A CHARACTERIZATION THEOREM FOR LIGHTLIKE HYPERSONSURFACES OF SEMI-RIEMANNIAN MANIFOLDS OF QUASI-CONSTANT CURVATURES

Dae Ho Jin

Abstract. In this paper, we study lightlike hypersurfaces $M$ of semi-Riemannian manifolds $\bar{M}$ of quasi-constant curvatures. Our main result is a characterization theorem for screen homothetic Einstein lightlike hypersurfaces of a Lorentzian manifold of quasi-constant curvature subject such that its curvature vector field $\zeta$ is tangent to $\bar{M}$.

1. Introduction

B.Y. Chen and K. Yano [2] introduced the notion of a Riemannian manifold of quasi-constant curvature as a Riemannian manifold $(\bar{M}, \bar{g})$ endowed with the curvature tensor $\bar{R}$ satisfying the following equation:

$$\bar{g}(\bar{R}(X,Y)Z,W) = \alpha\{\bar{g}(Y,Z)\bar{g}(X,W) - \bar{g}(X,Z)\bar{g}(Y,W)\} + \beta\{\bar{g}(X,W)\theta(Y)\theta(Z) - \bar{g}(X,Z)\theta(Y)\theta(W) + \bar{g}(Y,Z)\theta(X)\theta(W) - \bar{g}(Y,W)\theta(X)\theta(Z)\},$$

for any vector fields $X$, $Y$, $Z$ and $W$ of $\bar{M}$, where $\alpha$ and $\beta$ are smooth functions and $\theta$ is a 1-form associated with a non-vanishing smooth unit vector field $\zeta$ by

$$\theta(X) = \bar{g}(X, \zeta),$$

$\zeta$ is called the curvature vector field of $\bar{M}$. It is well known that if the curvature tensor $\bar{R}$ is of the form (1.1), then $\bar{M}$ is conformally flat. If $\beta = 0$, then $\bar{M}$ is a space of constant curvature $\alpha$.

A non-flat Riemannian manifold $\bar{M}$ of dimension $n(>2)$ is called a quasi-Einstein manifold [1] if its Ricci tensor $\bar{Ric}$ satisfies the condition

$$\bar{Ric}(X,Y) = a\bar{g}(X,Y) + b\phi(X)\phi(Y),$$

where $a$ and $b$ are smooth functions such that $b \neq 0$ and $\phi$ is a non-vanishing 1-form such that $\bar{g}(X,U) = \phi(X)$ for any vector field $X$, where $U$ is a unit vector field

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vector field. If $b = 0$, then $\bar{M}$ is an Einstein manifold. It is easily to see that every Riemannian manifold of quasi-constant curvature is quasi-Einstein.

The classification of Einstein lightlike hypersurfaces $M$ in semi-Riemannian manifolds $\bar{M}$ was studied by K.L. Duggal and D.H. Jin [5]. Their main results focused on the geometry of Einstein lightlike hypersurfaces $M$ of a Lorentzian space form $\bar{M}(c)$ of constant curvature $c$, whose shape operator is conformal to the shape operator of its screen distribution by some non-zero constant $\varphi$, which is called the conformal factor. Such a $M$ is called screen homothetic. The reason for this geometric restriction on $M$ was due to the fact that such a class admits a canonical integrable screen distribution and a symmetric induced Ricci tensor of $M$. Authors proved a characterization theorem for screen homothetic Einstein lightlike hypersurfaces of a Lorentzian space form as it follow:

**Theorem 1.1.** Let $M$ be a screen homothetic Einstein lightlike hypersurface of a Lorentzian space form $\bar{M}^{m+2}(c)$, $m > 2$, such that $\text{Ric} = \kappa g$. Then $c = 0$, i.e., $\bar{M}$ is flat manifold, and $M$ is locally a product manifold $\mathcal{C} \times M_1 \times M_2$, where $\mathcal{C}$ is a null curve tangent to the radical distribution, and $M_1$ and $M_2$ are leaves of some integrable distributions of $M$ such that

1. If $\kappa \neq 0$, then either $M_1$ or $M_2$ is an $m$-dimensional totally umbilical Einstein Riemannian space form which is isometric to a sphere or a hyperbolic space according to the sign of $\kappa$ and the other is a point.
2. If $\kappa = 0$, then $M_1$ is an $(m - 1)$ or an $m$-dimensional Euclidean space and $M_2$ is a non-null curve or a point.

After that, D.H. Jin [6] generalized the above Duggal-Jin’s characterization theorem for screen conformal Einstein lightlike hypersurfaces of Lorentzian space forms in which the conformal factor is non-vanishing smooth function $\varphi$.

The objective of this paper is to generalize the above characterization theorem for screen homothetic Einstein lightlike hypersurfaces of a Lorentzian manifold of quasi-constant curvature. We prove a characterization theorem for screen homothetic lightlike hypersurfaces $M$ of a Lorentzian manifold $\bar{M}$ of quasi-constant curvature subject such that the curvature vector field $\zeta$ of $\bar{M}$, defined by (1.2), is tangent to $M$.

### 2. Lightlike hypersurface

It is well-known [3] that the normal bundle $TM^\perp$ of the lightlike hypersurfaces $(M, g)$ of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is a subbundle of the tangent bundle $TM$ and coincides with the radical distribution $\text{Rad}(TM) = TM \cap TM^\perp$. Thus there exists a non-degenerate complementary vector bundle $S(TM)$ of $\text{Rad}(TM)$ in $TM$, which is called a screen distribution, such that

$$TM = \text{Rad}(TM) \oplus_{\text{orth}} S(TM),$$

(2.1)

where $\oplus_{\text{orth}}$ denotes the orthogonal direct sum. We denote such a lightlike hypersurface by $M = (M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth
functions on $M$ and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle $E$ over $M$. It is well-known [3] that, for any null section $\xi$ of $\text{Rad}(TM)$ on a coordinate neighborhood $U \subset M$, there exists a unique null section $N$ of a unique lightlike vector bundle $tr(TM)$ in $S(TM)^\perp$ satisfying
\[ g(\xi, N) = 1, \quad g(N, N) = g(N, X) = 0, \quad \forall X \in \Gamma(S(TM)|_U). \]
Then the tangent bundle $TM$ of $\tilde{M}$ is decomposed as follows:
\[
TM = TM \oplus tr(TM) = \{ \text{Rad}(TM) \oplus tr(TM) \} \oplus_{\text{orth}} S(TM). \tag{2.2}
\]
We call $tr(TM)$ and $N$ the transversal vector bundle and the null transversal vector field of $M$ with respect to $S(TM)$ respectively.

In the sequel, we take $X, Y, Z, W \in \Gamma(TM)$, unless otherwise specified. Let $\nabla$ be the Levi-Civita connection of $M$ and $P$ the projection morphism of $TM$ on $S(TM)$ with respect to the decomposition (2.1). Then the local Gauss and Weingartan formulas for $M$ and $S(TM)$ are given respectively by
\[
\begin{align*}
\bar{\nabla}_XY &= \nabla XY + B(X, Y)N, \tag{2.3} \\
\bar{\nabla}_XN &= -A_N X + \tau(X)N; \tag{2.4} \\
\bar{\nabla}_XPY &= \nabla^*_X PY + C(X, PY)\xi, \tag{2.5} \\
\bar{\nabla}_X\xi &= -A^*_\xi X - \tau(X)\xi, \tag{2.6}
\end{align*}
\]
where $\nabla$ and $\nabla^*$ are the linear connections on $TM$ and $S(TM)$ respectively, $B$ and $C$ are the local second fundamental forms on $TM$ and $S(TM)$ respectively, $A_N$ and $A^*_\xi$ are the shape operators on $TM$ and $S(TM)$ respectively and $\tau$ is a 1-form on $TM$. Since $\nabla$ is torsion-free, $\nabla$ is also torsion-free and $B$ is symmetric. From the fact $B(X, Y) = g(\nabla XY, \xi)$, we know that $B$ is independent of the choice of the screen distribution $S(TM)$ and
\[
B(X, \xi) = 0. \tag{2.7}
\]
The induced connection $\nabla$ of $M$ is not metric and satisfies
\[
(\nabla_X g)(Y, Z) = B(X, Y) \eta(Z) + B(X, Z) \eta(Y), \quad \eta(X) = g(\xi, X). \tag{2.8}
\]
But the induced connection $\nabla^*$ on $S(TM)$ is metric. The above two local second fundamental forms $B$ and $C$ are related to their shape operators by
\[
\begin{align*}
B(X, Y) &= g(A^*_\xi X, Y), \quad \bar{g}(A^*_\xi X, N) = 0, \tag{2.9} \\
C(X, PY) &= g(A_N X, PY), \quad \bar{g}(A_N X, N) = 0. \tag{2.10}
\end{align*}
\]
From (2.9), $A^*_\xi$ is $S(TM)$-valued and self-adjoint on $TM$ such that
\[
A^*_\xi \xi = 0. \tag{2.11}
\]
Denote by $\bar{R}$, $R$ and $R^*$ the curvature tensors of the connections $\nabla$, $\nabla$ and $\nabla^*$ respectively. Using the Gauss-Weingarten formulas for $M$ and $S(TM)$, we
obtain the Gauss-Codazzi equations for $M$ and $S(TM)$ such that
\begin{equation}
\bar{g}(\bar{R}(X,Y)Z, PW) = \bar{g}(\bar{R}(X,Y)Z, PW) = g(R(X,Y)Z, PW)
+ B(X, Z)C(Y, PW) - B(Y, Z)C(X, PW),
\end{equation}
\begin{equation}
\bar{g}(\bar{R}(X,Y)Z, \xi) = (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z)
+ B(Y, Z)\tau(X) - B(X, Z)\tau(Y),
\end{equation}
\begin{equation}
\bar{g}(\bar{R}(X,Y)Z, N) = \bar{g}(\bar{R}(X,Y)Z, N),
\end{equation}
\begin{equation}
g(\bar{R}(X,Y)\xi, N) = g(A_\xi^X A_\xi^Y) - g(A_\xi^Y A_\xi^X) - 2d\tau(X, Y),
\end{equation}
\begin{equation}
g(\bar{R}(X,Y)PZ, PW) = g(R^*(X,Y)PZ, PW)
+ C(X, PZ)B(Y, PW) - C(Y, PZ)B(X, PW),
\end{equation}
\begin{equation}
g(\bar{R}(X,Y)PZ, N) = (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ)
+ C(X, PZ)\tau(Y) - C(Y, PZ)\tau(X).
\end{equation}

The Ricci tensor $\bar{Ric}$ of $M$ is defined by
\[\bar{Ric}(X, Y) = \text{trace}[Z \rightarrow \bar{R}(Z, X)Y],\] for any $X, Y \in \Gamma(TM)$. Let $\dim M = m + 2$. Locally, $\bar{Ric}$ is given by
\begin{equation}
\bar{Ric}(X, Y) = \sum_{i=1}^{m+2} \epsilon_i \bar{g}(\bar{R}(E_i, X)Y, E_i),
\end{equation}
where $\{E_1, \ldots, E_{m+2}\}$ is an orthonormal frame field of $TM$.

### 3. Screen homothetic lightlike hypersurfaces

Now we consider an induced quasi-orthonormal frame field $\{\xi; W_a\}$ on $M$, where $\text{Rad}(TM) = \text{Span}\{\xi\}$ and $S(TM) = \text{Span}\{W_a\}$ be the corresponding frame field on $M$. By using (2.18), we get
\begin{equation}
\bar{Ric}(X, Y) = \sum_{a=1}^{m} \epsilon_a \bar{g}(\bar{R}(W_a, X)Y, W_a) + \bar{g}(\bar{R}(\xi, X)Y, N)
\end{equation}
\begin{equation}
+ \bar{g}(\bar{R}(N, X)Y, \xi), \quad \forall X, Y \in \Gamma(TM),
\end{equation}
where $\epsilon_a (\equiv \pm 1)$ denotes the causal character of respective vector field $W_a$. Let $R^{(0,2)}$ denote the induced Ricci type tensor of type $(0, 2)$ on $M$ given by
\begin{equation}
R^{(0,2)}(X, Y) = \text{trace}[Z \rightarrow R(Z, X)Y], \quad \forall X, Y \in \Gamma(TM).
\end{equation}

Using the induced quasi-orthonormal frame field $\{\xi; W_a\}$ on $M$, we obtain
\begin{equation}
R^{(0,2)}(X, Y) = \sum_{a=1}^{m} \epsilon_a g(R(W_a, X)Y, W_a) + \bar{g}(R(\xi, X)Y, N).
\end{equation}

Substituting (2.12) and (2.14) in (3.1) an using (2.9) and (2.10), we obtain
\begin{equation}
R^{(0,2)}(X, Y) = R\bar{ic}(X, Y) + B(X, Y)trA_X - g(A_X A_\xi Y)
- \bar{g}(R(\xi, Y)X, N), \quad \forall X, Y \in \Gamma(TM).
\end{equation}
This shows that $R^{(0,2)}$ is not symmetric. The tensor field $R^{(0,2)}$ is called its induced Ricci tensor [4], and denote it by $\text{Ric}$, if it is symmetric. If $R^{(0,2)}$ is an induced Ricci tensor $\text{Ric}$ of $M$ and $\text{Ric} = \kappa g$, then $M$ is called an Einstein manifold. In this case, if $m > 1$, then we show that $\kappa$ is a constant.

Using (2.15), (3.4) and the first Bianchi’s identity, we obtain

$$R^{(0,2)}(X,Y) - R^{(0,2)}(Y,X) = 2d\tau(X,Y), \quad \forall X, Y \in \Gamma(TM).$$

From this equation, we have the following theorem:

**Theorem 3.1.** [3, 4] Let $M$ be a lightlike hypersurface of a semi-Riemannian manifold $M$. Then the Ricci type tensor $R^{(0,2)}$ is symmetric if and only if the 1-form $\tau$ is closed, i.e., $d\tau = 0$, on any coordinate neighborhood $U \subset M$.

**Remark 1.** In case $d\tau = 0$, by the cohomology theory there exist a smooth function $l$ such that $\tau = dl$. Thus we get $\tau(X) = X(l)$. If we take $\xi = \gamma \xi$, then we have $\tau(X) = \tilde{\tau}(X) + X(ln\gamma)$. Setting $\gamma = \exp(l)$ in this equation, we get $\tilde{\tau}(X) = 0$. We call the pair $\{\xi, N\}$ such that the corresponding 1-form $\tau$ vanishes the canonical null pair of $M$. Although $S(TM)$ is not unique and the lightlike geometry depends on its choice but it is canonically isomorphic to the factor vector bundle $S(TM)^\mathbb{R} = TM/Rad(TM)$ due to Kupeli [8]. Thus all $S(TM)$ are mutually isomorphic. In the sequel, we deal with only lightlike hypersurfaces $M$ equipped with the canonical null pair $\{\xi, N\}$.

Let $M$ be a lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$ of quasi-constant curvature. We may assume that the curvature vector field $\zeta$ of $\bar{M}$ is a spacelike unit tangent vector field of $M$. In this case, if $\zeta$ belongs to $\text{Rad}(TM)$, then $\zeta = e\xi$, where $e = \theta(N) \neq 0$. From this fact, we have $1 = \tilde{g}(\zeta, \zeta) = e^2 g(\xi, \xi) = 0$. It is a contradiction. This enables one to choose a screen distribution $S(TM)$ which contains $\zeta$ due to (2.1). This implies that if $\zeta$ is tangent to $M$, then it belongs to $S(TM)$ which we assume in this paper.

**Definition 1.** A lightlike hypersurface $M$ of a semi-Riemannian manifold $\bar{M}$ is screen conformal [4, 5, 6] if the shape operators $A_\nu^\ast$ and $A_\xi^\ast$ are related by $A_\nu = \varphi A_\xi^\ast$, or equivalently, the second fundamental forms $B$ and $C$ satisfy

$$C(X, PY) = \varphi B(X,Y),$$

where $\varphi$ is a non-vanishing smooth function on a coordinate neighborhood $U$ in $M$. If $\varphi$ is a non-zero constant, then we say that $M$ is screen homothetic.

**Example 1.** Let $(\mathbb{R}^7, \tilde{g}_0)$ be a 7-dimensional semi-Euclidean space of index 2 with signature $(-, -, +, +, +, +, +)$ of the canonical basis

$$\{\partial x_1, \partial x_2, \ldots, \partial x_6, \partial x_7 = \zeta\}.$$

Consider a lightlike hypersurface of $\mathbb{R}^7$, defined by

$$X(u_1, u_2, u_3, u_4, u_5, t) = (u_1 + u_2 + u_3, u_1, u_2, u_3, u_4, u_5, t),$$
whose radical distribution $\text{Rad}(TM)$ is spanned by
\[ \xi = \partial_1 - \partial_2 + \partial_3 + \partial_4. \]
We consider a complementary vector bundle $F^*$ of $TM^\perp$ in $S(TM)^\perp$ and take
\[ V^* = \partial_1 - \partial_2 \in \Gamma(F^*), \quad V^* \neq 0, \text{ such that } \bar{g}_0(\xi, V^*) \neq 0. \]
Then the transversal vector bundle is given by $tr(TM) = \text{Span}\{N\}$, where
\[ N = \frac{1}{\bar{g}_0(\xi, V^*)} \left\{ V^* - \frac{\bar{g}_0(V^*, V^*)}{2\bar{g}_0(\xi, V^*)} \xi \right\} = \frac{1}{4}(\partial_1 - \partial_2 - \partial_3 - \partial_4). \]
It follows that the corresponding screen distribution $S(TM)$ is spanned by
\[ \{W_1 = \partial_1 + \partial_2, \quad W_2 = \partial_3 - \partial_4, \quad W_3 = \partial_5, \quad W_4 = \partial_6, \quad W_5 = \partial_7 = \zeta\}. \]
Taking the covariant derivative to $N$ along $R_2$, we get
\[ \bar{\nabla}_X N = \frac{1}{4} \bar{\nabla}_X \xi, \quad \text{since } \bar{\nabla}_X V^* = 0. \]
Using Gauss and Weingarten formulae, we obtain
\[ -A_N X + \tau(X)N = -\frac{1}{4}(A_N^* X + \tau(X)\xi). \]
Taking the scaler product with $\xi$ and $N$ to this, we get $\tau(X) = 0$, which gives
\[ A_N X = \frac{1}{4} A_N^* X, \quad \forall X \in \Gamma(TM). \]
Thus $M$ is a screen homothetic lightlike hypersurface of conformal factor $\varphi = \frac{1}{4}$.

**Theorem 3.2.** Let $M$ be a screen conformal lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$ of quasi-constant curvature. If $\zeta$ is tangent to $M$, then the tensor field $R^{(0,2)}$ is an induced symmetric Ricci tensor of $M$.

**Proof.** Replacing $W$ by $N$ to (1.1) and using the fact $\theta(N) = 0$, we have
\[ \bar{g}(\bar{R}(X,Z)N, N) = \alpha \{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\} + \beta \{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}\theta(Z). \]
Replacing $Z$ by $\xi$ to (3.6) and using $\theta(\xi) = 0$, we have $\bar{g}(\bar{R}(X,Y)\xi, N) = 0$. Comparing this result with (2.15) and using the fact $A_N = \varphi A_N^*$, we show that $d\tau = 0$. Thus, by Theorem 3.1, we have our assertion. □

**Theorem 3.3.** Let $M$ be a screen homothetic lightlike hypersurface of a semi-Riemannian manifold $M$ of quasi-constant curvature. If $\zeta$ is tangent to $M$, then the functions $\alpha$ and $\beta$ vanish identically. Thus $M$ is a flat manifold.

**Proof.** Using (1.1), (2.14) and (3.6), we have
\[ \bar{g}(\bar{R}(X,Y)\xi, N) = \alpha g(X,Y) + \beta \theta(X)\theta(Y), \quad \text{for } \alpha g(X,Y) + \beta \theta(X)\theta(Y). \]
Substituting the last two equations into (3.4), we have
\[ R^{(0,2)}(X,Y) = \{ma + \beta\}g(X,Y) + \beta(m-1)\theta(X)\theta(Y) \]  
\[ + \quad B(X,Y)\text{tr}A_N - g(A_N X, A_N^\ast Y). \]  
(3.9)

As \( d\tau = 0 \), we can take a canonical null pair such that \( \tau = 0 \) due to Remark 1. Replacing \( W \) by \( \xi \) to (1.1) and using (2.13) and the fact \( \theta(\xi) = 0 \), we have
\[ (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = 0. \]  
(3.10)

Assume that \( M \) is screen homothetic. Substituting (3.5) into (2.17) and using (3.10), we get
\[ \bar{g}(R(X,Y)PZ,N) = 0. \]  
From this result and the fact \( \bar{g}(R(X,Y)\xi,N) = 0 \), we show that, for all \( Z \in \Gamma(TM), \)
\[ \bar{g}(R(X,Y)Z,N) = 0. \]  
(3.11)

Replacing \( X \) by \( \xi \) and \( Z \) by \( X \) to this and comparing with (3.8), we have
\[ \beta\theta(X)\theta(Y) = -\alpha g(X,Y), \quad \forall X, Y \in \Gamma(TM). \]  
(3.11)

Taking \( X = Y = \zeta \) to (3.11), we get \( \beta = -\alpha \). Substituting (3.11) into (3.9) and using the fact \( \beta = -\alpha \), we obtain
\[ \text{Ric}(X,Y) = \varphi\{B(X,Y)\text{tr}A_N^\ast - g(A_N^\ast X, A_N^\ast Y)\}. \]  
(3.12)

Substituting (3.11) into (1.1) and using (2.12), (2.13) and (3.5), we have
\[ R(X,Y)Z = \alpha(g(X,Z)Y - g(Y,Z)X) \]  
\[ + \varphi\{B(Y,Z)A_N^\ast X - B(X,Z)A_N^\ast Y\}. \]  
(3.13)

Substituting (3.13) and \( \bar{g}(R(\xi,Y)X,N) = 0 \) into (3.3), we also have
\[ \text{Ric}(X,Y) = -(m-1)\alpha g(X,Y) + \varphi\{B(X,Y)\text{tr}A_N - g(A_N^\ast X, A_N^\ast Y)\}. \]  
(3.14)

Comparing (3.12) and (3.14), we obtain \( \alpha = 0 \) as \( m > 1 \). As \( \beta = -\alpha \), we also have \( \beta = 0 \). Thus \( \bar{M} \) is a flat manifold. □

By the characterization theorem of Duggal-Jin [5] (Theorem 1.1 in this paper), we have the following result:

**Theorem 3.4.** Let \( M \) be a screen homothetic Einstein lightlike hypersurface of a Lorentzian manifold \( \bar{M}^{m+2}, m > 2 \), of quasi-constant curvature such that \( \text{Ric} = \kappa g \). If the curvature vector field \( \zeta \) of \( \bar{M} \) is tangent to \( M \), then \( \bar{M} \) is flat manifold and \( M \) is locally a product manifold \( \mathcal{C} \times M_1 \times M_2 \), where \( \mathcal{C} \) is a null curve tangent to the radical distribution, and \( M_1 \) and \( M_2 \) are leaves of some integrable distributions of \( M \) such that

1. If \( \kappa \neq 0 \), then either \( M_1 \) or \( M_2 \) is an \( m \)-dimensional totally umbilical Einstein Riemannian space form which is isometric to a sphere or a hyperbolic space according to the sign of \( \kappa \) and the other is a point.
2. If \( \kappa = 0 \), then \( M_1 \) is an \( (m-1) \) or an \( m \)-dimensional Euclidean space and \( M_2 \) is a non-null curve or a point.
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Department of Mathematics, Dongguk University, Gyeongju 780-714, Republic of Korea
E-mail address: jindh@dongguk.ac.kr