ERROR ESTIMATES FOR A SEMI-DISCRETE MIXED DISCONTINUOUS GALERKIN METHOD WITH AN INTERIOR PENALTY FOR PARABOLIC PROBLEMS

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Abstract. In this paper, we consider a semi-discrete mixed discontinuous Galerkin method with an interior penalty to approximate the solution of parabolic problems. We define an auxiliary projection to analyze the error estimate and obtain optimal error estimates in $L^\infty(L^2)$ for the primary variable $u$, optimal error estimates in $L^2(L^2)$ for $u_t$, and suboptimal error estimates in $L^\infty(L^2)$ for the flux variable $\sigma$.

1. Introduction

Discontinuous Galerkin methods with interior penalties which generalized Nitsche method in [11] were introduced to approximate the solutions of elliptic or parabolic problems by several authors [1, 6, 19]. The discontinuous Galerkin methods are widely used for many partial differential equations because of its advantages such as the mesh adaptivity and the local mass conservativeness. There are now a lot of forms and names of the discontinuous Galerkin method. For more details, we refer to [2, 3] and the literatures cited therein.

Riviere and Wheeler [18] introduced semidiscrete and fully discrete locally conservative discontinuous Galerkin methods for nonlinear parabolic equations. They obtained optimal error estimates in $L^2(H^1)$ and suboptimal error estimates in $L^\infty(L^2)$ for semidiscrete approximations and optimal error estimates in $\ell^2(H^1)$ and suboptimal error estimates in $\ell^\infty(L^2)$ for fully discrete approximations. Ohm et. al [12, 13] obtained optimal error estimates in $L^\infty(L^2)$ for semidiscrete approximations and optimal error estimates in $\ell^\infty(L^2)$ for fully discrete approximations which improved the results of Riviere and Wheeler [18]. And using Crank-Nicolson method for time stepping, Ohm et. al [14] introduced fully discrete discontinuous Galerkin method for nonlinear parabolic equations.

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and obtained optimal error estimates in $\ell^\infty(L^2)$ for both spatial and temporal
directions.

Raviart and Thomas [17] and Nedelec [10] introduced mixed finite element
methods to approximate both primary variable and its flux variable, simulta-
neously. These mixed finite element methods requiring the inf-sup conditions
are widely used for elliptic or parabolic problems [5, 7, 9]. And Pani [15] intro-
duced $H^1$-Galerkin mixed finite element method without inf-sup conditions for
parabolic problems. Applications of $H^1$-Galerkin mixed finite element method
can be seen in [8, 16].

Chen [3] introduced a family of mixed discontinuous finite element methods
for second-order elliptic equations. Chen and Chen [4] developed a theory for
stability and convergence for mixed discontinuous finite element methods in a
general form for second-order partial differential problems.

In this paper, we consider a semi-discrete mixed discontinuous Galerkin
method with an interior penalty to approximate the solution of parabolic prob-
lems and obtain error estimates for both primary variable and its flux vari-
able, simultaneously. In Section 2, we introduce a model problem, semi-discrete
mixed discontinuous Galerkin method with an interior penalty for the model
problem, and some projections with approximation properties. In Section 3,
we define auxiliary projections and give some estimates for the auxiliary pro-
jections which will be used in Section 4. And in Section 4, we obtain optimal
error estimates in $L^\infty(L^2)$ for the primary variable $u$, optimal error estimates in
$L^2(L^2)$ for $u_t$, and suboptimal error estimates in $L^\infty(L^2)$ for the flux variable
$\sigma$.

2. A model problem and finite element spaces

We consider the following parabolic problem

\[
\begin{align*}
    u_t - \nabla \cdot (a(x)\nabla u) &= f, & \text{in } \Omega \times (0, T], \\
    u &= g_D, & \text{on } \partial \Omega_D \times (0, T], \\
    a(x)\nabla u \cdot n &= g_N, & \text{on } \partial \Omega_N \times (0, T], \\
    u(x, 0) &= u^0(x), & \text{in } \Omega, \\
\end{align*}
\]

(2.1)

where $\Omega \subset \mathbb{R}^d$, $1 \leq d \leq 3$, is an open bounded convex domain with the boundary
$\partial \Omega = \partial \Omega_D \cup \partial \Omega_N$, $\partial \Omega_D \cap \partial \Omega_N = \phi$ and $n$ is the unit outward normal vector to
$\partial \Omega$. Here $a$ is a symmetric, positive definite bounded tensor. And $f \in L^2(\Omega)$,
$u^0 \in L^2(\Omega)$, $g_D \in H^{1/2}(\partial \Omega_D)$, and $g_N \in H^{-1/2}(\partial \Omega_N)$ are given functions.
Letting \( \sigma = a(x)\nabla u \), we obtain the mixed formulation of (2.1)
\[
\begin{align*}
  u_t - \nabla \cdot \sigma &= f, & \text{in } \Omega \times (0, T], \\
  \sigma &= a(x)\nabla u, & \text{in } \Omega \times (0, T], \\
  u &= g_D, & \text{on } \partial \Omega_D \times (0, T], \\
  \sigma \cdot n &= g_N, & \text{on } \partial \Omega_N \times (0, T], \\
  u(x, 0) &= u^0(x), & \text{in } \Omega.
\end{align*}
\]

(2.2)

To introduce the mixed discontinuous Galerkin finite element method for the problem (2.1), let \( \{T_h\}_{h>0} \) be a sequence of a regular quasi-uniform partitions of \( \Omega \) and each subdomain \( T \in T_h \) be a triangle or a quadrilateral (a 3-simplex or 3-rectangle) if \( d = 2 \) (if \( d = 3 \), respectively). Let \( h_T \) be the diameter of \( T \) and \( h = \max_{T \in T_h} h_T \). From the assumptions of regularity and quasi-uniformity, there exist constants \( \rho \) and \( \gamma \) such that each \( T \) contains a ball of radius \( \rho h_T \) and \( h \leq \gamma h_T \) for all \( T \in T_h \). Two adjacent elements in \( T_h \) are not required to be matched, i.e., a vertex of one element can lie on the edge or face of another element. For a given \( T_h \), let \( E_h^I \) denote the set of all interior boundaries \( e \) of \( T_h \), \( E_h^D \) and \( E_h^N \) be the sets of boundaries \( e \) on \( \partial \Omega_D \) and \( \partial \Omega_N \), respectively, \( E_h^B = E_h^D \cup E_h^N \) the set of the boundaries \( e \) on \( \partial \Omega \), \( E_h^{ID} = E_h^I \cup E_h^D \), and \( E_h = E_h^I \cup E_h^B \). For \( e \in E_h^B \), \( n \) is the unit outward normal vector to \( \partial \Omega \). For \( e \in E_h^I \), with \( e = T_1 \cap T_2 \) and \( T_1, T_2 \in T_h \), the direction of \( n \) is associated with the definition of jump across \( e \).

For \( \ell \geq 0 \), we define
\[
H^\ell(T_h) = \{ v \in L^2(\Omega) : v|_T \in H^\ell(T), \ T \in T_h \},
\]
\[
H^\ell(T_h) = \{ w \in (L^2(\Omega))^d : w|_T \in H^\ell(T) = (H^\ell(T))^d, \ T \in T_h \}
\]
with
\[
\|v\|_\ell = \left( \sum_{T \in T_h} \|v\|_{H^\ell(T)}^2 \right)^{1/2},
\]
\[
\|w\|_\ell = \left( \sum_{T \in T_h} \|w\|_{H^\ell(T)}^2 \right)^{1/2}.
\]

We simply write \( \| \cdot \| \) when \( \ell = 0 \). For \( v \in H^\ell(T_h) \) with \( \ell > \frac{1}{2} \), the jump of \( v \) across \( e = \partial T_1 \cap \partial T_2 \in E_h^I \) is defined by
\[
[v] = v|_{T_2 \cap e} - v|_{T_1 \cap e}.
\]
The average of \( v \) on \( e = \partial T_1 \cap \partial T_2 \in E_h^I \) is defined as
\[
\{v\} = \frac{1}{2} \left( v|_{T_1 \cap e} + v|_{T_2 \cap e} \right).
As a convention, for \(e \in \mathcal{E}_h\), the jump and the average are defined as follows:

\[
\{v\} = v|_e, \quad [v] = \begin{cases} 
0, & e \in \mathcal{E}_h^D, \\
\nu, & e \in \mathcal{E}_h^N.
\end{cases}
\]

Let \(V = H^1(T_h)\) and \(W = \{w \in H^1(T_h) \mid \nabla \cdot w \in L^2(\Omega)\}\). And let \(V_h = \{v \in V \mid v|_T \in P_k(T), \ T \in T_h\}\) and \(W_h = \{w \in W \mid w|_T \in \mathcal{P}_k(T), \ T \in T_h\}\) be the finite element spaces of \(V\) and \(W\), respectively, where \(P_k(T)\) the set of polynomials of total degree \(\leq k\) defined on \(T\) and \(\mathcal{P}_k(T) = (P_k(T))^d\). They are defined locally on each element \(T \in T_h\), so that \(W_h(T) = W_h|_T\) and \(V_h(T) = V_h|_T\). Neither continuity constraint nor boundary values are imposed on \(W_h \times V_h\).

Now the corresponding semi-discrete mixed discontinuous Galerkin method with an interior penalty of (2.1) is: Find \(u_h \in V_h\) and \(\sigma_h \in W_h\) such that

\[
\begin{aligned}
((u_h)_t, v) + \sum_T (\sigma_h, \nabla v)_T - \sum_{e \in \mathcal{E}_h^D} (\{\sigma_h \cdot n\}, [v])_e + J(u_h, v) \\
= \sum_{e \in \mathcal{E}_h^N} (g_N, v)_e + \sum_{e \in \mathcal{E}_h^D} h^{-1}_e (g_D, v)_e + (f, v), \quad \forall v \in V_h,
\end{aligned}
\]

(2.3)

and

\[
\begin{aligned}
(\alpha(x)\sigma_h, \tau) - \sum_T (\nabla u_h, \tau)_T + \sum_{e \in \mathcal{E}_h^D} (\{\tau \cdot n\}, [u_h])_e \\
= \sum_{e \in \mathcal{D}} (g_D, \tau \cdot n)_e, \quad \forall \tau \in W_h,
\end{aligned}
\]

(2.4)

where \(J(u, v) = \sum_{e \in \mathcal{E}_h^N} h^{-1}_e \int [u][v]|_e ds, \ h_e = |e|, \ \alpha(x) = a(x)^{-1}, \ (, )\) denote an \(L^2\) inner product on \(\Omega\), \((, )_T\) an \(L^2\) inner product on \(T\), and \((, )_e\) an \(L^2\) inner product on \(e\). We define the following bilinear forms as follows:

\[
A(q, r) = (\alpha(x)q, r), \quad \forall q, r \in W
\]

\[
B(\tau, v) = \sum_T (\tau, \nabla v)_T - \sum_{e \in \mathcal{E}_h^D} (\{\tau \cdot n\}, [v])_e, \quad \forall \tau \in W, v \in V,
\]

(2.5)

\[
C(u, v) = J(u, v) + \lambda(u, v), \quad \forall u, v \in V,
\]

where \(\lambda\) is a positive real number. And we define the following broken norms on \(V\) and \(W\) as follows:

\[
\begin{aligned}
\|v\|_C^2 &= J(v, v) + \lambda\|v\|^2, \\
\|v\|_S^2 &= \|v\|_1^2 + J(v, v), \\
\|
\sigma\|_W^2 &= \|
\sigma\|^2 + \sum_{T \in T_h} h^2_T \|
\nabla\cdot\n\sigma\|^2_T, \\
\|
\tau\|_A^2 &= A(\tau, \tau),
\end{aligned}
\]

(2.6)
where \( \| \cdot \|_1 \) denotes \( H^1 \) norm on \( V \) and \( \| \cdot \| \) denotes \( L^2 \) norm on \( V \) or \( W \). Notice that \( \| v \|_C \leq \| v \|_S \) for sufficiently small \( \lambda \). And also we define the following linear functionals on \( V \) as follows:

\[
F(v) = (f, v), \\
G_N(v) = \sum_{e \in \mathcal{E}_h^N} (g_N, v)_e, \\
G_D^1(\tau) = \sum_{e \in \mathcal{E}_h^D} (g_D, \tau \cdot n)_e, \\
G_D^2(v) = h_e^{-1} (g_D, v)_e.
\]  

Then (2.3)-(2.4) can be rewritten into the system

\[
((u_h)_t, v) + B(\sigma_h, v) + C(u_h, v) - \lambda (u_h, v) = G_N(v) + G_D^2(v) + F(v), \quad \forall v \in V_h, 
\]  

and

\[
A(\sigma_h, \tau) - B(\tau, u_h) = G_D^1(\tau), \quad \forall \tau \in W_h. 
\]

Obviously, the solution \((u, \sigma)\) of the problem (2.2) satisfy the system

\[
(u_t, v) + B(\sigma, v) + C(u, v) - \lambda (u, v) = G_N(v) + G_D^2(v) + F(v), \quad \forall v \in V, 
\]  

and

\[
A(\sigma, \tau) - B(\tau, u) = G_D^1(\tau), \quad \forall \tau \in W. 
\]

Let \( P_h : V \to V_h \) and \( \Pi_h : W \to W_h \) denote the projections satisfying the following approximation properties:

\[
\| v - P_h v \|_i \leq Kh^{r-i} \| v \|_r, \quad \forall v \in V \cap H^r(T), \quad i \leq r \leq k + 1, \quad i = 0, 1, \\
\| w - \Pi_h w \| \leq Kh^{r} \| w \|_r, \quad \forall w \in W \cap H^r(T), \quad 1 \leq r \leq k + 1, \\
\| \nabla \cdot (w - \Pi_h w) \| \leq Kh^r \| \nabla \cdot w \|_r, \quad \forall w \in W \cap H^r(T), \quad 0 \leq r \leq k. 
\]

**Lemma 2.1.** For any \( u, v \in V \) and any \( \sigma, \tau \in W \), the followings hold:

1. \( A(\sigma, \tau) \leq K \| \sigma \|_A \| \tau \|_A, \quad A(\sigma, \tau) \leq K \| \sigma \|_W \| \tau \|_W; \)
2. \( B(\sigma, v) \leq K \| \sigma \|_W \| v \|_S; \)
3. \( C(u, v) \leq K \| u \|_C \| v \|_C, \quad C(u, v) \leq K \| u \|_S \| v \|_S. \)

**Proof.** The proofs of (1) and (3) are trivial. So we will prove (2) only.

(2) Let \( v \in V \) and \( \sigma \in W \). Then

\[
B(\sigma, v) = \sum_T (\sigma, \nabla v)_T - \sum_{e \in \mathcal{E}_h^D} ([\sigma \cdot n], [v])_e \\
\leq \| \sigma \|_{(L^2(\Omega))^d} \left( \sum_T \| \nabla v \|^2_{L^2(T)} \right)^{1/2} 
\]
\[
+ \left( \sum_{e \in \epsilon_D^h} h_e \| \sigma \cdot n \|_{L^2(e)}^2 \right)^{1/2} \left( \sum_{e \in \epsilon_D^h} h_e^{-1} \| \nu \|_{L^2(e)}^2 \right)^{1/2}
\leq \| \sigma \|_{(L^2(\Omega))^d} \| \nabla v \|_{(L^2(\Omega))^d}^2
+ K \left[ \| \sigma \|_{(L^2(\Omega))^d}^2 + \left( \sum_{T} h_T^2 \| \nabla \cdot \sigma \|_{(L^2(T))^d}^2 \right) \right]^{1/2} J(v, v)^{1/2}
\leq K \| \sigma \|_{W} \| v \|_{S}.
\]

This completes the proof. \qed

Lemma 2.2. For any \( v \in V_h \) and any \( \tau \in W_h \), the followings hold:
(1) \( A(\tau, \tau) \geq K \| \tau \|_{W}^2 \);
(2) \( C(v, v) \geq K \| v \|_{S}^2 \), for \( \lambda > 0 \).

Proof. The proofs of these results are trivial from the given conditions on \( a \) and \( \lambda > 0 \). \qed

3. Auxiliary projections and some estimates

For given \((u, \sigma) \in V \times W\), we define \((\tilde{u}, \tilde{\sigma}) \in V_h \times W_h\) such that
\[
B(\sigma - \tilde{\sigma}, v) + C(u - \tilde{u}, v) = 0, \quad \forall v \in V_h
\] (3.1)
and
\[
A(\sigma - \tilde{\sigma}, \tau) - B(\tau, u - \tilde{u}) = 0, \quad \forall \tau \in W_h.
\] (3.2)

Due to [4], the unique existence of \((\tilde{u}, \tilde{\sigma}) \in V_h \times W_h\) follows from Lemmas 2.1 and 2.2.

Lemma 3.1. For any \( u \in V \cap H^{k+1}(T_h) \) and any \( \sigma \in W \cap H^{k+1}(T_h) \), we have
\[
\| u - \tilde{u} \|_C + \| \sigma - \tilde{\sigma} \|_A \leq Kh^{k}(\| \sigma \|_{k+1} + \| u \|_{k+1}).
\]

Proof. From (3.1)-(3.2), together with \( v = v_h \) and \( \tau = \tau_h \), we obtain the following system
\[
B(\Pi_h \sigma - \tilde{\sigma}, v_h) + C(P_h u - \tilde{u}, v_h),
= B(\Pi_h \sigma - \tilde{\sigma}, v_h) + C(P_h u - u, v_h),
\] (3.3)
\[
A(\Pi_h \sigma - \tilde{\sigma}, \tau_h) - B(\tau_h, P_h u - \tilde{u})
= A(\Pi_h \sigma - \tilde{\sigma}, \tau_h) - B(\tau_h, P_h u - u).
\] (3.4)
Let \( v_h = P_h u - \tilde{u} \) and \( \tau_h = \Pi_h \sigma - \tilde{\sigma} \) in (3.3)-(3.4). Then adding both sides of (3.3)-(3.4), we get

\[
\|P_h u - \tilde{u}\|_C^2 + \|\Pi_h \sigma - \tilde{\sigma}\|_A^2
= C(P_h u - \tilde{u}, P_h u - \tilde{u}) + A(\Pi_h \sigma - \tilde{\sigma}, \Pi_h \sigma - \tilde{\sigma})
= B(\Pi_h \sigma - \sigma, P_h u - \tilde{u}) + C(P_h u - u, P_h u - \tilde{u})
+ A(\Pi_h \sigma - \sigma, \Pi_h \sigma - \tilde{\sigma}) - B(\Pi_h \sigma - \tilde{\sigma}, P_h u - u)
\]
(3.5)

By (2.12), we have for \( \epsilon > 0 \)

\[
I_1 = B(\Pi_h \sigma - \sigma, P_h u - \tilde{u})
= \sum_T \left( \Pi_h \sigma - \sigma, \nabla (P_h u - \tilde{u}) \right)_T - \sum_{e \in \mathcal{E}_h} \left( \{\Pi_h \sigma - \sigma\} \cdot n, [P_h u - \tilde{u}] \right)_e
\leq K h^{-1} \|\sigma - \Pi_h \sigma\| \left( \sum_T h_T^2 \|\nabla (P_h u - \tilde{u})\|_T^2 \right)^{1/2}
+ K \left( \sum_{e \in \mathcal{E}_h} h_e^4 \|\{\Pi_h \sigma - \sigma\} \cdot n\|_e^2 \right)^{1/2} \left( \sum_{e \in \mathcal{E}_h} h_e^{-1} \|P_h u - \tilde{u}\|_e^2 \right)^{1/2}
\leq K h^{-1} \|\sigma - \Pi_h \sigma\| \|P_h u - \tilde{u}\|
+ K (\|\Pi_h \sigma - \sigma\|^2 + h^2 \|\nabla \cdot (\Pi_h \sigma - \sigma)\|^2) \frac{1}{2} J(P_h u - \tilde{u}, P_h u - \tilde{u})^{1/2}
\leq K h^{-2} \|\sigma - \Pi_h \sigma\|^2 + h^2 \|\nabla \cdot (\Pi_h \sigma - \sigma)\|^2 + \epsilon \|P_h u - \tilde{u}\|_C^2
\leq K h^{2k} \|\sigma\|_{k+1}^2 + \epsilon \|P_h u - \tilde{u}\|_C^2.
\]

Since

\[
C(P_h u - u, P_h u - u)
= \sum_{e \in \mathcal{E}_h} h_e^{-1} \|P_h u - u\|_e^2 + \lambda \|P_h u - u\|^2
\leq K \sum_T \left[ h_T^{-1} \|P_h u - u\|_T^2 + \|\nabla (P_h u - u)\|_T^2 \right] + \lambda \|P_h u - u\|^2
\leq K h^{2k} \|u\|_{k+1}^2,
\]

by (2.12), we get for \( \epsilon > 0 \)

\[
I_2 = C(P_h u - u, P_h u - \tilde{u})
\leq C(P_h u - u, P_h u - u) + \epsilon \|P_h u - \tilde{u}\|_C^2
\leq K h^{2k} \|u\|_{k+1}^2 + \epsilon \|P_h u - \tilde{u}\|_C^2.
\]
And by (2.12), we have the following estimates: for \( \epsilon > 0 \)
\[
I_3 = A(\Pi_h \sigma, \Pi_h \sigma - \tilde{\sigma}) \leq K \| \Pi_h \sigma - \sigma \|_A^2 + \epsilon \| \Pi_h \sigma - \tilde{\sigma} \|_A^2
\leq K h^{2(k+1)} \| \sigma \|_{k+1}^2 + \epsilon \| \Pi_h \sigma - \tilde{\sigma} \|_A^2
\]
and
\[
I_4 = -B(\Pi_h \sigma - \tilde{\sigma}, P_h u - u)
= - \sum_T \left( \Pi_h \sigma - \tilde{\sigma}, \nabla (P_h u - u) \right)_T + \sum_{e \in \mathcal{E}_h^D} \left( \{ (\Pi_h \sigma - \tilde{\sigma}) \cdot n \}, [P_h u - u] \right)_e
\leq \| \Pi_h \sigma - \tilde{\sigma} \| \| \nabla (u - P_h u) \|
+ \left( \sum_{e \in \mathcal{E}_h^D} \int_e \| \Pi_h \sigma - \tilde{\sigma} \|_e^2 \right)^{\frac{1}{2}} J(u - P_h u, u - P_h u)^{\frac{1}{2}}
\leq K \left[ \| \nabla (u - P_h u) \|^2 + J(u - P_h u, u - P_h u) \right] + \epsilon \| \Pi_h \sigma - \tilde{\sigma} \|_A^2
\leq K h^{2k} \| u \|_{k+1}^2 + \epsilon \| \Pi_h \sigma - \tilde{\sigma} \|_A^2.
\]
Therefore, substituting the bounds for \( I_1 - I_4 \) into (3.5) and taking \( \epsilon > 0 \) sufficiently small, we obtain
\[
\| P_h u - \tilde{u} \|_C^2 + \| \Pi_h \sigma - \tilde{\sigma} \|_A^2 \leq K h^{2k}(\| \sigma \|_{k+1}^2 + \| u \|_{k+1}^2).
\tag{3.6}
\]
Thus, using (2.12) and the triangular inequality, we get
\[
\| u - \tilde{u} \|_C + \| \sigma - \tilde{\sigma} \|_A \leq K h^k(\| \sigma \|_{k+1} + \| u \|_{k+1}),
\tag{3.7}
\]
which completes the proof. \( \square \)

**Lemma 3.2.** For any \( u \in V \cap H^{k+1}(T_h) \) and any \( \sigma \in W \cap H^{k+1}(T_h) \),
\[
\| \sigma - \tilde{\sigma} \|_W \leq K h^{k}(\| \sigma \|_{k+1} + \| u \|_{k+1}).
\]

**Proof.** From (2.12), the local inverse property, and Lemma 3.1, we have
\[
\| \nabla \cdot (\sigma - \tilde{\sigma}) \| \leq \| \nabla \cdot (\sigma - \Pi_h \sigma) \| + \| \nabla \cdot (\Pi_h \sigma - \tilde{\sigma}) \|
\leq K h^k \| \sigma \|_{k+1} + K h^{-1} \| \Pi_h \sigma - \tilde{\sigma} \|
\leq K h^k \| \sigma \|_{k+1} + K h^{-1}(\| \Pi_h \sigma - \sigma \| + \| \sigma - \tilde{\sigma} \|)
\leq K h^k \| \sigma \|_{k+1} + K h^{-1} \| \sigma - \tilde{\sigma} \|
\leq K h^{k-1}(\| \sigma \|_{k+1} + \| u \|_{k+1}).
\tag{3.8}
\]
Therefore, using the definition of \( \| \cdot \|_W \), Lemma 3.1 and (3.8), we get
\[
\| \sigma - \tilde{\sigma} \|_W^2 = \| \sigma - \tilde{\sigma} \|_{}^2 + \sum_T h_T^2 \| \nabla \cdot (\sigma - \tilde{\sigma}) \|_T^2
\]
\[ \leq K \left( \| \sigma - \tilde{\sigma} \|^2_A + h^2 \| \nabla \cdot (\sigma - \tilde{\sigma}) \|^2 \right) \]
\[ \leq K h^{2k} \left( \| \sigma \|^2_{k+1} + \| u \|^2_{k+1} \right), \]
which completes the proof. \hfill \Box

Lemma 3.3. For any \( u_t \in V \cap H^{k+1}(T_h) \) and any \( \sigma_t \in W \cap H^{k+1}(T_h) \),
\[ \| u_t - \tilde{u}_t \|_C + \| \sigma_t - \tilde{\sigma}_t \|_A \leq K h^k \left( \| u_t \|_{k+1} + \| \sigma_t \|_{k+1} \right), \]
\[ \| \sigma_t - \tilde{\sigma}_t \|_W \leq K h^k \left( \| u_t \|_{k+1} + \| \sigma_t \|_{k+1} \right). \]

Proof. The proofs of these results are similar to those of Lemma 3.1 and Lemma 3.2. \hfill \Box

Lemma 3.4. For any \( u \in V \cap H^{k+1}(T_h) \) and any \( \sigma \in W \cap H^{k+1}(T_h) \),
\[ \| u - \tilde{u} \| \leq K h^{k+1} \left( \| u \|_{k+1} + \| \sigma \|_{k+1} \right), \]
\[ \| u_t - \tilde{u}_t \| \leq K h^{k+1} \left( \| u_t \|_{k+1} + \| \sigma_t \|_{k+1} \right). \]

Proof. Define \( \phi \in H^2(\Omega) \) and \( \psi \in (H^1(\Omega))^d \) satisfying
\[ \begin{align*}
\nabla \phi - \alpha(x) \psi &= 0, \quad \text{in } \Omega, \\
- \nabla \cdot \psi + \lambda \phi &= u - \tilde{u}, \quad \text{in } \Omega, \\
\phi &= 0, \quad \text{on } \partial \Omega_D, \\
\psi \cdot n &= 0, \quad \text{on } \partial \Omega_N.
\end{align*} \tag{3.9} \]
Then, by the property of elliptic regularity, we have
\[ \| \phi \|_2 + \| \psi \|_1 \leq K \| u - \tilde{u} \|. \]
By (3.9), the integration by parts, and the definition of \( B(\cdot, \cdot) \), we obviously have
\[ \begin{align*}
(- \nabla \cdot \psi, u - \tilde{u}) &= (\psi, \nabla (u - \tilde{u})) \\
&- \sum_{e \in E_h^I} \left( \{ \psi \cdot n \}, \{ u - \tilde{u} \}_e \right) + \left( \{ \psi \cdot n \}, [u - \tilde{u}]_e \right) \\
&- \sum_{e \in E_h^D} \left( \psi \cdot n, u - \tilde{u} \right)_e - \sum_{e \in E_h^N} \left( \psi \cdot n, u - \tilde{u} \right)_e \\
&= B(\psi, u - \tilde{u}). \tag{3.10} \end{align*} \]
Since $\phi \in H^2(\Omega) \subset C(\Omega)$, $\psi \in (H^1(\Omega))^d$, $\phi = 0$ on $\partial \Omega_D$, and $\psi \cdot n = 0$ on $\partial \Omega_N$, we get the followings:

$$(\sigma - \bar{\sigma}, \nabla \phi) = (\sigma - \bar{\sigma}, \nabla \phi) - \sum_{e \in \mathcal{E}_h^I} \left(\{ (\sigma - \bar{\sigma}) \cdot n \}, [\phi] \right)_e$$

$$- \sum_{e \in \mathcal{E}_h^D} \left( (\sigma - \bar{\sigma}) \cdot n, \phi \right)_e$$

$$= B(\sigma - \bar{\sigma}, \phi)$$

and

$$J(u - \bar{u}, \phi) = \sum_{e \in \mathcal{E}_h^{ID}} h_e^{-1} ([u - \bar{u}], [\phi])_e = 0.$$  \hspace{1cm} (3.12)

By (3.9)-(3.12), and the definition of $A(\cdot, \cdot)$, we get

$$\|u - \bar{u}\|^2 = (-\nabla \cdot \psi, u - \bar{u}) + \lambda(\phi, u - \bar{u})$$

$$+ (\sigma - \bar{\sigma}, \nabla \phi) - (\sigma - \bar{\sigma}, \alpha(x) \psi)$$

$$= B(\psi, u - \bar{u}) + \lambda(\phi, u - \bar{u}) + B(\sigma - \bar{\sigma}, \phi) - A(\sigma - \bar{\sigma}, \psi)$$

$$= B(\psi - \Pi_h \psi, u - \bar{u}) + B(\Pi_h \psi, u - \bar{u})$$

$$+ \lambda(\phi - P_h \phi, u - \bar{u}) + \lambda(P_h \phi, u - \bar{u}) + B(\sigma - \bar{\sigma}, \phi - P_h \phi)$$

$$+ B(\sigma - \bar{\sigma}, P_h \phi) - A(\sigma - \bar{\sigma}, \psi - \Pi_h \psi) - A(\sigma - \bar{\sigma}, \Pi_h \psi).$$

Notice that by (2.12), we get

$$\|u - P_h u\|_S^2 = \|u - P_h u\|_T^2 + \sum_{e \in \mathcal{E}_h^{ID}} h_e^{-1} \int_e \|[u - P_h u]\|^2 ds$$

$$\leq Kh^{2k} \|u\|_{k+1}^2$$

and for $v \in V_h$

$$B(\psi - \Pi_h \psi, v)$$

$$= \sum_T (\psi - \Pi_h \psi, \nabla v)_T - \sum_{e \in \mathcal{E}_h^{ID}} \left(\{ (\psi - \Pi_h \psi) \cdot n \}, [v] \right)_e$$

$$= - \sum_{e \in \mathcal{E}_h^{ID}} \left(\{ (\psi - \Pi_h \psi) \cdot n \}, [v] \right)_e$$

$$\leq Kh \|\psi\|_1 \|v\|_C.$$ 

By applying (3.1), (3.2), (3.12), (3.14), and (3.15) to (3.13), we get

$$\|u - \bar{u}\|^2 = B(\psi - \Pi_h \psi, u - \bar{u}) + \lambda(\phi - P_h \phi, u - \bar{u}) + B(\sigma - \bar{\sigma}, \phi - P_h \phi)$$

$$- A(\sigma - \bar{\sigma}, \psi - \Pi_h \psi) - J(u - \bar{u}, P_h \phi)$$

$$= B(\psi - \Pi_h \psi, u - P_h u) + B(\psi - \Pi_h \psi, P_h u - \bar{u})$$

$$+ \lambda(\phi - P_h \phi, u - \bar{u}) + B(\sigma - \bar{\sigma}, \phi - P_h \phi)$$
and hence for sufficiently small $h > 0$ we have

$$\|u - \bar{u}\| \leq Kh (h^k \|u\|_{k+1} + \|P_h u - \bar{u}\|_C + \|\sigma - \bar{\sigma}\|_W + \|u - \bar{u}\|_C).$$

Therefore, by Lemma 3.2, (3.6), and (3.7), we obtain

$$\|u - \bar{u}\| \leq K h^{k+1} (\|u\|_{k+1} + \|\sigma\|_{k+1}),$$

which completes the proof of the first result. The proof of the second result is similar to one of the first result. □

4. Error estimates

Theorem 4.1. If $(u, \sigma) \in (V \cap H^{k+1}(T_h)) \times (W \cap H^{k+1}(T_h))$ is the solution of (2.2) and $(u_h, \sigma_h) \in V_h \times W_h$ is the solution of (2.3)-(2.4), then

$$\|u - u_h\|_{L^\infty(L^2)} + h \|\sigma - \sigma_h\|_{L^\infty(L^2)} \leq Kh^{k+1} \left( \|u\|_{L^2(H^{k+1})} + \|u_t\|_{L^2(H^{k+1})} + \|\sigma\|_{L^2(H^{k+1})} + \|\sigma_t\|_{L^2(H^{k+1})} \right).$$

Proof. From (2.8)-(2.11), we obtain the system of error equations

$$(u_t - (u_h)_t, v_h) + B(\sigma - \sigma_h, v_h) + C(u - u_h, v_h) = \lambda(u - u_h, v_h), \forall v_h \in V_h,$$

$$A(\sigma - \sigma_h, \tau_h) - B(\tau_h, u - u_h) = 0, \forall \tau_h \in W_h.$$  

And using (3.1)-(3.2) in the system of error equations, we get

$$\begin{align*}
(\bar{u}_t - (u_h)_t, v_h) + B(\bar{\sigma} - \sigma_h, v_h) + C(\bar{u} - u_h, v_h) \\
= (\bar{u}_t - u_t, v_h) + \lambda(u - u_h, v_h), \forall v_h \in V_h
\end{align*}$$

(4.1)
and

\[ A(\tilde{\sigma} - \sigma_h, \tau_h) - B(\tau_h, \tilde{u} - u_h) = 0, \quad \forall \tau_h \in W_h. \]  

(4.2)

Letting \( v_h = \tilde{u} - u_h, \quad \tau_h = \tilde{\sigma} - \sigma_h \) in (4.1)-(4.2), we get

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \| \tilde{u} - u_h \|^2 + \| \tilde{u} - u_h \|_C^2 + \| \tilde{\sigma} - \sigma_h \|_A^2 \\
\leq \| \tilde{u}_t - u_t \| \| \tilde{u} - u_h \| + \lambda \| u - u_h, \tilde{u} - u_h \| \\
\leq \| \tilde{u}_t - u_t \| \| \tilde{u} - u_h \| + \lambda ( \| u - \tilde{u} \| + \| \tilde{u} - u_h \| ) \| \tilde{u} - u_h \| \\
\leq K \left[ \| u_t - \tilde{u}_t \|^2 + \| u - \tilde{u} \|^2 + \| \tilde{u} - u_h \|^2 \right].
\end{align*}
\]

(4.3)

Now we integrate both sides of (4.3) with respect to \( t \) from 0 to \( t \leq T \) to get

\[
\begin{align*}
\frac{1}{2} \| (\tilde{u} - u_h)(t) \|^2 + \int_0^t \| \tilde{u} - u_h \|_C^2 + \| \tilde{\sigma} - \sigma_h \|_A^2 \, ds \\
\leq \frac{1}{2} \| (\tilde{u} - u_h)(0) \|^2 \\
+ K \int_0^t \| (u_t - \tilde{u}_t)(s) \|^2 + \| (u - \tilde{u})(s) \|^2 + \| (\tilde{u} - u_h)(s) \|^2 \, ds
\end{align*}
\]

and hence by Gronwall's inequality we get

\[
\begin{align*}
\| \tilde{u} - u_h \|_{L^\infty(L^2)} + \| \tilde{u} - u_h \|_{L^2(C)} + \| \tilde{\sigma} - \sigma_h \|_{L^2(A)} \\
\leq K \left( \| u - \tilde{u} \|_{L^2(L^2)} + \| u_t - \tilde{u}_t \|_{L^2(L^2)} + \| (\tilde{u} - u_h)(0) \| \right) \\
\leq K h^{k+1} \left( \| u \|_{L^2(H^{k+1})} + \| u_t \|_{L^2(H^{k+1})} + \| \sigma \|_{L^2(H^{k+1})} + \| \sigma_t \|_{L^2(H^{k+1})} \right).
\end{align*}
\]

Therefore, we have

\[
\begin{align*}
\| u - u_h \|_{L^\infty(L^2)} + h \| \sigma - \sigma_h \|_{L^\infty(L^2)} \\
\leq K h^{k+1} \left( \| u \|_{L^2(H^{k+1})} + \| u_t \|_{L^2(H^{k+1})} + \| \sigma \|_{L^2(H^{k+1})} + \| \sigma_t \|_{L^2(H^{k+1})} \right).
\end{align*}
\]

This completes the proof. \( \square \)

**Theorem 4.2.** If \((u, \sigma) \in (V \cap H^{k+1}(T_h)) \times (W \cap H^{k+1}(T_h))\) is the solution of (2.2) and \((u_h, \sigma_h) \in V_h \times W_h\) is the solution of (2.3)-(2.4), then

\[
\| u_t - (u_h)_t \|_{L^2(L^2)} \\
\leq K h^{k+1} \left( \| u \|_{L^2(H^{k+1})} + \| u_t \|_{L^2(H^{k+1})} + \| \sigma \|_{L^2(H^{k+1})} + \| \sigma_t \|_{L^2(H^{k+1})} \right).
\]

**Proof.** Differentiating (4.2) with respect to \( t \), we obtain

\[
A(\tilde{\sigma}_t - (\sigma_h)_t, \tau_h) - B(\tau_h, \tilde{u}_t - (u_h)_t) = 0, \quad \forall \tau_h \in W_h.
\]

(4.4)
Letting \( v_h = \tilde{u}_t - (u_h)_t \), \( \tau_h = \tilde{\sigma} - \sigma_h \) in (4.1) and (4.4) and adding the resulting equations, we get
\[
||\tilde{u}_t - (u_h)_t||^2 + C(\tilde{u} - u_h, \tilde{u}_t - (u_h)_t) + A(\tilde{\sigma}_t - (\sigma_h)_t, \tilde{\sigma} - \sigma_h)
= (\tilde{u}_t - u_t, \tilde{u}_t - (u_h)_t) + \lambda (u - u_h, \tilde{u}_t - (u_h)_t).
\]
Since
\[
C(\tilde{u} - u_h, \tilde{u}_t - (u_h)_t) = J(\tilde{u} - u_h, \tilde{u}_t - (u_h)_t) + \lambda (\tilde{u} - u_h, \tilde{u}_t - (u_h)_t)
= \frac{1}{2} \frac{d}{dt} \sum_{e \in \mathcal{E}_h^I} h_e^{-1} ||\tilde{u} - u_h||^2_e + \frac{\lambda}{2} \frac{d}{dt} ||\tilde{u} - u_h||^2
= \frac{1}{2} \frac{d}{dt} ||\tilde{u} - u_h||^2_C
\]
and
\[
A(\tilde{\sigma}_t - (\sigma_h)_t, \tilde{\sigma} - \sigma_h) = \frac{1}{2} \frac{d}{dt} ||\alpha^{\frac{1}{2}}(x)(\tilde{\sigma} - \sigma_h)||^2,
\]
we obtain
\[
||\tilde{u}_t - (u_h)_t||^2 + \frac{1}{2} \frac{d}{dt} \left( ||\tilde{u} - u_h||^2_C + ||\alpha^{\frac{1}{2}}(x)(\tilde{\sigma} - \sigma_h)||^2 \right)
\leq ||\tilde{u}_t - u_t|| ||\tilde{u}_t - (u_h)_t|| + \lambda ||u - u_h|| ||\tilde{u}_t - (u_h)_t||
\]
and so
\[
||\tilde{u}_t - (u_h)_t||^2 + \frac{d}{dt} \left( ||\tilde{u} - u_h||^2_C + ||\alpha^{\frac{1}{2}}(x)(\tilde{\sigma} - \sigma_h)||^2 \right)
\leq K(||\tilde{u}_t - u_t||^2 + ||u - u_h||^2).
\]
Now we integrate both sides of the above inequality with respect to \( t \) from 0 to \( t \leq T \) to get
\[
||\tilde{u}_t - (u_h)_t||^2_{L^2(L^2)} + \sup_{[0,T]} ||\tilde{u} - u_h||^2_C + ||\alpha^{\frac{1}{2}}(x)(\tilde{\sigma} - \sigma_h)||^2_{L^\infty(L^2)}
\leq K(||\tilde{u}_t - u_t||^2_{L^2(L^2)} + ||u - u_h||^2_{L^2(L^2)})
\]
and so, by Theorem 4.1 and Lemma 3.4, we get
\[
||\tilde{u}_t - (u_h)_t||^2_{L^2(L^2)} + ||\tilde{u} - u_h||^2_{L^\infty(L^2)} + ||(\tilde{\sigma} - \sigma_h)||^2_{L^\infty(L^2)}
\leq K(||\tilde{u}_t - u_t||^2_{L^2(L^2)} + ||u - u_h||^2_{L^2(L^2)})
\leq K h^{2(k+1)} \left( ||\sigma||^2_{L^2(H^{k+1})} + ||\sigma_t||^2_{L^2(H^{k+1})} + ||u||^2_{L^2(H^{k+1})} + ||u_t||^2_{L^2(H^{k+1})} \right).
\]
Therefore, using the triangular inequality and Lemma 3.4, we have
\[
||u_t - (u_h)_t||_{L^2(L^2)}
\leq K h^{k+1} \left( ||\sigma||^2_{L^2(H^{k+1})} + ||\sigma_t||^2_{L^2(H^{k+1})} + ||u||_{L^2(H^{k+1})} + ||u_t||_{L^2(H^{k+1})} \right).
\]
This completes the proof. \( \square \)
References


