SOME REMARKS ON NON-SYMPLECTIC AUTOMORPHISMS OF K3 SURFACES OVER A FIELD OF ODD CHARACTERISTIC

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Abstract. In this paper, we present a simple proof of Corollary 3.3 in [5] using the fact that for a K3 surface of finite height over a field of odd characteristic, the height is a multiple of the non-symplectic order. Also we prove for a non-symplectic CM K3 surface defined over a number field, the Frobenius invariant of the reduction over a finite field is determined by the congruence class of residue characteristic modulo the non-symplectic order of the K3 surface.

1. Introduction

Let \( k \) be an algebraically closed field of odd characteristic \( p \). Let \( W \) be the ring of Witt vectors of \( k \) and \( K \) be the fraction field of \( W \). Assume \( X \) is a K3 surface over \( k \). The second crystalline cohomology of \( X \), \( H^2_{\text{cris}}(X/W) \) is a free \( W \)-module of rank 22 equipped with a canonical Frobenius linear morphism

\[
F : H^2_{\text{cris}}(X/W) \to H^2_{\text{cris}}(X/W).
\]

Let \( H^2_{\text{cris}}(X/K) = H^2_{\text{cris}}(X/W) \otimes K \) be the rational crystalline cohomology. If all the Newton slopes of \( F \)-isocrystal \( (H^2_{\text{cris}}(X/K), F) \) are 1, we say \( X \) is supersingular. If \( X \) is not supersingular, there exists an integer \( h \) between 1 and 10 such that the slopes of \( H^2_{\text{cris}}(X/K) \) are \( 1 - 1/h, 1, 1 + 1/h \) of length \( h, 22 - h, h \) respectively. In this case, we say \( X \) is of height \( h \).

If \( X \) is of finite height \( h \), the Dieudonné module of the formal Brauer group of \( X \) is

\[
D(\widehat{Br}_X) = W[F, V]/(FV = p, F = V^{h-1}).
\]

Here \( D(\widehat{Br}_X) \) is a free \( W \)-module of rank \( h \) equipped with a Frobenius linear operator \( F \) and a Frobenius-inverse linear operator \( V \). The Dieudonné module

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\( \mathbb{D}(\mathcal{B}_r X) \) is isomorphic to \( H^2(X, \mathcal{W} \mathcal{O}_X) \) ([1]). \( H^2_{\text{cris}}(X/W) \) has an \( F \)-crystal decomposition according to the Newton slopes

\[
H^2_{\text{cris}}(X/W) = H^2_{\text{cris}}(X/W)[1-1/h] \oplus H^2_{\text{cris}}(X/W)[1] \oplus H^2_{\text{cris}}(X/W)[1+1/h].
\]

Here

\[
H^2_{\text{cris}}(X/W)[1-1/h] = \mathbb{D}(\mathcal{B}_r X)
\]

and

\[
H^2_{\text{cris}}(X/W)[1+1/h] = \text{Hom}(H^2(X, \mathcal{W} \mathcal{O}_X), W(p^2)).
\]

By the cup product pairing, \( H^2_{\text{cris}}(X/W)[1-1/h] \) and \( H^2_{\text{cris}}(X/W)[1+1/h] \) are isotropic and dual to each other. \( H^2_{\text{cris}}(X/W)[1] \) is unimodular. The discriminant of the \( \mathbb{Z}_p \)-lattice \( H^2_{\text{cris}}(X/W)[1] \) is \( (-1)^{h+1} \) ([10]). The image of the cycle map

\[
NS(X) \otimes W \hookrightarrow H^2_{\text{cris}}(X/W)
\]
sits inside \( H^2_{\text{cris}}(X/W)[1] \). Therefore the Picard number of \( X \), the rank of \( NS(X) \) is at most \( 22 - 2h \). Let \( T_{\text{cris}}(X) \) be the orthogonal complement of the embedding \( NS(X) \hookrightarrow H^2_{\text{cris}}(X/W) \). We call \( T_{\text{cris}}(X) \) the crystalline transcendental lattice of \( X \). By the above observation, we can see \( H^2_{\text{cris}}(X/W)[1-1/h] \oplus H^2_{\text{cris}}(X/W)[1+1/h] \) is a direct factor of \( T_{\text{cris}}(X) \). From the exact sequence of sheaves on \( X \)

\[
0 \to W \mathcal{O}_X \overset{\psi}{\to} W \mathcal{O}_X \to \mathcal{O}_X \to 0,
\]

we have a canonical morphism

\[
H^2_{\text{cris}}(X/W)[1-1/h] = \mathbb{D}(\mathcal{B}_r X) \cong H^2(X, \mathcal{W} \mathcal{O}_X) \to H^2(X, \mathcal{O}_X).
\]

Let

\[
\chi_{\text{cris},X} : \text{Aut } X \to O(T_{\text{cris}}(X))
\]

and

\[
\rho_X : \text{Aut } X \to \text{Gl}(H^0(X, \Omega^2_{X/k}))
\]

be the representation of \( \text{Aut } X \) on the crystalline transcendental lattice and on global 2 forms. The images of \( \chi_{\text{cris},X} \) and \( \rho_X \) are finite and there is a canonical projection ([5])

\[
\text{Im } \chi_{\text{cris},X} \to \text{Im } \rho_X.
\]

If \( X \) is a supersingular K3 surface over \( k \), the rank of \( NS(X) \) is 22 ([2], [6]). The cycle map \( NS(X) \otimes W \hookrightarrow H^2_{\text{cris}}(X) \) is an embedding of \( W \)-lattices of same rank and we have

\[
NS(X) \otimes W \subset H^2_{\text{cris}}(X/W) \subset NS(X)^* \otimes W.
\]

The quotient \( H^2_{\text{cris}}(X/W)/(NS(X) \otimes W) \) is a \( \sigma \) dimensional \( k \)-space for an integer \( \sigma \) between 1 and 10. We say \( \sigma \) is the Artin-invariant of \( X \). It is known that \( NS(X) \) is completely determined by \( p \) and \( \sigma \) ([12]). We denote the Neron-Severi lattice of a supersingular K3 surface of Artin invariant \( \sigma \) over a field of
characteristic $p$ by $\Lambda_{p,\sigma}$. For a lattice $L$, let $d(L)$ be the discriminant of $L$. The discriminant $d(\Lambda_{p,\sigma})$ is $-p^{2\sigma}$. It is also known that there is a decomposition

$$\Lambda_{p,\sigma} \otimes \mathbb{Z}_p = E_0(p) \oplus E_1.$$  

Here $E_0$ and $E_1$ are unimodular $\mathbb{Z}_p$-lattices of rank $2\sigma$ and $22 - 2\sigma$ respectively. And $d(E_0) = (-1)^\sigma \delta$ and $d(E_1) = (-1)^{\sigma+1} \delta$, where $\delta$ is a non-square unit of $\mathbb{Z}_p$.

Note that a unimodular $\mathbb{Z}_p$-lattice is uniquely determined up to isomorphism by the rank and the discriminant, square or non-square.

We denote the characteristic polynomial over an indeterminate $T$ of a linear endomorphism $L$ by $\varphi(L)$. When $\alpha \in \text{Aut}X$ is an automorphism of a K3 surface of $X$, $\varphi(\alpha^*|H^2_{\text{cris}}(X/W))$ is a polynomial with integer coefficients ([4], 3.7.3). If $X$ is of finite height, $\varphi(\chi_{\text{cris},X})$ is also an integral polynomial. An automorphism $\alpha \in \text{Aut}X$ is symplectic if $\rho_X(\alpha) = 1$. An automorphism $\alpha \in \text{Aut}X$ is purely non-symplectic if $\alpha$ is of finite order and the order of $\alpha$ is equal to the order of $\rho_X(\alpha)$. For $\alpha \in \text{Aut}X$, we say the order of $\rho_X(\alpha)$ is the non-symplectic order of $\alpha$. Also, we call the order of $\text{Im}\rho_X$ the non-symplectic order of $X$. An automorphism $\alpha \in \text{Aut}X$ is tame if $\alpha$ is of finite order and the order of $\alpha$ is not divisible by $p$. An automorphism of finite order which is divisible by $p$ is called a wild automorphism. It is known that if $p > 11$, there is no wild automorphism ([3]). When $X$ is of finite height, an automorphism $\alpha \in \text{Aut}X$ is weakly tame if the order of $\chi_{\text{cris},X}(\alpha)$ is not divisible by $p$.

Every tame automorphism is weakly tame. Since $\chi_{\text{cris},X}(\alpha)$ is of finite order, all roots of $\varphi(\chi_{\text{cris},X}(\alpha))$ are roots of unity. Hence if a primitive $n$-th root of unity is an eigenvalue of $\chi_{\text{cris},X}(\alpha)$, the rank of $T_{\text{cris}}(X)$ is at least $\phi(n)$ because $\varphi(\chi_{\text{cris},X}(\alpha)) \in \mathbb{Z}[T]$. Therefore when $X$ is of finite height and $p \geq 23$, every automorphism of $X$ is weakly tame.

If $X$ is of height $h$, $\alpha$ is a weakly tame automorphism of $X$ and $\rho(\alpha) = \xi$ is of order $n$, then all the eigenvalues of $\alpha^*|H^2_{\text{cris}}(X/W)[1-h] \oplus H^2_{\text{cris}}(X/W)[1+h]$ are $\xi^{11}, \xi^{\pm 1}, \ldots, \xi^{\pm 11}$ up to multiplicity ([5], Theorem 2.9). By this and an argument based on the Tate conjecture for K3 surfaces ([9], [6]), we also proved the following.

**Proposition** ([5], Corollary 3.3) Let $k$ be an algebraically closed filed of odd characteristic $p$. Let $X$ be a K3 surface over $k$ equipped with an automorphism $\alpha \in \text{Aut}(X)$ such that the order of $\rho_X(\alpha)$ is $N > 2$. We assume the rank of the Neron-Severi group of $X$ is at least $22 - \phi(N)$. If $p^m \equiv -1$ modulo $N$ for some $m$, $X$ is supersingular. If $p^m \not\equiv -1$ modulo $N$ for any $m$ and the order of $p$ in $(\mathbb{Z}/N\mathbb{Z})^*$ is $n$, $X$ is of height $n$.

In the next section, we present a simple proof of the above proposition. For that, we prove the following theorem.
Theorem 2.1. Let $k$ be an algebraically closed field of odd characteristic $p$. Assume $X$ is a K3 surface over $k$ and $N > 2$ is the non-symplectic order of $X$. If $p^m \not\equiv -1$ modulo $N$ for any $m$, $X$ is of finite height. In this case, if $n$ is the order of $p$ in $(\mathbb{Z}/N)^*$, the height of $X$ is a multiple of $n$.

When $X$ is a complex algebraic K3 surface, the transcendental lattice of $X$ is the orthogonal complement of the embedding

$$NS(X) \hookrightarrow H^2(X,\mathbb{Z}).$$

We denote the transcendental lattice of $X$ by $T(X)$. When $\rho(X)$ is the Picard number of $X$, the signature of $T(X)$ is $(2, 20 - \rho(X))$. If $N$ is the non-symplectic order of $X$, the rank of $T(X)$ is a multiple of $\phi(N)$ ([7]). We say $X$ is a non-symplectic CM K3 surface of order $N$ if $\text{rank} T(X)$ is equal to $\phi(N)$. A non-symplectic CM K3 surface gives a CM point in a moduli Shimura variety, so it has a model over a number field ([11]). It seems like that there are only few non-symplectic CM K3 surfaces. Also it is known that there is a unique non-symplectic CM K3 surface of order $N$ for many $N$. In a previous work ([5]), if $X$ is a non-symplectic CM K3 surface of order $N$ with $\phi(N) > 10$ and $\text{Im} \rho_X$ is generated by a purely non-symplectic automorphism, we proved that the height and the Artin invariant of a reduction of $X$ over a field of odd characteristic $p$ is determined by the congruence class of $p$ modulo $N$. In the next section, we give a generalization of this result for an arbitrary non-symplectic CM K3 surface.

2. Results

Theorem 2.1. Let $k$ be an algebraically closed field of odd characteristic $p$. Assume $X$ is a K3 surface over $k$ and $N > 2$ is the non-symplectic order of $X$. If $p^m \not\equiv -1$ modulo $N$ for any $m$, $X$ is of finite height. In this case, if $n$ is the order of $p$ in $(\mathbb{Z}/N)^*$, the height of $X$ is a multiple of $n$.

Proof. If $X$ is a supersingular K3 surface of Artin-invariant $\sigma$, the non-symplectic order of $X$ divides $p^\sigma + 1$ ([8], Theorem 2.1). Hence under the assumption, $X$ is not supersingular. Let us choose an automorphism $\alpha \in \text{Aut} X$ such that $\text{Im} \rho_X$ is generated by $\rho_X(\alpha)$. If the order of $\chi_{\text{cris}, X}(\alpha)$ is $p^r M$ for $r, M \in \mathbb{N}$ with $p \nmid M$, $\alpha^{p^r}$ also generates $\text{Im} \rho_X$ and is weakly tame. After replacing $\alpha$ by $\alpha^{p^r}$, we may assume $\alpha$ is weakly tame. Suppose the height of $X$ is $h$. If $\xi = \rho_X(\alpha)$, by ([5], Theorem 2.9), all the eigenvalues of $\alpha^* | H^2_{\text{cris}}(X/W)_{[1-1/h]}$ are $\xi^{-1}, \xi^{-1/p}, \ldots, \xi^{-1/p^{h-1}}$ up to multiplicity. On the other hand, if $\alpha^* x = \lambda x$ for some $x \in H^2_{\text{cris}}(X/W)_{[1-1/h]}$, $\alpha^*(Vx) = V(\alpha^* x) = V(\lambda x) = \lambda^{1/p} Vx$.

Therefore for any $i \in \mathbb{Z}$, $\xi^{-1/p^i}$ is an eigenvalue of $\alpha^* | H^2_{\text{cris}}(X/W)_{[1-1/h]}$ and the rank of eigenspace for the eigenvalue $\xi^{-1/p^i}$ is equal to the rank of
There is an embedding $X \hookrightarrow \text{Aut}(\mathcal{O}_K)$.

**Proof.** By Theorem 2.1, $X$ is of finite height and the height of $X$ is a multiple of $n$. By the assumption, the rank of $T_{\text{cris}}(X)$ is $\phi(N)$ is an integral polynomial of degree $\phi(n)$ and a primitive $N$-th root of unity is an eigenvalue of $\chi_{\text{cris},X}(\alpha)$ is the $N$-th cyclotomic polynomial. It follows that every eigenvalue of $\chi_{\text{cris},X}(\alpha)$ occurs only one time. Considering ([5], Theorem 2.9), the height of $X$ is at most $n$. Therefore the height of $X$ is $n$. □

Let $X$ be a non-symplectic CM K3 surface of order $N$. We may assume $X$ is defined over a number field $F$. Let us fix a smooth projective integral model $X_R$ of $X$ over an integer ring $R$ with $\text{Spec} \, R \subset \text{Spec} \, \mathcal{O}_F$. For each place $v \in \text{Spec} \, R$, let $p_v$ be the residue characteristic of $v$. We assume $p_v \nmid N$ and $p_v$ is unramified in $F$ for any $v \in \text{Spec} \, R$. We denote the reduction of $X_R$ over the residue field $k(v)$ by $X_v$.

**Theorem 2.3.** If $p_v^m \equiv -1 \mod N$ for any $m \in \mathbb{Z}$, $X_v$ is of finite height and the height of $X_v$ is the order of $p_v$ in $(\mathbb{Z}/N\mathbb{Z})^*$. If $p_v^m \equiv -1$ for some $m \in \mathbb{Z}$, $X_v$ is supersingular. Moreover if $p_v$ does not divide $d(\text{NS}(X))$, the Artin-invariant of $X_v$ is the half of the order of $p_v$ in $(\mathbb{Z}/N\mathbb{Z})^*$.

**Proof.** There is an embedding $\text{NS}(X) \hookrightarrow \text{NS}(X_v)$, so the rank of $\text{NS}(X_v)$ is at least $22 - \phi(N)$. Let $N_v$ be the non-symplectic order of $X_v$. Then $N_v$ is a multiple of $N$. If $p_v^m \equiv -1 \mod N$ for any $m \in \mathbb{Z}$, $p_v^m \equiv -1 \mod N_v$, so $X_v$ is of finite height. Since the rank of $T(X_v)$ is at least $\phi(N_v)$, $\phi(N) = \phi(N_v)$ and $N_v = N$ or $N_v = 2N$. In any case, the order of $p$ in $(\mathbb{Z}/N_v\mathbb{Z})^*$ is equal to the order of $p$ in $(\mathbb{Z}/N\mathbb{Z})^*$. By Corollary 2.2, the height of $X_v$ is the order of $p_v$ in $(\mathbb{Z}/N\mathbb{Z})^*$. Assume the order of $\xi = \rho_X(\alpha)$ is $N$ for some $\alpha \in \text{Aut} \, X$. Let $T_{\text{cris}}(X)$ be the orthogonal complement of the embedding $\text{NS}(X) \otimes W \hookrightarrow \text{NS}(X_v) \otimes W \hookrightarrow H^2_{\text{cris}}(X_{\text{et}}, W)$. Since $H^2_{\text{cris}}(X/W)$ is canonically isomorphic to $H^2_{\text{cris}}(X/R) \otimes W$, $\alpha^*|T_{\text{cris}}(X)$ is of finite order. If $X_v$ is of finite height, $T_{\text{cris}}(X_v) \subset T_{\text{cris}}(X)$ and all the eigenvalues of $\alpha^*|T_{\text{cris}}(X_v)$ are distinct. If $p_v^m \equiv -1 \mod N$, $\xi^{-1/p_v^m} = \xi$ occurs as an eigenvalue of $\alpha^*|H^2_{\text{cris}}(X/W)_{[1-1/\xi]}$. But $\xi$ is also an eigenvalue of $\alpha^*|H^2_{\text{cris}}(X/W)_{[1+1/\xi]}$ and it is a contradiction. Therefore $X_v$ is supersingular.
If \( p \) does not divide \( d(\text{NS}(X)) \), \( \text{NS}(X) \otimes W \) is unimodular and the Artin-invariant of \( X_\nu \) is at most \( \phi(N)/2 \). Also when \( \sigma \) is the Artin-invariant of \( X_\nu \), \( N \) divides \( p^\sigma + 1 \), so \( p^\sigma \equiv -1 \) modulo \( N \). We have an isomorphism \( \text{NS}(X_\nu)^* / \text{NS}(X_\nu) = T_{\text{cris}}(X)^* / T_{\text{cris}}(X) \) compatible with the action of \( \text{Aut} X \). By ([5], theorem 2.9), if \( \sigma \) is greater than the half of the order of \( p_\nu \) in \((\mathbb{Z}/N)^*\), \( \xi \) appears more than twice in the eigenvalues of \( \alpha^*|T_{\text{cris}}(X) \). Therefore the Artin-invariant of \( X_\nu \) is the half of the order of \( p_\nu \) in \((\mathbb{Z}/N)^*\).

**Remark 2.4.** If a non-symplectic CM K3 surface of order \( N \), \( X \) has a reduction of height \( \phi(N)/2 \) or of Artin invariant \( \phi(N)/2 \), the Legendre symbol \( \left( \frac{d(\text{NS}(X))}{p} \right) \) is constant for all primes \( p \) in a congruence class modulo \( N \).

**References**


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