A SOLUTION OF EINSTEIN’S FIELD EQUATIONS FOR
THE THIRD CLASS IN $X_4$

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Abstract. The main goal in the present paper is to obtain a solution of Einstein’s unified field equations for the third class in $X_4$.

1. Introduction

Einstein([1]) proposed a new unified field theory that would include both gravitation and electromagnetism. Hlavatý([5]) gave the mathematical foundation of the Einstein’s unified field theory in a 4-dimensional generalized Riemannian space $X_4$ (i.e., space-time) for the first time. Since then this theory had been generalized in a generalized Riemannian manifold $X_n$, the so-called Einstein’s n-dimensional unified field theory, and many consequences of this theory has been obtained by a number of mathematicians. However, it has been unable yet to represent a general n-dimensional Einstein’s connection in a surveyable tensorial form, probably due to the complexity of the higher dimensions. The purpose of the present paper is to obtain a necessary and sufficient condition for a connection with a new torsion tensor be an Einstein’s connection in $X_n$. In the next, we obtain a solution of Einstein’s field equations for the third class in $X_4$. The obtained results and discussions in the present paper will be useful for the 4-dimensional considerations of the unified field theory.

2. Preliminary

Let $X_n$ be an n-dimensional generalized Riemannian manifold covered by a system of real coordinate neighborhoods \{U; $x^k$\}, where, here and

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in the sequel, Greek indices run over the range \( \{1, 2, \cdots, n\} \) and follow the summation convention. The algebraic structure on \( X_n \) is imposed by a basic real non-symmetric tensor \( g_{\lambda\mu} \), which may be split into its symmetric part \( h_{\lambda\mu} \) and skew-symmetric part \( k_{\lambda\mu} \):

\[
g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu},
\]

where we assume that

\[
(2.2) \quad (a) \ G = \det((g_{\lambda\mu})) \neq 0, \quad (b) \ H = \det((h_{\lambda\mu})) \neq 0.
\]

Since \( \det((h_{\lambda\mu})) \neq 0 \), we may define a unique tensor \( h^{\lambda\nu} (= h^{\nu\lambda}) \) by

\[
(2.3) \quad h_{\lambda\mu}h^{\lambda\nu} = \delta^\nu_\mu.
\]

We use the tensors \( h^{\lambda\nu} \) and \( h_{\lambda\mu} \) as tensors for raising and/or lowering indices for all tensors defined on \( X_n \) in the usual manner. Then we may define new tensors by

\[
(2.4) \quad (a) \ k^{\alpha}_\mu = k_{\lambda\mu}h^{\lambda\alpha}, \quad (b) \ k_\lambda^{\alpha} = k_{\lambda\mu}h^{\mu\alpha}, \quad (c) \ k_{\lambda\beta}^{\alpha\beta} = k_{\lambda\mu}h^{\lambda\alpha}h^{\mu\beta}.
\]

The manifold \( X_n \) is assumed to be connected by a general real connection \( \Gamma^\nu_{\lambda\mu} \) which may also be split into its symmetric part \( \Lambda^\nu_{\lambda\mu} \) and skew-symmetric part \( S_{\lambda\mu} \), called the torsion tensor of \( \Gamma^\nu_{\lambda\mu} \):

\[
(2.5) \quad (a) \ \Lambda^\nu_{\lambda\mu} = \Gamma^\nu_{\lambda\mu} = \frac{1}{2}(\Gamma^\nu_{\lambda\mu} + \Gamma^\nu_{\mu\lambda}), \quad (b) \ S_{\lambda\mu}^\nu = \Gamma^\nu_{[\lambda\mu]} = \frac{1}{2}(\Gamma^\nu_{\lambda\mu} - \Gamma^\nu_{\mu\lambda}).
\]

The Einstein's \( n \)-dimensional unified field theory in \( X_n \) is governed by the following set of equations:

\[
(2.6) \quad \partial_\nu g_{\lambda\mu} - g_{\alpha\mu}\Gamma^\alpha_{\lambda\nu} - g_{\lambda\alpha}\Gamma^\alpha_{\alpha\mu} = 0 \quad (\partial_\nu = \frac{\partial}{\partial x^\nu}),
\]

and

\[
(2.7) \quad (a) \ S_\lambda = S_{\lambda\alpha}^\alpha = 0, \quad (b) \ R_{[\lambda\mu]} = \partial_\nu P_\mu, \quad (c) \ R_{\lambda\mu} = 0,
\]

where \( P_\mu \) is an arbitrary vector, called the Einstein's vector, and \( R_{\lambda\mu} \) is the contracted curvature tensor \( R^\alpha_{\lambda\mu\alpha} \) of the curvature tensor \( R^\nu_{\lambda\mu\nu} \):

\[
(2.8) \quad R^\nu_{\lambda\mu\nu} = \partial_\nu \Gamma^\nu_{\lambda\mu} - \partial_\nu \Gamma^\nu_{\mu\lambda} + \Gamma^\alpha_{\lambda\nu}\Gamma^\alpha_{\alpha\mu} - \Gamma^\alpha_{\lambda\mu}\Gamma^\alpha_{\alpha\nu}.
\]

The equation (2.6) is called the Einstein's equation, and the solution \( \Gamma^\nu_{\lambda\mu} \) of the Einstein's equation is called an Einstein's connection. And the vector \( S_\lambda \), defined by (2.7)(a), is called the torsion vector.
3. An Einstein’s connection in $X_n$

The following theorem was proved by Hlavatý ([5]).

**Theorem 3.1.** In $X_n$, if the Einstein’s equation (2.6) admits a solution $\Gamma^\lambda_{\mu\nu}$, then this solution must be of the form

\[ \Gamma^\nu_{\lambda\mu} = \{\lambda^{\nu}_{\mu}\} + 2h^\nu_{\lambda\alpha} S_\alpha (\lambda^\beta_{\mu\beta}) + S^\nu_{\lambda\mu} \]

where $\{\lambda^{\nu}_{\mu}\}$ are the Christoffel symbols defined by $h_{\lambda\mu}$. 

**Remark 3.2.** In virtue of Theorem 3.1, the equation (3.1) reduces the investigation of $\Gamma^\nu_{\lambda\mu}$ to the study of its torsion tensor $S^\nu_{\lambda\mu}$. Hence in order to know an Einstein’s connection $\Gamma^\nu_{\lambda\mu}$, it is necessary and sufficient to know its torsion tensor $S^\nu_{\lambda\mu}$. For this, we introduce a new torsion tensor $S^\nu_{\lambda\mu}$ given by

\[ S^\nu_{\lambda\mu} = 2\delta^\nu_{\lambda\beta} k^\beta_{\mu\alpha} Y^\alpha + k^\nu_{\lambda\beta} Y^\beta \]

for some nonzero vector $Y^\lambda$.

**Theorem 3.3.** In $X_n$, if the connection (3.1) is a connection such that its torsion tensor is of the form (3.2) for some nonzero vector $Y^\lambda$, then the connection is given by

\[ S^\nu_{\lambda\mu} = 2\delta^\nu_{\lambda\beta} k^\beta_{\mu\alpha} Y^\alpha + k^\nu_{\lambda\beta} Y^\beta \]

\[ \Gamma^\nu_{\lambda\mu} = \{\lambda^{\nu}_{\mu}\} + 2h^\nu_{\lambda\alpha} S_\alpha (\lambda^\beta_{\mu\beta}) + S^\nu_{\lambda\mu} \]

**Proof.** Since the torsion tensor of the connection (3.1) is of the form (3.2), we obtain

\[ 2h^\nu_{\lambda\alpha} S_\alpha (\lambda^\beta_{\mu\beta}) = 0 \]

by a straightforward computation. Substituting (3.2) and (3.4) into (3.1), we obtain (3.3). 

**Theorem 3.4.** In $X_n$, the connection (3.3) is an Einstein’s connection if and only if the vector $Y^\lambda$ defining (3.3) satisfies the following condition

\[ \nabla_\nu k^\mu_{\lambda\mu} = 2h^\nu_{\lambda\alpha} k^\beta_{\mu\alpha} Y^\alpha - 2k^\nu_{\lambda\beta} Y^\beta \]

where $\nabla_\omega$ is the symbolic vector of the covariant derivative with respect to $\{\lambda^{\nu}_{\mu}\}$.

**Proof.** The connection (3.3) is an Einstein’s connection if and only if the connection (3.3) satisfies the Einstein’s equation (2.6). Substituting (2.1) and (3.3) into (2.6), and making use of $\nabla_\nu h_{\lambda\mu} = 0$, we obtain

\[ \nabla_\nu k^\mu_{\lambda\mu} - 2h^\nu_{\lambda\alpha} k^\beta_{\mu\alpha} Y^\alpha + 2k^\nu_{\lambda\beta} Y^\beta = 0 \]
by a straightforward computation. Hence the connection (3.3) is an Einstein’s connection if and only if the vector $Y_\lambda$ defining (3.3) satisfies the condition (3.5).

**Theorem 3.5.** In $X_n$, if the connection (3.3) is an Einstein’s connection, then its torsion vector satisfies the following relation:

$$S_\lambda = \nabla_\alpha k_\lambda^\alpha$$

(3.7)

**Proof.** Contracting for (3.2) for $\mu$ and $\nu$, we obtain

$$S_\lambda = S_\lambda^\alpha = (2 - n) k_\lambda^\alpha Y^\alpha.$$  

(3.8)

Next, multiplying $h^{\mu\alpha}$ on both sides of (3.5), and contracting for $\nu$ and $\alpha$, we obtain

$$\nabla_\alpha k_\lambda^\alpha = (2 - n) k_\lambda^\alpha Y^\alpha.$$  

(3.9)

The results (3.8) and (3.9) imply the relation (3.7).

4. **A solution of field equations for the third class in $X_4$**

In this section we shall display a solution for the third class of (2.6) and (2.7) in $X_4$. Assume $h_{\lambda\mu}$ to be of the form

$$((h_{\lambda\mu})) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$  

(4.1)

Define two vectors by

$$(a) \ A_\lambda : (0, 0, 1, -1), \quad (b) \ B_\lambda : (\phi, \psi, 0, 0),$$

where $\phi = \phi(x_1, x_2, x_3, x_4)$ and $\psi = \psi(x_1, x_2, x_3, x_4)$ are nonzero real-valued functions to be determined. Now, we define a basic tensor $g_{\lambda\mu}$ in $X_4$ by

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu},$$  

(4.3)

where $h_{\lambda\mu}$ is defined by (4.1), and $k_{\lambda\mu}$ is defined by

$$k_{\lambda\mu} = 2 A_\lambda [B_\mu],$$

(4.4)

that is,

$$((k_{\lambda\mu})) = \begin{pmatrix} 0 & 0 & -\phi & \phi \\ 0 & 0 & -\psi & \psi \\ \phi & \psi & 0 & 0 \\ -\phi & -\psi & 0 & 0 \end{pmatrix},$$  

(4.5)
A solution of Einstein’s field equations in $X_4$ which is obviously of the third class:

\[(4.6)\quad (a) \text{Det}(k_{\lambda\mu}) = 0, \quad (b) K = \frac{1}{4} k_{\alpha\beta} k^{\alpha\beta} = 0.\]

Then all the Christoffel symbols $\{\lambda^\nu_{\mu}\}$ vanish. Hence the components of the first covariant derivatives with respect to $\{\lambda^\nu_{\mu}\}$ are ordinary derivatives, and $H^\nu_{\lambda\mu\nu} = 0$. Furthermore,

\[(4.7)\quad (a) \ A^\lambda(= h^\lambda_{\nu} A_{\mu}) : (0, 0, 1, 1), \quad (b) \ B^\lambda(= h^\lambda_{\nu} B_{\mu}) : (\phi, \psi, 0, 0)\]

\[(4.8)\quad (a) \ A_\alpha A^\alpha = A_\alpha B^\alpha = 0, \quad (b) \ k_{\lambda\alpha} A^\alpha = 0, \quad (c) \ \partial_\lambda A_\mu = 0.\]

The following theorem is immediate consequences of Theorem 3.3 and Theorem 3.4, in virtue of (2.7)(a), (3.8), and (4.3).

**Theorem 4.1.** In $X_4$, for the basic tensor $g_{\lambda\mu}$ given by (4.3), the connection (3.3) is given by

\[(4.9) \ \Gamma^\nu_{\lambda\mu} = 2 \delta^\nu_{[\lambda} k_{\mu]\alpha} Y^\alpha + k_{\lambda\mu} Y^\nu,\]

and this connection (4.9) is a solution of (2.6) and (2.7)(a) if and only if the vector $Y^\nu$ defining (4.9) satisfies the following conditions

\[(4.10)\quad (a) \ k_{\mu\alpha} Y^\alpha = 0, \quad (b) \ \partial_\nu k_{\lambda\mu} = -2 k_{\nu[\lambda} Y_{\mu]}\]

If these conditions (4.10) are satisfied, then the connection (4.9) is given by

\[(4.11) \ \Gamma^\nu_{\lambda\mu} = k_{\lambda\mu} Y^\nu,\]

which is an Einstein’s connection with zero torsion vector.

**Remark 4.2.** In $X_4$, since the tensor $k_{\lambda\mu} \neq 0$ is skew-symmetric, we know from elementary algebra that the rank of the matrix $((k_{\lambda\mu}))$ can be either four or two. In virtue of (4.6)(a), in our case the rank must be two. Therefore, the homogeneous equations (4.10)(a) have at least two distinct solutions $Y^\nu_1 : (0, 0, 1, 1)$ and $Y^\nu_2 : (\psi, -\phi, 0, 0)$. Every linear combination

\[(4.12) \quad Y^\nu = \rho Y^\nu_1 + \eta Y^\nu_2 : (\eta \psi, -\eta \phi, \rho, \rho)\]

with scalars $\rho$, $\eta$ is also a solution of (4.10)(a). On the other hand, if (4.12) is a solution of the condition (4.10)(b), then, in virtue of $k_{12} = 0$ and $Y_\lambda = h_{\lambda\nu} Y^\nu : (\eta \psi, -\eta \phi, \rho, -\rho)$, we obtain

\[(4.13) \quad 0 = \partial_3 k_{12} = -2 k_{3[1} Y_{2]} = -\phi(-\eta \phi) + \psi(\eta \psi) = \eta(\phi^2 + \psi^2),\]
which implies that $\eta = 0$. Therefore the solutions of the conditions (4.10) are of the form:

(4.14) $$ Y^\nu = \rho Y_1^\nu = \rho A^\nu, $$

for some nonzero real-valued function $\rho = \rho(x_1, x_2, x_3, x_4)$ to be determined.

**Theorem 4.3.** In $X_4$, the vector $Y^\nu = \rho A^\nu$ given by (4.14) is a solution of the conditions (4.10) if and only if the vector $B_\lambda$ given by (4.2)(b) satisfies the following condition

(4.15) $$ \partial_\omega B_\mu = \rho A_\omega B_\mu. $$

**Proof.** Suppose that (4.15) is satisfied. Differentiating both sides of (4.4), and making use of (4.8)(c) and (4.15), we obtain

(4.16) $$ \partial_\omega k_{\lambda\mu} = A_\lambda (\rho A_\omega B_\mu) - A_\mu (\rho A_\omega B_\lambda) = -k_{\omega\lambda}(\rho A_\mu) + k_{\omega\mu}(\rho A_\lambda). $$

Hence, in virtue of Remark 4.2 and the relation (4.16), the vector $Y^\nu = \rho A^\nu$ is a solution of (4.10). Conversely, suppose that the vector $Y^\nu = \rho A^\nu$ is a solution of (4.10). Since $B_\mu = k_{3\mu}$, we obtain

(4.17) $$ \partial_\omega B_\mu = \partial_\omega k_{3\mu} = -k_{\omega3}Y_\mu + k_{\omega\mu}Y_3 = -(A_\omega B_3 - A_3 B_\omega)(\rho A_\mu) + (A_\omega B_\mu - A_\mu B_\omega)(\rho A_3) = \rho A_\omega B_\mu, $$
in virtue of (4.4), (4.10)(b), and (4.2). Hence the condition (4.15) is satisfied.

**Theorem 4.4.** In $X_4$, the condition (4.15) is satisfied if and only if the functions $\rho$, $\phi$, and $\psi$, given in (4.15), satisfy the following conditions, respectively,

(4.18) $$ \partial\phi/\partial x^1 = 0, \quad \partial\phi/\partial x^2 = 0, \quad \partial\phi/\partial x^3 = \rho \phi, \quad \partial\phi/\partial x^4 = -\rho \phi, $$

(4.19) $$ \partial\psi/\partial x^1 = 0, \quad \partial\psi/\partial x^2 = 0, \quad \partial\psi/\partial x^3 = \rho \psi, \quad \partial\psi/\partial x^4 = -\rho \psi $$

(4.20) $$ \partial\rho/\partial x^1 = 0, \quad \partial\rho/\partial x^2 = 0, \quad \partial\rho/\partial x^3 + \partial\rho/\partial x^4 = 0. $$

**Proof.** In virtue of (4.15), we obtain

(4.21) $$ \partial\phi/\partial x^\omega = \partial_\omega B_1 = \rho A_\omega B_1 = \rho \phi A_\omega, $$

which imply (4.18), in virtue of (4.2)(a). Similarly, we obtain (4.19). Next, differentiating both sides of (4.21), we obtain

(4.22) $$ \frac{\partial^2 \phi}{\partial x^\nu \partial x^\omega} = \frac{\partial \rho}{\partial x^\nu} \phi A_\omega + \rho^2 \phi A_\nu A_\omega, $$
A solution of Einstein’s field equations in $X_4$, and we also obtain

$$\frac{\partial^2 \phi}{\partial x^\omega \partial x^\nu} = \frac{\partial \rho}{\partial x^\omega} \phi A_\nu + \rho^2 \phi A_\omega A_\nu. \tag{4.23}$$

Hence we obtain

$$0 = \frac{\partial^2 \phi}{\partial x^\nu \partial x^\omega} - \frac{\partial^2 \phi}{\partial x^\omega \partial x^\nu} = (\frac{\partial \rho}{\partial x^\nu} A_\omega - \frac{\partial \rho}{\partial x^\omega} A_\nu) \phi, \tag{4.24}$$

which implies that, since $\phi \neq 0$,

$$\frac{\partial \rho}{\partial x^\nu} A_\omega - \frac{\partial \rho}{\partial x^\omega} A_\nu = 0. \tag{4.25}$$

From the above condition (4.25), we obtain that if $\nu = 1$ and $\omega = 3$, then $\partial \rho/\partial x^1 = 0$, if $\nu = 2$ and $\omega = 3$, then $\partial \rho/\partial x^2 = 0$, and if $\nu = 3$ and $\omega = 4$, then $\partial \rho/\partial x^3 + \partial \rho/\partial x^4 = 0$. Hence we obtain (4.20). Obviously, the converse is true.

**Theorem 4.5.** In $X_4$, for the basic tensor $g_{\lambda \mu}$ given by (4.3), the connection (4.11) which is a solution of (2.6) and (2.7)(a) is given by

$$\Gamma^\nu_{\lambda \mu} = 2 \rho A_{[\lambda} B_{\mu]} A^\nu, \tag{4.26}$$

where $\rho$ satisfies the condition (4.20). And the curvature tensor $R^\omega_{\lambda \mu \nu}$ with respect to this connection (4.26) is given by

$$R^\omega_{\lambda \mu \nu} = 2 \{ (\partial_{[\mu} \rho) B_{\nu]} A_\lambda - (\partial_{[\mu} \rho) A_{\nu]} B_\lambda \} A^\omega + 2 \rho^2 A_{[\mu} B_{\nu]} A_\lambda A^\omega, \tag{4.27}$$

and its contracted curvature tensor $R_{\lambda \mu}$ satisfies

$$R_{\lambda \mu} = 0. \tag{4.28}$$

**Proof.** Substituting (4.4) and (4.14) into (4.11), we obtain (4.26), in virtue of Remark 4.2, Theorem 4.3, and Theorem 4.4. Substituting (4.26) into (2.8), we obtain (4.27) by a straightforward computation. In the next, Contracting for (4.27) for $\omega$ and $\nu$, we obtain

$$R_{\lambda \mu} = -2 (\partial_{[\omega} \rho) A^\alpha A_{[\lambda} B_{\mu]} - 2 \rho^2 A_{[\mu} B_{\nu]} A_\lambda A^\omega, \tag{4.29}$$

On the other hand, in virtue of (4.7)(a) and (4.20), we obtain

$$\partial_{[\lambda} \rho) A^\alpha = \partial \rho/\partial x^3 + \partial \rho/\partial x^4 = 0 \tag{4.30}$$

Hence we obtain (4.28).

**Remark 4.6.** The set of the functions $\phi$ satisfying (4.18) is not empty. For example, when $\rho = $ constant, the function

$$\phi(x_1, x_2, x_3, x_4) = e^{\rho(x^3 - x^4)} \tag{4.31}$$

satisfies (4.18). Similarly, we can define the function $\psi$ satisfying (4.19).
Conclusion. In virtue of Theorem 4.5, if $X_4$ is endowed with a non-symmetric tensor $g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}$ such that (4.1) and (4.5), where $\phi$ and $\psi$ satisfy the conditions (4.18) and (4.19), respectively. Then a solution $\Gamma_{\lambda\mu}^\nu$ of (2.6) and (2.7)(a) is given by (4.26), where $\rho$ satisfies the condition (4.20). In the next, since the contracted curvature tensor $R_{\lambda\mu}$ with respect to the connection (4.26) satisfies $R_{\lambda\mu} = 0$, the field equation (2.7)(c) is satisfied automatically, and the field equation (2.7)(b) is equivalent to $\partial_{\lambda} P_{\mu} = 0$. Since the field equation (2.7)(b) is satisfied by a vector $P_{\mu} = \partial_{\mu} P$, the vector $P_{\mu} = \partial_{\mu} P$ is an Einstein’s vector.

References


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