A NEW GENERALIZED RESOLVENT AND APPLICATION
IN BANACH MAPPINGS

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Abstract. In this paper, we introduce a new generalized resolvent in a
Banach space and discuss its some properties. Using these properties, we
obtain an iterative scheme for finding a point which is a fixed point of
relatively weak nonexpansive mapping and a zero of monotone mapping.
Furthermore, strong convergence of the scheme to a point which is a fixed
point of relatively weak nonexpansive mapping and a zero of monotone
mapping is proved.

1. Introduction

Let $E$ be a real Banach space with dual $E^*$. We denote by $J$ the normalized
duality mapping from $E$ into $2^{E^*}$, defined by

$$Jx := \{f^* \in E^*: \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if $E^*$
is strictly convex then $J$ is single-valued and if $E$ is uniformly smooth then $J$
is uniformly continuous on bounded subsets of $E$. Moreover, if $E$ is a reflexive
and strictly convex Banach space with a strictly convex dual, then $J^{-1}$ is single
valued, one-to-one, surjective, and it is the duality mapping from $E^*$ into $E$
and thus $JJ^{-1} = I_{E^*} = I^*$ and $J^{-1}J = I_E = I$ (see [3]). We note that in a
Hilbert space $H$, $J$ is the identity mapping. Let $E$ be a smooth, reflexive, and
strictly convex Banach space. We define the function $V_2 : E \times E \to \mathbb{R}$ by

$$V_2(y, x) = \|x\|^2 - 2\langle Jy, x \rangle + \|y\|^2,$$  \hspace{1cm} (1.1)

for $\forall x \in E, y \in E$. Let $C$ be a nonempty closed convex subset of $E$. For an
arbitrary point $x$ of $E$, consider the set $\{z \in C : V_2(z, x) = \min_{y \in C} V_2(y, x)\}$. 

Received May 15, 2013; Accepted January 14, 2014.
2000 Mathematics Subject Classification. 47H05,47H10,47J05.
Key words and phrases. Generalized resolvent; Generalized projection; Uniformly smooth
Banach space; Weak nonexpansive mapping; Monotone mapping.

This work was financially supported by The Natural Science Foundation for Young Fund of
HeBei Province (No. A2010000191) and The Planning Guide of Research and Development
of Science and Technology of Hebei Province, Baoding City (No. 12ZJ001).
It is known that this set is always a singleton (see [7]). Let \( \Pi_C \) be a mapping of \( E \) onto \( C \) satisfying

\[
V_2(\Pi_C x, x) = \min_{y \in C} V_2(y, x).
\]

(1.2)

Such a mapping \( \Pi_C \) is called the generalized projection.

Applying the definitions of \( V_2 \) and \( J \), a functional \( V : E^* \times E \to R \) is defined by the formula:

\[
V(x^*, y) = V_2(J^{-1}x^*, y), \quad \forall x^* \in E^*, y \in E.
\]

In the following, we shall make use of the following lemmas.

**Lemma 1.1.** ([1]) Let \( E \) be a real smooth Banach space, \( A : E \to 2^{E^*} \) be a maximal monotone mapping, then \( A^{-1}0 \) is a closed and convex subset of \( E \) and the graph of \( A \), \( G(A) \), is demiclosed in the following sense: \( \forall x_n \in D(A) \) with \( x_n \to x \) in \( E \), and \( \forall y_n \in Ax_n \) with \( y_n \to y \) in \( E \) imply that \( x \in D(A) \) and \( y \in Ax \).

**Lemma 1.2.** ([7]) Let \( C \) be a nonempty closed and convex subset of a real reflexive, strictly convex, and smooth Banach space \( E \) and let \( x \in E \). Then \( y \in C \)

\[
V_2(y, \Pi_C x) + V_2(\Pi_C x, x) \leq V_2(y, x).
\]

**Lemma 1.3.** ([7]) Let \( C \) be a convex subset of a real smooth Banach space \( E \). Let \( x \in E \) and \( x_0 \in C \). Then \( V_2(x_0, x) = \inf \{ V_2(z, x) : z \in C \} \) if and only if

\[
\langle z - x_0, Jx_0 - Jx \rangle \geq 0.
\]

**Lemma 1.4.** ([4]) Let \( E \) be a real smooth and uniformly convex Banach space and let \( \{x_n\} \) and \( \{y_n\} \) be two sequences of \( E \). If either \( \{x_n\} \) or \( \{y_n\} \) is bounded and \( V_2(x_n, y_n) \to 0 \) as \( n \to \infty \), then \( x_n - y_n \to 0 \), as \( n \to \infty \).

Let \( E^* \) be a smooth Banach space and let \( D^* \) be a nonempty closed convex subset of \( E^* \). A mapping \( R^* : D^* \to D^* \) is called generalized nonexpansive if \( F(R^*) \neq 0 \) and

\[
V(R^*x^*, J^{-1}y^*) \leq V(x^*, J^{-1}y^*), \quad \forall x^* \in D^*, y^* \in F(R^*),
\]

where \( F(R^*) \) is the set of fixed points of \( R^* \).

Let \( C \) be a nonempty closed convex subset of \( E \), and let \( T \) be a mapping from \( C \) into itself. We denote by \( F(T) \) the set of fixed points of \( T \). A point of \( p \) in \( C \) is said to be a strong asymptotic fixed point of \( T \) if \( C \) contains a sequence \( \{x_n\} \) which converges strongly to \( p \) such that the strong \( \lim_{n \to \infty} (Tx_n - x_n) = 0 \). The set of strong asymptotic fixed points of \( T \) will be denoted by \( \tilde{F}(T) \). A mapping \( T \) from \( C \) into itself is called weak relatively nonexpansive if \( \tilde{F}(T) = F(T) \) and \( V_2(p, Tx) \leq V_2(p, x) \) for all \( x \in C \) and \( p \in F(T) \). (see [8])

In this paper, motivated by Alber [7], Iiduka and Takahashi [6] and Habtu [2], we first introduce the generalized resolvent and discuss its properties. Secondly, we give an iterative scheme for finding a point which is a fixed point of relatively weak nonexpansive mapping and a zero of monotone mapping.
Finally we show its convergence.

2. Second section

Let $E^*$ be a reflexive and smooth Banach space and let $B \subset E \times E^*$ be a maximal monotone operator. For each $\lambda > 0$ and $x \in E$, consider the set

$$J_\lambda x^* := \{ z^* \in E^* : x^* \in z^* + \lambda BJ^{-1}(z^*) \}.$$  

If $z_1^* + \lambda w_1^* = x^*, z_2^* + \lambda w_2^* = x^*, w_1^* \in BJ^{-1}(z_1^*), w_2^* \in BJ^{-1}(z_2^*)$, then we have from the monotonicity of $B$ that

$$\langle w_1^* - w_2^*, J^{-1}(z_1^*) - J^{-1}(z_2^*) \rangle \geq 0$$

and hence

$$\left( \frac{x^* - z_1^*}{\lambda} - \frac{x^* - z_2^*}{\lambda}, J^{-1}(z_1^*) - J^{-1}(z_2^*) \right) \geq 0.$$  

So, we obtain

$$\langle x^* - z_1^* - (x^* - z_2^*), J^{-1}(z_1^*) - J^{-1}(z_2^*) \rangle \geq 0.$$  

and hence

$$\langle z_2^* - z_1^*, J^{-1}(z_1^*) - J^{-1}(z_2^*) \rangle \geq 0.$$  

This implies $z_1^* = z_2^*$. Then $J_\lambda x^*$ consists of one point. We also denote the domain and the range of $J_\lambda x^*$ by $D(J_\lambda) = R(I^* + \lambda BJ^{-1})$ and $R(J_\lambda) = D(BJ^{-1})$, respectively, where $I^*$ is the identity on $E^*$. Such a $J_\lambda : E^* \to E^*$ is called the generalized resolvent of $B$ and is denoted by

$$J_\lambda = (I^* + \lambda BJ^{-1})^{-1}.$$  

(2.1)

We get some properties of $J_\lambda$ and $(BJ^{-1})^{-1}0$.

**Proposition 2.1.** Let $E^*$ be a reflexive and strictly convex Banach space with a Fréchet differentiable norm and let $B \subset E \times E^*$ be a maximal monotone operator with $B^{-1}0 \neq 0$. Then the following hold:

1. $D(J_\lambda) = E^*$ for each $\lambda > 0$.
2. $(BJ^{-1})^{-1}0 = F(J_\lambda)$ for each $\lambda > 0$, where $F(J_\lambda)$ is the set of fixed points of $J_\lambda$.
3. $(BJ^{-1})^{-1}0$ is closed.
4. $J_\lambda : E^* \to E^*$ is generalized nonexpansive for each $\lambda > 0$.

**Proof.** (1) From the maximality of $B$, we have

$$R(J + \lambda B) = E^*, \forall \lambda > 0.$$  

Hence, for each $x^* \in E^*$, there exists $x \in E$ such that $x^* \in Jx + \lambda Bx$. Since $E$ is reflexive and strictly convex, then $J$ is bijective. Therefore, there exists $z^* \in E^*$ such that $x = J^{-1}(z^*)$. Therefore, we have

$$x^* \in JJ^{-1}(z^*) + \lambda BJJ^{-1}(z^*) = z^* + \lambda BJJ^{-1}(z^*) \subset R(I^* + \lambda BJ^{-1}) = D(J_\lambda).$$  

This implies $E^* \subset D(J_\lambda)$. $D(J_\lambda) \subset E^*$ is clear. So, we have $D(J_\lambda) = E^*$. 

Therefore we get

\[ x^* \in F(J_\lambda) \iff J_\lambda^* x^* = x^* \iff x^* \in x^* + \lambda BJ^{-1}(x^*) \]

\[ \iff 0 \in \lambda BJ^{-1}(x^*) \iff 0 \in BJ^{-1}(x^*) \iff x^* \in (BJ^{-1})^{-1} 0. \]

(3) Let \( \{x_n^*\} \subset (BJ^{-1})^{-1} 0 \) with \( x_n^* \to x^*. \) From \( x_n^* \in (BJ^{-1})^{-1} 0, \) we have \( J^{-1}(x_n^*) \in B^{-1} 0. \) Since \( J^{-1} \) is norm to norm continuous, and \( B^{-1} 0 \) is closed, we have that \( J^{-1}(x_n^*) \to J^{-1}(x^*) \in B^{-1} 0. \) This implies \( x^* \in (BJ^{-1})^{-1} 0. \) That is, \( (BJ^{-1})^{-1} 0 \) is closed.

(4) Let \( x^* \in E^*, y^* \in E^*, z^* \in E^* \) and \( \lambda > 0. \) By definition (1.1) and calculated that

\[ V(x^*, J^{-1} z^*) + V(z^*, J^{-1} y^*) = \|x^*\|^2 + \|z^*\|^2 - 2\langle x^*, J^{-1} z^* \rangle \]

\[ + \|y^*\|^2 + \|z^*\|^2 - 2\langle z^*, J^{-1} y^* \rangle \]

\[ = V(x^*, J^{-1} y^*) + 2\langle z^* - x^*, J^{-1} z^* - J^{-1} y^* \rangle, \]

we have that

\[ V(x^*, J^{-1} y^*) = V(x^*, J^{-1} z^*) + V(z^*, J^{-1} y^*) + 2\langle x^* - z^*, J^{-1} z^* - J^{-1} y^* \rangle. \]

Let \( x^* \in E^*, y^* \in F(J_\lambda) \) and \( \lambda > 0. \) From above formula, we have

\[ V(x^*, J^{-1} y^*) = V(x^*, J^{-1} J_\lambda x^*) + V(J_\lambda^* x^*, J^{-1} y^*) + 2\langle x^* - J_\lambda^* x^*, J^{-1} J_\lambda x^* - J^{-1} y^* \rangle. \]

Since \( \frac{x^* - J_\lambda^* x^*}{\lambda} \in BJ^{-1}(J_\lambda x^*) \) and \( 0 \in BJ^{-1}(y^*), \) we have

\[ \langle x^* - J_\lambda^* x^*, J^{-1} J_\lambda^* x^* - J^{-1} y^* \rangle \geq 0. \]

Therefore we get

\[ V(x^*, J^{-1} y^*) \geq V(x^*, J^{-1} J_\lambda^* x^*) + V(J_\lambda^* x^*, J^{-1} y^*) \geq V(J_\lambda^* x^*, J^{-1} y^*). \]

That is, \( J_\lambda^* \) is generalized nonexpansive on \( E^*. \)

Theorem 2.2. \((5)\) Let \( E \) be a Banach space and let \( A \subset E \times E^* \) be a maximal monotone operator with \( A^{-1} 0 \neq \emptyset. \) If \( E^* \) is strictly convex and has a Fréchet differentiable norm, then, for each \( x \in E, \) \( \lim_{\lambda \to \infty} (J + \lambda A)^{-1} J(x) \) exists and belongs to \( A^{-1} 0. \)

Using Theorem 2.2, we get the following result.

Theorem 2.3. Let \( E^* \) be a uniformly convex Banach space with a Fréchet differentiable norm and let \( B \subset E \times E^* \) be a maximal monotone operator with \( B^{-1} 0 \neq \emptyset. \) Then the following hold:

(1) For each \( x^* \in E^*, \) \( \lim_{\lambda \to \infty} J_\lambda^* x^* \) exists and belongs to \( (BJ^{-1})^{-1} 0. \)

(2) If \( R^* x^* := \lim_{\lambda \to \infty} J_\lambda^* x^* \) for each \( x^* \in E^*, \) then \( R^* \) is a sunny generalized nonexpansive retraction of \( E^* \) onto \( (BJ^{-1})^{-1} 0. \)
Proof. (1) Defining a mapping $Q_\lambda$ from $E$ to $E$ by

$$Q_\lambda x := (I + \lambda J^{-1}B)^{-1}x, \quad \forall x \in E, \lambda > 0,$$

we have, for $\forall x^* \in E^*, \lambda > 0$, $J^*_\lambda x^* = JQ_\lambda J^{-1}(x^*)$. In fact, define

$$x^*_\lambda := JQ_\lambda J^{-1}(x^*) = [J(I + \lambda J^{-1}B)J^{-1}]^{-1}(x^*).$$

Then, we have

$$x^* \in J(I + \lambda J^{-1}B)J^{-1}(x^*_\lambda) = (I^* + \lambda BJ^{-1})x^*_\lambda$$

and hence $x^*_\lambda = J^*_\lambda x^*$. From Theorem 2.1, we get

$$\lim_{\lambda \to \infty} Q_\lambda J^{-1}(x^*) = u \in B^{-1}0.$$

If $E^*$ is uniformly convex, then $E$ has a Fréchet differentiable norm. So, then $J$ is norm to norm continuous. Since $B^{-1}0$ is closed, we have

$$\lim_{\lambda \to \infty} J^*_\lambda x^* = \lim_{\lambda \to \infty} JQ_\lambda J^{-1}(x^*) = Ju \in JB^{-1}0 = (BJ^{-1})^{-1}0.$$

(2) Defining a mapping $R^*$ from $E^*$ to $E^*$ by

$$R^*x^* := \lim_{\lambda \to \infty} J^*_\lambda x^* \quad \forall x^* \in E^*.$$

Let $u^* \in (BJ^{-1})^{-1}0 = F(J^*_\lambda x^*)$. Then $R^*u^* = \lim_{\lambda \to \infty} J^*_\lambda u^* = \lim_{\lambda \to \infty} u^* = u^*$. Therefore $R^*$ is a retraction of $E^*$ onto $(BJ^{-1})^{-1}0$. Since $x^* \in J^*_\lambda x^* + \lambda BJ^{-1}(J^*_\lambda x^*)$, we have

$$\left\langle x^* - J^*_\lambda x^*, J^{-1}(J^*_\lambda x^*) - J^{-1}(z^*) \right\rangle \geq 0, \quad \forall z^* \in (BJ^{-1})^{-1}0,$$

and hence

$$(x^* - J^*_\lambda x^*, J^{-1}(J^*_\lambda x^*) - J^{-1}(z^*)) \geq 0.$$

Letting $\lambda \to 0$, we get

$$\langle x^* - R^*x^*, J^{-1}(R^*x^*) - J^{-1}(z^*) \rangle \geq 0, \quad \forall z^* \in (BJ^{-1})^{-1}0.$$

From Proposition 2.1, $R^*$ is sunny and generalized nonexpansive. This implies that $R^*$ is a sunny generalized nonexpansive retraction of $E^*$ onto $(BJ^{-1})^{-1}0$. □

Now we construct an iterative scheme which converges strongly to a point which is a fixed point of relatively weak nonexpansive mapping and a zero of monotone mapping.

**Theorem 2.4.** Let $E^*$ be a uniformly convex Banach space and uniformly smooth Banach space. Let $A \subseteq E \times E^*$ be a maximal monotone operator. Let $C$ be a nonempty closed convex subset of $E$. Let $T : C \to C$ be a relatively weak
nonexpansive mapping with $A^{-1}0 \cap F(T) \neq \emptyset$. Assume that $0 \leq \alpha_n < a < 1$ is a sequence of real numbers. Then the sequence $\{x_n\}$ generated by

$$
\begin{align*}
x_0 &\in C, \quad \lambda_n \to +\infty, \\
y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)J_{\lambda_n}^* Jx_n), \quad J_{\lambda_n}^* = (I^* + \lambda_n A J^{-1})^{-1}, \\
z_n &= Ty_n, \\
H_0 &= \{v \in C: V_2(v, z_0) \leq V_2(v, y_0) \leq V_2(v, x_0)\}, \\
H_n &= \{v \in H_{n-1} \cap W_{n-1} : V_2(v, z_n) \leq V_2(v, y_n) \leq V_2(v, x_n)\}, \\
W_0 &= C, \\
W_n &= \{v \in H_{n-1} \cap W_{n-1} : \langle v - x_n, Jx_0 - Jx_n \rangle \leq 0\}, \\
x_{n+1} &= \Pi_{H_n \cap W_n}(x_0), \quad n \geq 1,
\end{align*}
$$

converges strongly to $\Pi_{A^{-1}0 \cap F(T)}(x_0)$, where $\Pi_{A^{-1}0 \cap F(T)}$ is the generalized projection from $E$ onto $A^{-1}0 \cap F(T)$.

**Proof.** We first show that $H_n$ and $W_n$ are closed and convex for each $n \geq 0$. From the definition of $H_n$ and $W_n$, it is obvious that $H_n$ is closed and convex for each $n \geq 0$. We show that $H_n$ is convex. Since

$$
H_n = \{v \in H_{n-1} \cap W_{n-1} : V_2(v, z_n) \leq V_2(v, y_n) \leq V_2(v, x_n)\}
$$

and that $V_2(v, y_n) \leq V_2(v, x_n)$ is equivalent to

$$
2\langle v, Jx_n - Jy_n \rangle + \|y_n\|^2 + \|x_n\|^2 \leq 0,
$$

$V_2(v, z_n) \leq V_2(v, y_n)$ is equivalent to

$$
2\langle v, Jy_n - Jz_n \rangle + \|z_n\|^2 + \|x_n\|^2 \leq 0,
$$

it follows that $H_n$ is convex.

Next, we show that $F =: A^{-1}0 \cap F(T) \subset H_n \cap W_n$ for each $n \geq 0$. Let $p \in F$, then relatively weak nonexpansiveness of $T$ and generalized nonexpansiveness of $J_{\lambda_n}^*$ give that

$$
V_2(p, z_n) &= V_2(p, Ty_n) \leq V_2(p, y_n) \\
&= V_2(p, J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n)J_{\lambda_n}^* Jx_0)) \\
&= \|p\|^2 + \|\alpha_n Jx_0 + (1 - \alpha_n)J_{\lambda_n}^* Jx_0\|^2 - 2\langle p, \alpha_n Jx_0 + (1 - \alpha_n)J_{\lambda_n}^* Jx_0 \rangle \\
&\leq \|p\|^2 - 2\alpha_n \langle p, Jx_0 \rangle - 2(1 - \alpha_n)\langle p, J_{\lambda_n}^* Jx_0 \rangle + \alpha_n \|Jx_0\|^2 + (1 - \alpha_n)\|J_{\lambda_n}^* Jx_0\|^2 \\
&= \alpha_n(\|p\|^2 - 2\alpha_n \langle p, Jx_0 \rangle + \|x_0\|^2) + (1 - \alpha_n)(\|p\|^2 - 2\langle p, J_{\lambda_n}^* Jx_0 \rangle + \|J_{\lambda_n}^* Jx_0\|^2) \\
&= \alpha_n V_2(p, x_0) + (1 - \alpha_n) \big( V_2(p, J^{-1}J_{\lambda_n}^* Jx_0) \\
&\leq \alpha_n V_2(p, x_0) + (1 - \alpha_n) V(p, J_{\lambda_n}^* Jx_0) \\
&\leq \alpha_n V_2(p, x_0) + (1 - \alpha_n) V(p, Jx_0) \\
&\leq \alpha_n V_2(p, x_0) + (1 - \alpha_n) V_2(p, x_0) = V_2(p, x_0).
$$

Thus, we give that $p \in H_0$. On the other hand it is clear that $p \in C$. Thus $F \subset H_0 \cap W_0$ and therefore, $x_1 = \Pi_{H_0 \cap W_0}$ is well defined. Suppose that $F \subset H_{n-1} \cap W_{n-1}$ and $\{x_n\}$ is well defined. Then the methods in (3.2) imply that $V_2(p, z_n) \leq V_2(p, y_n) \leq V_2(p, x_n)$ and that $p \in H_n$. Moreover, it follows from Lemma 1.3 that

$$
\langle p - x_n, Jx_n - Jx_0 \rangle \geq 0,
$$
which implies that \( p \in W_n \). Hence \( F \subset H_n \cap W_n \) and \( x_{n+1} = \Pi_{H_n \cap W_n} \) is well defined. Then by induction, \( F \subset H_n \cap W_n \) and the sequence generated by (3.1) is well defined for each \( n \geq 0 \).

Now we show that \( \{x_n\} \) is a bounded sequence and converges to a point of \( F \). Let \( p \in F \). Since \( x_{n+1} = \Pi_{H_n \cap W_n}(x_0) \) and \( H_n \cap W_n \subset H_{n-1} \cap W_{n-1} \) for all \( n \geq 1 \), we have

\[
V_2(x_n, x_0) \leq V_2(x_{n+1}, x_0)
\]

for all \( n \geq 0 \). Therefore, \( \{V_2(x_n, x_0)\} \) is nondecreasing. In addition, it follows from definition of \( W_n \) and Lemma 1.3 that \( x_n = \Pi_{W_n}(x_0) \). Therefore, by Lemma 1.2 we have

\[
V_2(x_n, x_0) = V_2(\Pi_{W_n}(x_0), x_0) \leq V_2(p, x_0) - V_2(p, x_n) \leq V_2(p, x_0),
\]

for each \( p \in F(T) \subset W_n \) for all \( n \geq 0 \). Therefore, \( \{V_2(x_n, x_0)\} \) is bounded. This together with (3.2) implies that the limit of \( \{V_2(x_n, x_0)\} \) exists. Put \( \lim_{n \to \infty} V_2(x_n, x_0) = d \). From Lemma 1.2, we have, for any positive integer \( m \), that

\[
V_2(x_{m+n}, x_n) = V_2(x_{n+m}, \Pi_{W_n}(x_0)) \leq V_2(x_{n+m}, x_0) - V_2(\Pi_{W_n}(x_0), x_0)
= V_2(x_{n+m}, x_0) - V_2(x_n, x_0),
\]

(3.3) for all \( n \geq 0 \). The existence of \( \lim_{n \to \infty} V_2(x_n, x_0) \) implies that \( \lim_{n \to \infty} V_2(x_{m+n}, x_n) = 0 \). Thus, Lemma 1.4 implies that

\[
x_{m+n} - x_n \to 0 \quad \text{as} \quad n \to \infty \tag{3.4}
\]

and hence \( \{x_n\} \) is a Cauchy sequence. Therefore, there exists a point \( q \in E \) such that \( x_n \to q \) as \( n \to \infty \). Since \( x_{n+1} \in H_n \), we have \( V_2(x_{n+1}, z_n) \leq V_2(x_{n+1}, x_n) \). Thus by Lemma 1.4 and (3.4) we get that

\[
x_{n+1} - z_n \to 0, \quad x_{n+1} - y_n \to 0 \quad \text{as} \quad n \to \infty \tag{3.5}
\]

and hence \( \|x_n - y_n\| \leq \|x_{n+1} - x_n\| + \|x_{n+1} - y_n\| \to 0 \) as \( n \to \infty \). Furthermore, since \( J \) is uniformly continuous on bounded sets, we have

\[
\lim_{n \to \infty} \|Jx_{n+1} - Jz_n\| = \lim_{n \to \infty} \|Jx_n - Jy_n\| = 0, \tag{3.6}
\]

which implies that

\[
\|Jx_{n+1} - JTy_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.7}
\]

Since \( J^{-1} \) is also uniformly norm-norm-continuous on bounded sets, we obtain

\[
\lim_{n \to \infty} \|x_{n+1} - Ty_n\| = \lim_{n \to \infty} \|J^{-1}Jx_{n+1} - J^{-1}JTy_n\| = 0. \tag{3.8}
\]

Therefore, from (3.5), (3.8) and \( \|y_n - Ty_n\| \leq \|x_{n+1} - Ty_n\| + \|x_n - y_n\| \), we obtain that \( \lim_{n \to \infty} \|y_n - Ty_n\| = 0 \). This together with the fact that \( \{x_n\} \) (and hence \( \{y_n\} \) ) converges strongly to \( q \in E \) and the definition of relatively weak nonexpansive mapping implies that \( q \in F(T) \). Furthermore, from (3.1) and (3.6), we have that \( (1 - \alpha_n)\|J^*_{\lambda_n}Jx_n - Jx_n\| = \|Jx_n - Jy_n\| \to 0 \) as \( n \to \infty \). Thus, \( \lim_{n \to \infty} J^*_{\lambda_n}Jx_n = \lim_{n \to \infty} Jx_n = Jq \in JA^{-1}0 = (AJ^{-1})^{-1}0 \), we obtain
that $q \in A^{-1}0$. Finally, we show that $q = \Pi_{A^{-1}0 \cap F(T)}(x_0)$ as $n \to \infty$. From Lemma 1.2, we have

$$V_2(q, \Pi_{A^{-1}0 \cap F(T)}(x_0)) + V_2(\Pi_{A^{-1}0 \cap F(T)}(x_0), x_0) \leq V_2(q, x_0).$$  \hspace{1cm} (3.9)

On the other hand, since $x_{n+1} = \Pi_{H_n \cap W_n}(x_0)$ and $F \subset H_n \cap W_n$ for all $n \geq 0$ we have by Lemma 1.2 that

$$V_2(\Pi_{A^{-1}0 \cap F(T)}(x_0), x_{n+1}) + V_2(x_{n+1}, x_0) \leq V_2(\Pi_{A^{-1}0 \cap F(T)}(x_0), x_0).$$ \hspace{1cm} (3.10)

Moreover, by the definition of $V_2(x, y)$ we get that

$$\lim_{n \to \infty} V_2(x_{n+1}, x_0) = V_2(q, x_0).$$ \hspace{1cm} (3.11)

Combining (3.9), (3.11) we obtain that $V_2(q, x_0) = V_2(\Pi_{A^{-1}0 \cap F(T)}(x_0), x_0)$. Therefore, it follows from the uniqueness of $\Pi_{A^{-1}0 \cap F(T)}(x_0)$ that $q = \Pi_{A^{-1}0 \cap F(T)}(x_0)$.

\[
\blacksquare
\]

**Remark 1.** If in Theorem 3.1 we have that $T = I$, the identity map on $E$ then we get the following:

**Corollary 2.5.** Let $E^*$ be a uniformly convex Banach space and uniformly smooth Banach space. Let $A \subset E \times E^*$ be a maximal monotone operator. Let $C$ be a nonempty closed convex subset of $E$ with $A^{-1}0 \neq \emptyset$. Assume that $0 \leq \alpha_n < a < 1$ is a sequence of real numbers. Then the sequence $\{x_n\}$ generated by

\[
\begin{align*}
x_0 & \in C, \quad \lambda_n \to +\infty, \\
y_n & = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) J^*_{\lambda_n} Jx_n), \quad J^*_{\lambda_n} = (I^* + \lambda_n AJ^{-1})^{-1}, \\
H_0 & = \{v \in C : V_2(v, y_0) \leq V_2(v, x_0) \}, \\
H_n & = \{v \in H_{n-1} \cap W_{n-1} : V_2(v, z_n) \leq V_2(v, y_n) \leq V_2(v, x_n) \}, \\
W_0 & = C, \\
W_n & = \{v \in H_{n-1} \cap W_{n-1} : (v - x_n, Jx_0 - Jx_n) \leq 0 \}, \\
x_{n+1} & = \Pi_{H_n \cap W_n}(x_n), \quad n \geq 1,
\end{align*}
\]

converges strongly to $\Pi_{A^{-1}0}$, where $\Pi_{A^{-1}0}$ is the generalized projection from $E$ onto $A^{-1}0$.

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