HILBERT 3-CLASS FIELD TOWERS OF IMAGINARY CUBIC FUNCTION FIELDS

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Abstract. In this paper we study the infiniteness of the Hilbert 3-class field tower of imaginary cubic function fields.

1. Introduction

Let $k = \mathbb{F}_q(T)$ be a rational function field over the finite field $\mathbb{F}_q$, $\infty = (1/T)$ and $\Lambda = \mathbb{F}_q[T]$. For a finite separable extension $F$ of $k$, write $\mathcal{O}_F$ for the integral closure of $\Lambda$ in $F$ and $H_F$ for the Hilbert class field of $F$ with respect to $\mathcal{O}_F$ (cf. [3]). Let $\ell$ be a prime number. Let $F_1^{(\ell)}$ be the Hilbert $\ell$-class field of $F_0^{(\ell)} = F$, i.e., $F_1^{(\ell)}$ is the maximal $\ell$-extension of $F$ inside $H_F$, and inductively, $F_n^{(\ell)}$ be the Hilbert $\ell$-class field of $F_{n-1}^{(\ell)}$ for $n \geq 1$. Then we obtain a sequence of fields

$$F_0^{(\ell)} = F \subset F_1^{(\ell)} \subset \cdots \subset F_n^{(\ell)} \subset \cdots,$$

which is called the Hilbert $\ell$-class field tower of $F$. We say that the Hilbert $\ell$-class field tower of $F$ is infinite if $F_n^{(\ell)} \neq F_{n+1}^{(\ell)}$ for each $n \geq 0$.

For a multiplicative abelian group $A$, write $r_\ell(A) = \dim_F(A/A^\ell)$, which is called the the $\ell$-rank of $A$. Let $\text{Cl}_F$ be the ideal class group of $\mathcal{O}_F$ and $\mathcal{O}_F^\times$ be the group of units of $\mathcal{O}_F$. In [4], Schoof proved that the Hilbert $\ell$-class field tower of $F$ is infinite if $r_\ell(\text{Cl}_F) \geq 2 + 2\sqrt{r_\ell(\mathcal{O}_F^\times)} + 1$.

Assume that $q$ is odd with $q \equiv 1 \mod 3$. By an imaginary cubic function field, we always mean a finite (geometric) cyclic extension $F$ over $k$ of degree 3 in which $\infty$ is ramified. In [1], Ahn and Jung studied the infiniteness of the Hilbert 2-class field tower of imaginary quadratic
function fields. The aim of this paper is to study the infiniteness of the Hilbert 3-class field tower of imaginary cubic function fields.

2. Preliminaries

2.1. Rédei matrix

Assume that $q$ is odd with $q \equiv 1 \mod 3$. Fix a generator $\gamma$ of $\mathbb{F}_q^*$. Write $\mathcal{P}$ for the set of all monic irreducible polynomials in $\mathbb{A}$. Any cubic function field $F$ can be written as $F = k(\sqrt[3]{D})$, where $D = aP_1^{r_1} \cdots P_t^{r_t}$ with $a \in \{1, \gamma\}$ and $P_i \in \mathcal{P}, r_i \in \{1, 2\}$ for $1 \leq i \leq t$. Then $F = k(\sqrt[3]{D})$ is imaginary if and only if $3 \nmid \deg D$. Let $\sigma$ be a generator of $G = \text{Gal}(F/k)$. Then we have

\begin{equation}
(2.1) \quad r_3(\text{Cl}_F) = \lambda_1(F) + \lambda_2(F),
\end{equation}

where $\lambda_i(F) = \dim_{\mathbb{F}_3} \left( \text{Cl}^{(1-\sigma)^i}_F / \text{Cl}^{(1-\sigma)^i}_F \right)$ for $i = 1, 2$. By the Genus theory, $\lambda_1(F) = t - 1$.

Put $\eta = \gamma^t \tau^{-1/3}$. Let $R'_F = (e_{ij})_{1 \leq i, j \leq t}$ be a $t \times t$ matrix over $\mathbb{F}_3$, where $e_{ij} \in \mathbb{F}_3$ is defined by $\eta^{r_{ij}} = (P_i/P_j)^3$ for $1 \leq i \neq j \leq t$ and the diagonal entries $e_{ii}$ are defined by the relation $\sum_{r=1}^{t} r_j e_{ij} = 0$ or $d_i + \sum_{r=1}^{t} r_j e_{ij} = 0$ according as $a = 1$ or $a = \gamma$. Let $d_i \in \mathbb{F}_3$ be defined by $\deg P_i \equiv d_i \mod 3$ for $1 \leq i \leq t$. Let $R_F$ be the $(t+1) \times t$ matrix over $\mathbb{F}_3$ obtained from $R'_F$ by adjoining $(d_1 \cdots d_t)$ in the last row. Then we have $\lambda_2(F) = t - \text{rank } R_F$ ([2, Corollary 3.8]). Let $\vartheta_F$ be 0 or 1 according as $a = 1$ or $a = \gamma$. Using the relation $\sum_{r=1}^{t} r_j e_{ij} = 0$ or $d_i + \sum_{r=1}^{t} r_j e_{ij} = 0$ according as $a = 1$ or $a = \gamma$, it can be shown that

\begin{equation}
(2.2) \quad \lambda_2(F) = t - 1 + \vartheta_F - \text{rank } R'_F.
\end{equation}

2.2. Martinet’s inequality

For a finite separable extension $F$ of $k$, write $S_\infty(F)$ for the set of all primes of $F$ lying above $\infty$.

**Proposition 2.1.** Let $E$ and $K$ be finite (geometric) separable extensions of $k$ such that $E/K$ is a cyclic extension of degree $t$, where $t$ is a prime number not dividing $q$. Let $\gamma_{E/K}$ be the number of prime ideals of $\mathcal{O}_K$ that ramify in $E$ and $\rho_{E/K}$ be the number of places $p_\infty$ in $S_\infty(K)$ that ramify or inert in $E$. If

\[ \gamma_{E/K} \geq |S_\infty(K)| - \rho_{E/K} + 3 + 2\sqrt{t|S_\infty(K)| + (1 - t)\rho_{E/K} + 1}, \]

then

\[ \lambda_2(F) = t - 1 + \vartheta_F - \text{rank } R'_F. \]
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\( \pi \) is infinite if \( r \in \mathbb{P} \).

\( \Pi \) is a imaginary cubic function field, where

we see that

Proposition 2.1 on

\( E/K \)

\( \Pi \) is infinite Hilbert 3-class field tower.

\( E/K \) has infinite Hilbert 3-class field tower. Hence

\( F \) also has infinite Hilbert 3-class field tower.

\( \Pi \) is odd with \( q \equiv 1 \mod 3 \). Let

\( F = k(\sqrt[3]{D}) \) be an imaginary cubic function field over \( k \). If there is a nonconstant monic polynomial

\( D' \) such that \( 3|\deg D' , \pi(D') \subset \pi(D) \)

and

\( \left( \frac{D}{F} \right)_3 = \left( \frac{D'}{F} \right)_3 = 1 \) for \( P_1, P_2, P_3 \in \pi(D) \setminus \pi(D') \), then \( F \) has infinite Hilbert 3-class field tower.

Proof. Put \( K = k(\sqrt[3]{D'}) \). By hypothesis, \( P_1, P_2, P_3 \) and \( \infty \) split completely in \( K \).

Hence, \( E := FK \) is contained in \( F^{(3)} \). Applying Proposition 2.1 on \( E/K \) with

\( \gamma_{E/K} \geq 9 \) and

\( |S_\infty(K)| = \rho_{E/K} = 3 \), we see that \( E \) has infinite Hilbert 3-class field tower. Hence \( F \) also has infinite Hilbert 3-class field tower.

Corollary 2.3. Assume that \( q \) is odd with \( q \equiv 1 \mod 3 \). Let

\( F = k(\sqrt[3]{D}) \) be an imaginary cubic function field over \( k \). If there are two distinct nonconstant monic polynomials

\( D_1, D_2 \) such that \( 3|\deg D_i, \pi(D_i) \subset \pi(D) \) for \( i = 1, 2 \) and

\( \left( \frac{D_i}{F} \right)_3 = \left( \frac{D_1}{F} \right)_3 = \left( \frac{D_2}{F} \right)_3 = 1 \) for some \( P_1, P_2 \in \pi(D) \setminus \pi(D_1 \cup \pi(D_2)) \), then \( F \) has infinite Hilbert 3-class field tower.

Proof. Put \( K = k(\sqrt[3]{D_1}, \sqrt[3]{D_2}) \). By hypothesis, \( P_1, P_2 \) and \( \infty \) split completely in \( K \).

Hence, \( E := FK \) is contained in \( F^{(3)} \). By applying Proposition 2.1 on \( E/K \) with

\( \gamma_{E/K} \geq 18 \) and

\( |S_\infty(K)| = \rho_{E/K} = 9 \), we see that \( E \) has infinite Hilbert 3-class field tower. Hence \( F \) also has infinite Hilbert 3-class field tower.

3. Hilbert 3-class field tower of imaginary cubic function field

Assume that \( q \) is odd with \( q \equiv 1 \mod 3 \). Let \( F = k(\sqrt[3]{D}) \) be an imaginary cubic function field, where

\( D = aP_1^{e_1} \cdots P_t^{e_t} \) with \( a \in \{1, \gamma\} \), \( P_i \in \mathcal{P} \), \( e_i \in \{1, 2\} \) for \( 1 \leq i \leq t \) and \( 3 \nmid \deg D \).

Since \( \mathcal{O}_F = \mathbb{F}_q \) and

\( r_3(\mathcal{O}_F) = 1 \), by Schoof’s theorem, the Hilbert 3-class field tower of \( F \) is infinite if

\( r_3(\mathcal{O}_F) = \lambda_1(F) + \lambda_2(F) \geq 5 \). By genus theory, we have

\( \lambda_1(F) = t - 1 \). Hence, if \( t \geq 6 \), then \( F \) has infinite Hilbert 3-class field tower.
3.1. Case $t = 4$

In this subsection we will consider the case $t = 4$ in detail.

**Theorem 3.1.** Assume that $q$ is odd with $q \equiv 1 \mod 3$. Let $F = k(\sqrt[3]{D})$ be an imaginary cubic function field with $D = aP_1^{r_1}P_2^{r_2}P_3^{r_3}P_4^{r_4}$. Let $\vartheta_F$ be 0 or 1 according as $a = 1$ or $a = \gamma$. If rank $R'_F \leq 1 + \vartheta_F$, then $F$ has infinite Hilbert 3-class field tower.

**Proof.** By (2.1) and (2.2), we have $r_3(Cl_F) = 6 + \vartheta_F - \text{rank } R'_F$. Since the Hilbert 3-class field tower of $F$ is infinite if $r_3(Cl_F) \geq 5$, the result follows immediately.

**Example 3.2.** Consider $k = \mathbb{F}_7(T)$. Then $\gamma = 3$ is a generator of $\mathbb{F}_7^*$ and $\eta = 2$. Let $P_1 = T$, $P_2 = T - 1$, $P_3 = T^2 + T - 1$ and $P_4 = T^2 - T - 1$, which are all monic irreducible polynomials in $\mathbb{A} = \mathbb{F}_7[T]$. We have $e_{12} = e_{13} = e_{14} = e_{23} = e_{24} = 0, e_{34} = 1$. Let $F = k(\sqrt[3]{D})$ with $D = P_1P_2^2P_3P_4$. Then deg $D = 7 \not\equiv 0 \mod 3$, and the matrix $R'_F$ is

$$
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 2 & 1 \\
0 & 0 & 1 & 2 
\end{pmatrix}
$$

whose rank is 1. Hence, $F$ has infinite Hilbert 3-class field tower.

**Theorem 3.3.** Assume that $q$ is odd with $q \equiv 1 \mod 3$. Let $F = k(\sqrt[3]{D})$ be an imaginary cubic function field with $D = aP_1^{r_1}P_2^{r_2}P_3^{r_3}P_4^{r_4}$. Then $F$ has infinite Hilbert 3-class field tower if one of the following conditions holds:

1. deg $P_3 \equiv 0 \mod 3$ and $(\frac{P_1}{P_3})_3 = (\frac{P_2}{P_3})_3 = (\frac{P_4}{P_3})_3 = 1$,
2. deg $P_3 \equiv \text{deg } P_4 \equiv 0 \mod 3$ and $(\frac{P_1}{P_2})_3 = (\frac{P_2}{P_3})_3 = (\frac{P_3}{P_2})_3 = 1$.

**Proof.** It follows immediately from Corollary 2.2 and Corollary 2.3.

**Example 3.4.** Let $k = \mathbb{F}_7(T)$. Let $P_1 = T$, $P_2 = T - 1$, $P_3 = T^2 + T - 1$ and $P_4 = T^3 + T - 1$, which are all monic irreducible polynomials in $\mathbb{A} = \mathbb{F}_7[T]$. We have $(\frac{P_1}{P_2})_3 = (\frac{P_2}{P_3})_3 = (\frac{P_3}{P_2})_3 = 1$. Let $D = \gamma P_1P_2P_3P_4$. Hence, the Hilbert 3-class field tower of $F = k(\sqrt[3]{D})$ by Theorem 3.3. But, the matrix $R'_F$ is

$$
\begin{pmatrix}
2 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 
\end{pmatrix}
$$

whose rank is 3. So Theorem 3.1 can’t guarantee the infiniteness of Hilbert 3-class field tower of $F$. 

3.2. Case \( t = 5 \)

In this subsection we will consider the case \( t = 5 \) in detail.

**Theorem 3.5.** Assume that \( q \) is odd with \( q \equiv 1 \mod 3 \). Let \( F = k(\sqrt[3]{D}) \) be an imaginary cubic function field with \( D = aP_1^3P_2^3P_3^3P_4^3P_5^5 \). Let \( R'_F = (e_{ij})_{1 \leq i, j \leq 5} \) be the \( 5 \times 5 \) matrix over \( \mathbb{F}_3 \) given in \( \S 2.1 \). If \( a = 1 \) with rank \( R'_F \leq 3 \) or \( a = \gamma \) with rank \( R'_F \leq 4 \), then \( F \) has infinite Hilbert 3-class field tower.

**Proof.** By (2.1) and (2.2), we have \( r_3(\text{Cl}_F) = 8 - \text{rank } R'_F \) or \( r_3(\text{Cl}_F) = 9 - \text{rank } R'_F \) according as \( a = 1 \) or \( a = \gamma \). Since the Hilbert 3-class field tower of \( F \) is infinite if \( r_3(\text{Cl}_F) \geq 5 \), the result follows immediately. \( \square \)

**Example 3.6.** Let \( k = \mathbb{F}_7(T) \). Let \( P_1 = T, P_2 = T - 1, P_3 = T^2 - T - 1, P_4 = T^3 + T - 1 \), \( P_5 = T^3 + T - 1 \), which are all monic irreducible polynomials in \( A = \mathbb{F}_7[T] \). We have \( \epsilon_{12} = \epsilon_{13} = \epsilon_{14} = \epsilon_{15} = \epsilon_{23} = \epsilon_{24} = \epsilon_{25} = \epsilon_{45} = 0, \epsilon_{34} = \epsilon_{35} = 1 \). Let \( F = k(\sqrt[3]{D}) \) with \( D = P_1P_2^3P_3P_4P_5 \). Then \( \deg D = 10 \equiv 0 \mod 3 \), and the matrix \( R'_F \) is

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 2
\end{pmatrix}
\]

whose rank is 2. Hence, \( F \) has infinite Hilbert 3-class field tower.

**Theorem 3.7.** Assume that \( q \) is odd with \( q \equiv 1 \mod 3 \). Let \( F = k(\sqrt[3]{D}) \) be an imaginary cubic function field with \( D = aP_1^3P_2^3P_3^3P_4^3P_5^5 \). Then \( F \) has infinite Hilbert 3-class field tower if one of the following conditions holds:

1. \( \deg P_5 \equiv 0 \mod 3 \) and \( (\frac{P_3}{P_1})_3 = (\frac{P_3}{P_2})_3 = (\frac{P_5}{P_2})_3 = 1 \),
2. \( \deg P_4 \equiv \deg P_5 \equiv 0 \mod 3 \) and \( (\frac{P_4}{P_1})_3 = (\frac{P_4}{P_2})_3 = (\frac{P_5}{P_2})_3 = 1 \),
3. \( \deg P_3 = \deg P_4 = \deg P_5 \equiv 0 \mod 3 \) and the rank of \( (\frac{e_{13}}{e_{23}}, \frac{e_{14}}{e_{24}}, \frac{e_{15}}{e_{25}}) \) is \( \leq 1 \).

**Proof.** (1) and (2) follow immediately from Corollary 2.2 and Corollary 2.3, respectively. For (3), by hypothesis, we can choose \( x, y, z, w \in \mathbb{F}_3 \) such that \( (\frac{x}{y}) \neq (0), (\frac{z}{w}) \neq (0), (\frac{e_{13}}{e_{23}}, \frac{e_{14}}{e_{24}})(\frac{x}{y}) = (0) \) and \( (\frac{e_{13}}{e_{23}}, \frac{e_{15}}{e_{25}})(\frac{x}{y}) = (0) \). Note that \( e_{ij} = e_{ji} \) for \( i \neq j \). We have

\[
\begin{align*}
(\frac{P_3^2P_4^5}{P_1})_3 & = \eta^{xe_{31} + ye_{41}} = 1, \\
(\frac{P_3^2P_5^4}{P_2})_3 & = \eta^{xe_{32} + ye_{42}} = 1, \\
(\frac{P_3^2P_5^w}{P_1})_3 & = \eta^{xe_{31} + xe_{51}} = 1, \\
(\frac{P_3^2P_5^w}{P_2})_3 & = \eta^{xe_{32} + xe_{52}} = 1.
\end{align*}
\]
Since $D_1 = P_3^x P_4^y$ and $D_2 = P_5^x P_5^u$ are nonconstant monic polynomials whose degree are divisible by 3, by Corollary 2.3, $F$ has infinite Hilbert 3-class field tower.

**Example 3.8.** Let $k = \mathbb{F}_7(T)$. Let $P_1 = T$, $P_2 = T - 1$, $P_3 = T^3 + T - 1$, $P_4 = T^3 - 3T + 1$ and $P_5 = T^3 - T - 2$, which are all monic irreducible polynomials in $\mathbb{A} = \mathbb{F}_7[T]$. We have $e_{13} = e_{14} = e_{23} = e_{24} = 0$ and $e_{15} = e_{25} = 2$. Let $D = P_1 P_2 P_3 P_4 P_5$. By Theorem 3.7, the Hilbert 3-class field tower of $F = k(\sqrt[3]{D})$ is infinite.

**References**


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