences in $(0, 1)$ such that $\alpha_n + \sum_{m=1}^{r} \beta_{n,m} + \sum_{m=1}^{r} \gamma_{n,m} = 1$. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_{1,1}\}$, $\{\beta_{1,2}\}$, \ldots, $\{\beta_{n,r}\}$, $\{\gamma_{1,1}\}$, $\{\gamma_{1,2}\}$, \ldots, $\{\gamma_{n,r}\}$ are satisfied

(a) $\lim \inf_{n \to \infty} \alpha_n \beta_{n,m} > 0$, and $\lim \inf_{n \to \infty} \alpha_n \gamma_{n,m} > 0$, $\forall m \in \{1, 2, \ldots, r\}$;

(b) $\sum_{m=1}^{r} \gamma_{n,m} t_{n} < 1.$
Then
\[
\lim_{n \to \infty} \|x_n - S_m x_n\| = \lim_{n \to \infty} \|x_n - T_m x_n\| = 0, \quad \forall m \in \{1, 2, \ldots, r\}.
\]

Proof. By the condition (b), we see that the sequence \(\{x_n\}\) generated by iterative process (3.1) is well defined. For \(p \in \mathcal{F}\), we see that
\[
\|x_n - p\| \leq \alpha_n \|x_{n-1} - p\| + \sum_{m=1}^{r} \beta_{n,m} \|S_m^n x_{n-1} - p\| + \sum_{m=1}^{r} \gamma_{n,m} \|T_m^n x_n - p\|
\]
\[
\leq (\alpha_n + \sum_{m=1}^{r} \beta_{n,m} k_n) \|x_{n-1} - p\| + \sum_{m=1}^{r} \gamma_{n,m} k_n \|x_n - p\|.
\]
In view of \(\liminf_{n \to \infty} \alpha_n \beta_{n,m} > 0\) and \(\alpha_n + \sum_{m=1}^{r} \beta_{n,m} + \sum_{m=1}^{r} \gamma_{n,m} = 1\), we see that there exists some positive integer \(n_1\) and a real number \(a\), where \(a \in (0,1)\), such that
\[
\sum_{m=1}^{r} \gamma_{n,m} \leq a, \quad \forall n \geq n_1.
\]
Since \(\sum_{m=1}^{\infty} (k_n - 1) < \infty\), there exists some positive integer \(n_2\) such that \(k_n \leq 1 + \frac{1-a}{2a}\), for all \(n \geq n_2\). It follows that
\[
\sum_{m=1}^{r} \gamma_{n,m} k_n \leq b < 1, \quad \forall n \geq n_3,
\]
where \(b = a(1 + \frac{1-a}{2a})\) and \(n_3 = \max\{n_1, n_2\}\). It follows that
\[
\|x_n - p\| \leq \frac{\alpha_n + \sum_{m=1}^{r} \beta_{n,m} k_n}{1 - \sum_{m=1}^{r} \gamma_{n,m} k_n} \|x_{n-1} - p\|
\]
\[
\leq (1 + \frac{\alpha_n + \sum_{m=1}^{r} \beta_{n,m} k_n + \sum_{m=1}^{r} \gamma_{n,m} k_n - 1}{1 - \sum_{m=1}^{r} \gamma_{n,m} k_n}) \|x_{n-1} - p\| \quad (3.4)
\]
\[
\leq (1 + \frac{k_n - 1}{1 - b}) \|x_{n-1} - p\|.
\]
It follows from Lemma 2.2 that \(\lim_{n \to \infty} \|x_n - p\|\) exists. This implies that the sequence \(\{x_n\}\) is bounded.

On the other hand, we find from Lemma 2.3 that
\[
\|x_n - p\|^2 \leq \alpha_n \|x_{n-1} - p\|^2 + \sum_{m=1}^{r} \beta_{n,m} \|S_m^n x_{n-1} - p\|^2 + \sum_{m=1}^{r} \gamma_{n,m} \|T_m^n x_n - p\|^2
\]
\[
- \alpha_n \beta_{n,m} g(\|x_{n-1} - S_m^n x_{n-1}\|)
\]
\[
\leq (\alpha_n + \sum_{m=1}^{r} \beta_{n,m} k_n) \|x_{n-1} - p\|^2 + \sum_{m=1}^{r} \gamma_{n,m} k_n \|x_n - p\|^2
\]
\[
- \alpha_n \beta_{n,m} g(\|x_{n-1} - S_m^n x_{n-1}\|), \quad \forall m \in \{1, 2, \ldots, N\}.
\]
This implies that
\[
\alpha_n\beta_{n,m}g(||x_{n-1} - S^n_{m}x_{n-1}||)
\]
\[
\leq (\alpha_n k_n + \sum_{m=1}^{r} \beta_{n,m}k_n)||x_{n-1} - p||^2 + \sum_{m=1}^{r} \gamma_{n,m}k_n||x_n - p||^2
\]
\[
- k_n||x_n - p||^2 + (k_n - 1)||x_n - p||^2
\]
\[
\leq (\alpha_n k_n + \sum_{m=1}^{r} \beta_{n,m}k_n)(||x_{n-1} - p||^2 - ||x_n - p||^2)
\]
\[
+ (k_n - 1)||x_n - p||^2, \quad \forall m \in \{1, 2, \ldots, r\}.
\]
Since \(\lim_{n \to \infty} ||x_n - p||\) exists, from the condition (a) we have that
\[
\lim_{n \to \infty} g(||x_{n-1} - S^n_{m}x_{n-1}||) = 0,
\]
for every \(m \in \{1, 2, \ldots, r\}\). It follows that
\[
\lim_{n \to \infty} ||x_{n-1} - S^n_{m}x_{n-1}|| = 0, \quad \forall m \in \{1, 2, \ldots, r\}.
\] (3.5)
From the Lemma 2.3, we obtain that
\[
||x_n - p||^2 \leq \alpha_n||x_{n-1} - p||^2 + \sum_{m=1}^{r} \beta_{n,m}||S^n_{m}x_{n-1} - p||^2 + \sum_{m=1}^{r} \gamma_{n,m}||T^n_{m}x_n - p||^2
\]
\[
- \alpha_n \gamma_{n,m}g(||x_{n-1} - T^n_{m}x_n||)
\]
\[
\leq (\alpha_n + \sum_{m=1}^{r} \beta_{n,m}k_n)||x_{n-1} - p||^2 + \sum_{m=1}^{r} \gamma_{n,m}k_n||x_n - p||^2
\]
\[
- \alpha_n \gamma_{n,m}g(||x_{n-1} - T^n_{m}x_n||), \quad \forall m \in \{1, 2, \ldots, r\}.
\]
This implies that
\[
\alpha_n \gamma_{n,m}g(||x_{n-1} - T^n_{m}x_n||)
\]
\[
\leq (\alpha_n k_n + \sum_{m=1}^{r} \beta_{n,m}k_n)||x_{n-1} - p||^2 + \sum_{m=1}^{r} \gamma_{n,m}k_n||x_n - p||^2
\]
\[
- k_n||x_n - p||^2 + (k_n - 1)||x_n - p||^2
\]
\[
\leq (\alpha_n k_n + \sum_{m=1}^{r} \beta_{n,m}k_n)(||x_{n-1} - p||^2 - ||x_n - p||^2)
\]
\[
+ (k_n - 1)||x_n - p||^2, \quad \forall m \in \{1, 2, \ldots, N\}.
\]
Since \(\lim_{n \to \infty} ||x_n - p||\) exists, from the condition (a) we have that
\[
\lim_{n \to \infty} g(||x_{n-1} - T^n_{m}x_n||) = 0,
\]
for every \(m \in \{1, 2, \ldots, r\}\). It follows that
\[
\lim_{n \to \infty} ||x_{n-1} - T^n_{m}x_n|| = 0, \quad \forall m \in \{1, 2, \ldots, r\}.
\] (3.6)
Notice that
\[ \|x_n - x_{n-1}\| = \sum_{m=1}^{r} \beta_{n,m} \|S_m^n x_{n-1} - x_{n-1}\| + \sum_{m=1}^{r} \gamma_{n,m} \|T_m^n x_n - x_{n-1}\|. \]

From the (3.5) and (3.6), we find that
\[ \lim_{n \to \infty} \|x_{n-1} - x_n\| = 0. \] (3.7)

Notice that
\[ \|x_n - T_m^n x_n\| \leq \|x_n - x_{n-1}\| + \|x_{n-1} - T_m^n x_n\|, \quad \forall m \in \{1, 2, \ldots, r\}. \]

This implies from (3.6), and (3.7) that
\[ \lim_{n \to \infty} \|x_n - T_m^n x_n\| = 0, \quad \forall m \in \{1, 2, \ldots, r\}. \] (3.8)

On the other hand, we have
\[ \|x_n - S_m^n x_n\| \leq \|x_n - x_{n-1}\| + \|x_{n-1} - S_m^n x_{n-1}\|
+ \|S_m^n x_{n-1} - S_m^n x_n\|, \quad \forall m \in \{1, 2, \ldots, r\}. \]

Since \(S_m\) is Lipschitz for every \(m \in \{1, 2, \ldots, r\}\), from (3.5) and (3.7) we know that
\[ \lim_{n \to \infty} \|x_n - S_m^n x_n\| = 0, \quad \forall m \in \{1, 2, \ldots, r\}. \] (3.9)

Notice that
\[
\begin{align*}
\|x_n - S_m^n x_n\| & \leq \|x_n - x_{n+1}\| + \|x_{n+1} - S_m^{n+1} x_{n+1}\|
+ \|S_m^{n+1} x_{n+1} - S_m^{n+1} x_n\| + \|S_m^n x_n - S_m^n x_{n+1}\|
\leq (1 + M)\|x_n - x_{n+1}\| + \|x_{n+1} - S_m^{n+1} x_{n+1}\|
+ M\|S_m^n x_n - x_n\|, \\
\end{align*}
\]

where \(M = \sup_{n \geq 1} \{k_n\}\). It follows from (3.7) and (3.9) that
\[ \lim_{n \to \infty} \|x_n - S_m^n x_n\| = 0, \quad \forall m \in \{1, 2, \ldots, r\}. \] (3.10)

On the other hand, we have
\[
\begin{align*}
\|x_n - T_m^n x_n\| & \leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_m^{n+1} x_{n+1}\|
+ \|T_m^{n+1} x_{n+1} - T_m^{n+1} x_n\| + \|T_m^{n+1} x_n - T_m^n x_n\|
\leq (1 + M)\|x_n - x_{n+1}\| + \|x_{n+1} - T_m^{n+1} x_{n+1}\|
+ M\|T_m^n x_n - x_n\|. \\
\end{align*}
\]

It follows from (3.7) and (3.8) that
\[ \lim_{n \to \infty} \|x_n - T_m^n x_n\| = 0, \quad \forall m \in \{1, 2, \ldots, r\}. \] (3.11)

This completes the proof. □

Now, we give the following weak convergence theorems with Opial’s condition.
Theorem 3.2. Let $C$ be a nonempty, closed, and convex subset of a uniformly convex Banach space $E$ which has Opial’s condition. Let $S_n, T_m : C \to C$ be asymptotically nonexpansive mapping with the sequence $\{s_{n,m}\}$ and $\{t_{n,m}\}$, for every $m \in \{1,2,\ldots,r\}$, where $r \geq 1$. Assume that $\mathcal{F} = \bigcap_{m=1}^r F(S_m) \cap \bigcap_{m=1}^r F(T_m)$ is nonempty. Let $t_n = \max\{t_{n,m} : 1 \leq m \leq r\}$ and $s_n = \max\{s_{n,m} : 1 \leq m \leq r\}$. Assume that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, where $k_n = \max\{s_n,t_n : 1 \leq m \leq r\}$. Let $\{x_n\}$ be a sequence generated by (3.1), where $\{\alpha_n\}, \{\beta_n,1\}, \{\beta_n,2\}, \ldots, \{\beta_n,r\}, \{\gamma_n,1\}, \{\gamma_n,2\}, \ldots, \{\gamma_n,r\}$ are real number sequences in $(0,1)$ such that $\alpha_n + \sum_{m=1}^r \beta_{n,m} + \sum_{m=1}^r \gamma_{n,m} = 1$. Assume that restrictions (a) and (b) as in Proposition 3.1 are satisfied. Then $\{x_n\}$ converges weakly to some point in $\mathcal{F}$.

Proof. Since $\{x_n\}$ is bounded, we find that there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to a point $\tilde{x} \in C$. It follows from Lemma 2.1 and Proposition 3.1 that $\tilde{x} \in \mathcal{F}$. Assume that there exists another subsequence $\{x_{n_t}\} \subset \{x_n\}$ such that $\{x_{n_t}\}$ converges weakly to a point $\hat{x} \in C$. It follows from Lemma 2.1 that $\hat{x} \in \mathcal{F}$. If $\tilde{x} \neq \hat{x}$, then

$$\lim_{n \to \infty} \|x_n - \tilde{x}\| = \liminf_{i \to \infty} \|x_{n_i} - \hat{x}\| < \liminf_{i \to \infty} \|x_{n_j} - \hat{x}\| = \limsup_{j \to \infty} \|x_{n_j} - \hat{x}\| = \lim_{n \to \infty} \|x_n - \hat{x}\|.$$

This is a contradiction. Hence $\tilde{x} = \hat{x}$. Hence every subsequence converges to same point $\tilde{x}$. This completes the proof. \hfill \Box

If $r = 1$, then Theorem 3.2 is reduced to the following.

Corollary 3.3. Let $C$ be a nonempty, closed and convex subset of a uniformly convex Banach space $E$ which has Opial’s condition. Let $S, T : C \to C$ be an asymptotically nonexpansive mappings with the sequences $\{s_n\}$ and $\{t_n\}$. Assume that $\mathcal{F} = F(S) \cap F(T)$ is nonempty. Assume that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, where $k_n = \max\{s_n,t_n : 1 \leq m \leq r\}$. Let $\{x_n\}$ be a sequence generated by the following

$$x_0 \in C, \quad x_n = \alpha_n x_{n-1} + \beta_n S^n x_{n-1} + \gamma_n T^n x_n, \quad \forall n \geq 1,$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real number sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Assume that the following restrictions imposed on the control sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are satisfied

(a) $\liminf_{n \to \infty} \alpha_n \beta_n > 0$ and $\liminf_{n \to \infty} \alpha_n \gamma_n > 0$;

(b) $\gamma_n t_n < 1$.

Then $\{x_n\}$ converges weakly to some point in $\mathcal{F}$.

If $S_n = I$, then Theorem 3.2 is reduced to the following.

Corollary 3.4. Let $C$ be a nonempty, closed, and convex subset of a uniformly convex Banach space $E$ which has Opial’s condition. Let $T_m : C \to C$
be an asymptotically nonexpansive mapping with the sequence \( \{t_{n,m}\} \), for every \( m \in \{1, 2, \ldots, r\} \), where \( r \geq 1 \). Assume that \( \mathcal{F} = \cap_{m=1}^{r} F(T_m) \) is nonempty.

Assume that \( \sum_{n=1}^{\infty} (t_n - 1) < \infty \), where \( t_n = \max\{t_{n,m} : 1 \leq m \leq r\} \).

Let \( \{x_n\} \) be a sequence generated by (3.2), where \( \{\alpha_n\}, \{\beta_{n,1}\}, \{\beta_{n,2}\}, \ldots, \{\beta_{n,r}\}, \{\gamma_{n,1}\}, \{\gamma_{n,2}\}, \ldots, \{\gamma_{n,r}\} \) are real number sequences in \((0, 1)\) such that \( \alpha_n + \sum_{m=1}^{r} \beta_{n,m} + \sum_{m=1}^{r} \gamma_{n,m} = 1 \). Assume that restrictions (a) and (b) in Proposition 3.1 are satisfied. Then \( \{x_n\} \) converges weakly to some point in \( \mathcal{F} \).

If \( T_m = 1 \), then Theorem 3.2 is reduced to the following.

**Corollary 3.5.** Let \( C \) be a nonempty, closed and convex subset of a uniformly convex Banach space \( E \) which has Opial’s condition. Let \( S_m : C \to C \) be an asymptotically nonexpansive mapping with the sequence \( \{s_{n,m}\} \), for every \( m \in \{1, 2, \ldots, r\} \), where \( r \geq 1 \) with \( \mathcal{F} = \cap_{m=1}^{r} F(S_m) \) is nonempty.

Assume that \( \sum_{n=1}^{\infty} (s_n - 1) < \infty \), where \( s_n = \max\{s_{n,m} : 1 \leq m \leq r\} \).

Let \( \{x_n\} \) be a sequence generated by (3.3), where \( \{\alpha_n\}, \{\beta_{n,1}\}, \{\beta_{n,2}\}, \ldots, \{\beta_{n,r}\}, \{\gamma_{n,1}\}, \{\gamma_{n,2}\}, \ldots, \{\gamma_{n,r}\} \) are real number sequences in \((0, 1)\) such that \( \alpha_n + \sum_{m=1}^{r} \beta_{n,m} + \sum_{m=1}^{r} \gamma_{n,m} = 1 \). Assume that the condition (a) in Proposition 3.1 are satisfied. Then \( \{x_n\} \) converges weakly to some point in \( \mathcal{F} \).

Next, we give a necessary and sufficient condition for the strong convergence of (3.1).

**Theorem 3.6.** Let \( C \) be a nonempty, closed and convex subset of a uniformly convex Banach space \( E \). Let \( S_m, T_m : C \to C \) be asymptotically nonexpansive mappings with the sequences \( \{s_{n,m}\} \) and \( \{t_{n,m}\} \), for every \( m \in \{1, 2, \ldots, r\} \), where \( r \geq 1 \). Assume that \( \mathcal{F} = \cap_{m=1}^{r} F(S_m) \cap \cap_{m=1}^{r} F(T_m) \) is nonempty.

Let \( t_n = \max\{t_{n,m} : 1 \leq m \leq r\} \) and \( s_n = \max\{s_{n,m} : 1 \leq m \leq r\} \).

Assume that \( \sum_{n=1}^{\infty} (s_n - 1) < \infty \), where \( k_n = \max\{s_n, t_n : 1 \leq m \leq r\} \).

Let \( \{x_n\} \) be a sequence generated by (3.1), where \( \{\alpha_n\}, \{\beta_{n,1}\}, \{\beta_{n,2}\}, \ldots, \{\beta_{n,r}\}, \{\gamma_{n,1}\}, \{\gamma_{n,2}\}, \ldots, \{\gamma_{n,r}\} \) are real number sequences in \((0, 1)\) such that \( \alpha_n + \sum_{m=1}^{r} \beta_{n,m} + \sum_{m=1}^{r} \gamma_{n,m} = 1 \). Assume that the conditions (a) and (b) in Proposition 3.1 are satisfied. Then \( \{x_n\} \) converges strongly to some point in \( \mathcal{F} \) if and only if

\[
\liminf_{n \to \infty} dist(x_n, \mathcal{F}) = 0.
\]

**Proof.** The necessity of the proof is obvious. We only show the sufficiency of the proof. Assume that \( \liminf_{n \to \infty} dist(x_n, \mathcal{F}) = 0 \). In view of (3.4), we know from Lemma 2.2 that \( \lim_{n \to \infty} dist(x_n, \mathcal{F}) \) exists. From the hypothesis, it follows that \( \lim_{n \to \infty} dist(x_n, \mathcal{F}) = 0 \).

Next, we show that the sequence \( \{x_n\} \) is Cauchy. For positive integers \( m, n \), where \( m > n \), we see from (3.4) that

\[
\|x_n - p\| \leq e^{h_n} \|x_{n-1} - p\|, \quad h_n = \frac{k_{n-1}}{1 - a}.
\]

This in turn implies that

\[
\|x_m - p\| \leq B \|x_n - p\|,
\]

where \( B = e^{h_n} \). It follows that

\[
\|x_n - x_m\| \leq \|x_n - p\| + \|x_m - p\| \leq (1 + B) \|x_n - p\|.
\]
Taking the infimum over all \( p \in \mathcal{F} \), we find that \( \{x_n\} \) is a Cauchy sequence in \( C \). Assume that \( \{x_n\} \) converges strongly to some \( \bar{q} \in C \). Since \( T_m \) and \( S_m \) are Lipschitz for each \( m \in \{1, 2, \ldots, N\} \), we know that \( \mathcal{F} \) is closed. This in turn implies that \( \bar{q} \in \mathcal{F} \). This completes the proof. \( \square \)

If \( S_m = I \), then Theorem 3.6 is reduced to the following.

**Corollary 3.7.** Let \( C \) be a nonempty, closed and convex subset of a uniformly convex Banach space \( E \). Let \( T_m : C \to C \) be an asymptotically nonexpansive mapping with the sequence \( \{t_{n,m}\} \), for every \( m \in \{1, 2, \ldots, r\} \), where \( r \geq 1 \). Assume that \( \mathcal{F} = \bigcap_{m=1}^{\infty} F(T_m) \) is nonempty. Assume that \( \sum_{n=1}^{\infty} (t_n - 1) < \infty \), where \( t_n = \max \{t_{n,m} : 1 \leq m \leq r\} \). Let \( \{x_n\} \) be a sequence generated by (3.2), where \( \{\alpha_n\}, \{\beta_{n,1}\}, \{\beta_{n,2}\}, \ldots, \{\beta_{n,r}\}, \{\gamma_{n,1}\}, \{\gamma_{n,2}\}, \ldots, \{\gamma_{n,r}\} \) are real number sequences in \( (0,1) \) such that \( \alpha_n + \sum_{m=1}^{r} \beta_{n,m} + \sum_{m=1}^{r} \gamma_{n,m} = 1 \). Assume that the conditions (a) and (b) in Proposition 3.1 are satisfied. Then \( \{x_n\} \) converges strongly to some point in \( \mathcal{F} \) if and only if \( \liminf_{n \to \infty} \text{dist}(x_n, \mathcal{F}) = 0 \).

If \( T_m = I \), then Theorem 3.6 is reduced to the following.

**Corollary 3.8.** Let \( C \) be a nonempty, closed and convex subset of a uniformly convex Banach space \( E \). Let \( S_m : C \to C \) be an asymptotically nonexpansive mapping with the sequence \( \{s_{n,m}\} \), for every \( m \in \{1, 2, \ldots, r\} \), where \( r \geq 1 \) is some positive integer. Assume that \( \sum_{n=1}^{\infty} (s_n - 1) < \infty \), where \( s_n = \max \{s_{n,m} : 1 \leq m \leq r\} \). Let \( \{x_n\} \) be a sequence generated by (3.2), where \( \{\alpha_n\}, \{\beta_{n,1}\}, \{\beta_{n,2}\}, \ldots, \{\beta_{n,r}\}, \{\gamma_{n,1}\}, \{\gamma_{n,2}\}, \ldots, \{\gamma_{n,r}\} \) are real number sequences in \( (0,1) \) such that \( \alpha_n + \sum_{m=1}^{r} \beta_{n,m} + \sum_{m=1}^{r} \gamma_{n,m} = 1 \). Assume that the condition (a) in Proposition 3.1 are satisfied. Then \( \{x_n\} \) converges strongly to some point in \( \mathcal{F} \) if and only if \( \liminf_{n \to \infty} \text{dist}(x_n, \mathcal{F}) = 0 \).

4. Applications

Finally, we consider the problem of approximation solutions of variational inequalities as an application of main results.

Let \( C \) be a nonempty, closed and convex subset of a smooth Banach space \( E \) and \( A : C \to E \) an operator. Find an \( x \in C \) such that

\[
\langle Ax, J(y - x) \rangle \geq 0, \quad \forall y \in C. \tag{4.1}
\]

In what follows, the symbol \( VI(C, A) \) stands for the solution set of the above inequality (4.1).

\( A \) is said to be accretive if

\[
\langle Ax - Ay, J(x - y) \rangle \geq 0, \quad \forall x, y \in C.
\]

\( A \) is said to be \( \alpha \)-inverse-strongly accretive if there exists a positive constant \( \alpha \) such that

\[
\langle Ax - Ay, J(x - y) \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.
\]
Let $K$ be a nonempty subset of $C$ and let $Q : C \rightarrow K$ be a mapping. $Q$ is said to be sunny if
\[ Qx = Q(Qx + t(x - Qx)) \]
whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. $Q$ is said to be retraction if $Q^2 = Q$. $Q$ is said to be a sunny nonexpansive retraction if $Q$ is sunny nonexpansive and a retraction onto $K$. A subset $K$ of $C$ is said to be a sunny nonexpansive retract of $C$ if there exists a sunny nonexpansive retraction from $C$ onto $K$.

The following results describe a characterization of sunny nonexpansive retractions on a smooth Banach space; see [22] and [23] for more details.

Let $C$ be a nonempty subset of a smooth Banach space $E$. Let $Q_C$ be a sunny nonexpansive retraction from $E$ onto $C$. Then the following are equivalent:

(a) $Q_C$ is sunny and nonexpansive;
(b) $\langle x - Qx, J(Qx - y) \rangle$, $\forall x \in C, y \in K$.

The following lemma can be found in [17] and [24].

**Lemma 4.1.** Let $E$ be a $q$-uniformly smooth Banach space with $q$-uniformly smoothness constant $C_q > 0$. Then the following holds
\[ \|x + y\|^q \leq \|x\|^q + q\langle y, J_y \rangle + C_q\|y\|^q, \quad \forall x, y \in E. \]

Now, we are in a position to give the main results of this section.

**Theorem 4.2.** Let $E$ be a uniformly convex and $q$-uniformly smooth Banach space with $q$-uniformly smoothness constant $C_q > 0$ and $C$ be a nonempty, closed and convex subset of $E$. Let $Q_C$ be a sunny nonexpansive retraction from $E$ onto $C$. Let $A_m : C \rightarrow E$ be an $a_m$-inverse-strongly accretive operator and $B_m : C \rightarrow E$ a $b_m$-inverse-strongly accretive operator, for every $m \in \{1, 2, \ldots, r\}$, where $r \geq 1$. Assume that $\mathcal{F} = \cap_{m=1}^r VI(C, A_m) \cap \cap_{m=1}^r VI(C, B_m)$ is nonempty. Let $\{x_n\}$ be a sequence generated by the following: $x_0 \in C$,
\[
x_n = \alpha_n x_{n-1} + \sum_{m=1}^{r} \beta_{n,m} Q_C(x_{n-1} - \mu_m A_m x_{n-1})
+ \sum_{m=1}^{r} \gamma_{n,m} Q_C(x_n - \nu_m B_m x_n), \quad \forall n \geq 1,
\]
where $\{\alpha_n\}, \{\beta_{n,1}\}, \{\beta_{n,2}\}, \ldots, \{\beta_{n,r}\}, \{\gamma_{n,1}\}, \{\gamma_{n,2}\}, \ldots, \{\gamma_{n,r}\}$ are real number sequences in $(0, 1)$ such that $\alpha_n + \sum_{m=1}^{r} \beta_{n,m} + \sum_{m=1}^{r} \gamma_{n,m} = 1$ and $\mu_1, \mu_2, \ldots, \mu_r, \nu_1, \nu_2, \ldots, \nu_r$ are real numbers such that $\mu_m \leq \left(\frac{\mu_m}{C_q}\right)^{\frac{1}{2}}$ and $\nu_m \leq \left(\frac{\nu_m}{C_q}\right)^{\frac{1}{2}}$, for every $m \in \{1, 2, \ldots, r\}$. Assume that the condition (a) in Proposition 3.1 are satisfied. If $E$ has Opial’s condition, then $\{x_n\}$ converges weakly to some point in $\mathcal{F}$.

**Proof.** From Lemma 2.7 of Aoyama, Iiduka and Takahashi [24], we find, for every $m \in \{1, 2, \ldots, r\}$, that $VI(C, A_m) = F(Q_C(I - \lambda A))$ and $VI(C, A_m) =
F(QC(I − λB)) for all λ > 0. Notice that QC(I − µmA) and QC(I − νmBm) are nonexpansive. Indeed, we find from Lemma 4.1 that

\[ \|Q_I(I - \mu_m A_m)x - Q_C(I - \mu_m A_m)y\|^q \]

\[ \leq \|x - y\|^q - q\mu_m\langle A_mx - A_my, J_q(x - y)\rangle + C_q\mu_m^q\|A_mx - A_my\|^q \]

\[ \leq \|x - y\|^q - q\mu_m\mu_m\|A_mx - A_my\|^q + C_q\mu_m^q\|A_mx - A_my\|^q \]

\[ = \|x - y\|^q - (q\mu_m\mu_m - C_q\mu_m^q)\|A_mx - A_my\|^q \]

\[ = \|x - y\|^q, \quad \forall x, y \in C. \]

This proves that QC(I − µmA) is nonexpansive, so is QC(I − µmB). Since nonexpansive mappings are asymptotically nonexpansive mappings with the sequence \{1\}, we can easily conclude from Theorem 3.2 the desired conclusion. This completes the proof. □

**Theorem 4.3.** Let E be a uniformly convex and q-uniformly smooth Banach space with q-uniformly smoothness constant Cq > 0 and C a nonempty, closed and convex subset of E. Let QC be a sunny nonexpansive retraction from E onto C. Let Am : C → E be an am-inverse-strongly accretive operator and Bm : E → E a bm-inverse-strongly accretive operator, for every m ∈ \{1, 2, ..., r\},

where \( r \geq 1 \). Assume that \( \mathcal{F} = \cap_{m=1}^{r} VI(C, A_m) \cap \cap_{m=1}^{r} VI(C, B_m) \) is nonempty and Cq ≤ λγ where \( \lambda = \min\{a, b, c\} \). Let \( \{x_n\} \) be a sequence generated by the following: \( x_0 \in C \),

\[ x_n = \alpha_n x_{n-1} + \beta_n \mu_m A_m x_{n-1} + \gamma_n \mu_m B_m x_n, \quad \forall n \geq 1, \]

where \( \{\alpha_n\}, \{\beta_n, 1\}, \{\beta_n, 2\}, \ldots, \{\beta_n, r\}, \{\gamma_n, 1\}, \{\gamma_n, 2\}, \ldots, \{\gamma_n, r\} \) are real number sequences in (0, 1) such that \( \alpha_n + \sum_{m=1}^{r} \beta_n, m + \sum_{m=1}^{r} \gamma_n, m = 1 \) and \( \mu_1, \mu_2, \ldots, \mu_r, \nu_1, \nu_2, \ldots, \nu_r \) are real numbers such that \( \mu_m \leq \left(\frac{\alpha_n, m}{C_q}\right) \) and \( \nu_m \leq \left(\frac{\alpha_n, m}{C_q}\right) \), for every \( m \in \{1, 2, \ldots, r\} \). Assume that the condition (a) in Proposition 3.1 are satisfied. Then \( \{x_n\} \) converges strongly to some point in \( \mathcal{F} \) if and only if \( \lim\inf_{n \to \infty} \text{dist}(x_n, \mathcal{F}) = 0 \).

**Proof.** Notice that QC(I − µmA), and QC(I − νmBm) are nonexpansive. We can immediately conclude from Theorem 3.6 the desired conclusion. This completes the proof. □

**References**


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