ON PUTINAR’S MATRICIAL MODEL OPERATOR OF RANK 2

JUN IK LEE*

Abstract. In this paper we study the Putinar’s matricial model operator of rank 2 and provide some evidences for the validity of the conjecture in [8].

1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces, let $\mathcal{L}(\mathcal{H}, \mathcal{K})$ be the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and write $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be normal if $T^*T = TT^*$, quasinormal if $T^*T^2 = TT^*T$, hyponormal if $T^*T \geq TT^*$, and subnormal if it has a normal extension, i.e., $T = N|_{\mathcal{H}}$, where $N$ is a normal operator on some Hilbert space $\mathcal{K}$ containing $\mathcal{H}$. In general it is quite difficult to determine the subnormality of an operator by definition. An alternative description of subnormality is given by the Bram-Halmos criterion, which states that an operator $T$ is subnormal if and only if

$$
\sum_{i,j} (T^i x_j, T^j x_i) \geq 0
$$

for all finite collections $x_0, x_1, \ldots, x_k \in \mathcal{H}$ ([3],[4, II.1.9]). It is easy to see that this is equivalent to the following positivity test:

$$
\begin{pmatrix}
I & T^* & \cdots & T^{*k} \\
T & T^*T & \cdots & T^{*k}T \\
\vdots & \vdots & \ddots & \vdots \\
T^k & T^*T^k & \cdots & T^{*k}T^k
\end{pmatrix} \geq 0 \quad \text{(all } k \geq 1 \text{)}.
$$

Received November 27, 2012; Accepted January 11, 2013.
2010 Mathematics Subject Classification: Primary 47A20, 47B20.
Key words and phrases: Putinar’s matricial model operator, subnormal operators, weakly subnormal operators, finite rank selfcommutators.
This research was supported by a 2012 Research Grant from Sangmyung University.
Condition (1.1) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity condition (1.1) for \( k = 1 \) is equivalent to the hyponormality of \( T \), while subnormality requires the validity of (1.1) for all \( k \). Let \([A, B] := AB - BA\) denote the commutator of two operators \( A \) and \( B \), and define \( T \) to be \( k\)-hyponormal whenever the \( k \times k \) operator matrix

\[
M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k
\]

is positive.

We now review a few essential facts concerning weak subnormality that we will need to begin with. Note that the operator \( T \) is subnormal if and only if there exist operators \( A \) and \( B \) such that \( \hat{T} := \left( \begin{array}{cc} T & A \\ \overline{B} & \overline{A}^* \end{array} \right) \) is normal, i.e.,

\[
\begin{cases}
[T^*, T] := T^*T - TT^* = AA^* \\
A^*T = BA^* \\
\end{cases}
\] (1.3)

An operator \( T \in \mathcal{L}(\mathcal{H}) \) is said to be weakly subnormal if there exist operators \( A \in \mathcal{L}(\mathcal{H}', \mathcal{H}) \) and \( B \in \mathcal{L}(\mathcal{H}') \) such that the first two conditions in (1.3) hold:

\[
[T^*, T] = AA^* \quad \text{and} \quad A^*T = BA^*,
\] (1.4)

or equivalently, there is an extension \( \hat{T} \) of \( T \) such that

\[
\hat{T}^*\hat{T}f = \hat{T}\hat{T}^*f \quad \text{for all} \ f \in \mathcal{H}.
\]

The operator \( \hat{T} \) is called a partially normal extension (briefly, p.n.e.) of \( T \). We also say that \( \hat{T} \) in \( \mathcal{L}(\mathcal{K}) \) is a minimal partially normal extension (briefly, m.p.n.e.) of \( T \) if \( \mathcal{K} \) has no proper subspace containing \( \mathcal{H} \) to which the restriction of \( \hat{T} \) is also a partially normal extension of \( T \). It is known ([6, Lemma 2.5 and Corollary 2.7]) that

\[
\hat{T} = \text{m.p.n.e.}(T) \iff \mathcal{K} = \sqrt{\{ \hat{T}^*h : h \in \mathcal{H}, \ n = 0, 1 \}},
\]

and the m.p.n.e.\((T)\) is unique. For convenience, if \( \hat{T} = \text{m.p.n.e.}(T) \) is also weakly subnormal then we write \( \hat{T}^{(2)} := \hat{T} \) and more generally, \( \hat{T}^{(n)} := \hat{T}^{(n-1)} \). It was ([6], [5]) shown that

\[
2\text{-hyponormal} \implies \text{weakly subnormal} \implies \text{hyponormal}
\] (1.5)

and the converses of both implications in (1.5) are not true in general. In particular, the following lemma is very useful in the sequel.
**Lemma 1.1.** ([6], [5]) Let $T \in \mathcal{L}(\mathcal{H})$.

(a) If $T$ is weakly subnormal then the operator $A$ in (1.4) can be taken as a positive operator;
(b) If $T$ is weakly subnormal then $\ker [T^*, T]$ is invariant for $T$;
(c) For any $k \geq 1$, $T$ is $(k + 1)$-hyponormal if and only if $T$ is weakly subnormal and $\hat{T} := \text{m.p.n.e.}(T)$ is $k$-hyponormal.

The self-commutator of an operator plays an important role in the study of subnormality. Subnormal operators with finite rank self-commutators have been extensively studied ([2], [9], [11], [16], [17], [18], [20], [21]). Particular attention has been paid to hyponormal operators with rank 1 or rank 2 self-commutators ([7], [10], [12], [13], [14], [16], [19], [22]). In particular, B. Morrel [10] showed that a pure subnormal operator with rank 1 self-commutator (pure means having no normal summand) is unitarily equivalent to a linear function of the unilateral shift.

It is worth noticing that in view of (1.5) and Lemma 1 (a), Morrel’s theorem gives that every weakly subnormal operator with rank 1 self-commutator is subnormal.

**2. The main results**

M. Putinar [15] gave a matricial model for the hyponormal operator $T \in \mathcal{L}(\mathcal{H})$ with finite rank self-commutator, in the cases where

$$\mathcal{H}_0 := \bigvee_{k=0}^{\infty} T^k(\text{ran}[T^*, T])$$

has finite dimension $d$ and

$$\mathcal{H} = \bigvee_{n=0}^{\infty} T^n \mathcal{H}_0.$$ 

Let $G_n := \bigvee_{k=0}^{n} T^k \mathcal{H}_0$ $(n \geq 0)$ and

$$\mathcal{H}_n := G_n \oplus G_{n-1} \quad (n \geq 1).$$

If $\dim(\mathcal{H}_n) = \dim(\mathcal{H}_{n+1}) = d$ $(n \geq 0)$, then $T$ has the following two-diagonal structure relative to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots$ ([15]):

$$T = \begin{pmatrix}
B_0 & 0 & 0 & 0 & \cdots \\
A_0 & B_1 & 0 & 0 & \cdots \\
0 & A_1 & B_2 & 0 & \cdots \\
0 & 0 & A_2 & B_3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},$$

where

$$[T^*, T] = ([B_0^*, B_0] + A_0^* A_0) \oplus 0_{\infty};$$

$$[B_{n+1}^*, B_{n+1}] + A_{n+1}^* A_{n+1} = A_n A_n^* \quad (n \geq 0);$$

$$A_n B_{n+1} = B_n A_n^* \quad (n \geq 0).$$
We will refer the operator (2.1) to the Putinar’s matricial model operator of rank \(d\). This model was also introduced in [7], [12], [19], [20], and etc.

In [8], using the Agler’s characterization of subnormality [1], the authors showed the following theorems:

**Theorem 2.1.** ([8]) Let \(T \in \mathcal{L}(H)\). If

1. \(T\) is a pure hyponormal operator;
2. \([T^*, T]\) is of rank 2; and
3. \(\ker[T^*, T]\) is invariant for \(T\),

then the following hold:

1. If \(T|_{\ker[T^*, T]}\) has the rank 1 self-commutator then \(T\) is subnormal;
2. If \(T|_{\ker[T^*, T]}\) has the rank 2 self-commutator then \(T\) is either a subnormal operator or the Putinar’s matricial model operator of rank 2.

**Theorem 2.2.** ([8]) The operator \(T\) in (2.1) is subnormal if \(B_n\) is normal for some \(n \geq 0\).

Also, they conjectured that:

**Conjecture 2.3.** ([8]) The Putinar’s matricial model operator of rank 2 is subnormal.

In this paper we examine the validity of the Conjecture 2.3, and we provide some affirmative evidences for the Conjecture 2.3. If \(A_0\) and \(A_1\) in (2.1) commute, we then have:

**Theorem 2.4.** Let \(T\) be the Putinar’s matricial model operator of rank 2. If \(A_0\) and \(A_1\) in (2.1) commute then \(T\) is either subnormal or is of the following form by a translation or a multiplication by an appropriate scalar: \(A_j = \begin{pmatrix} p_j & 0 \\ 0 & q_j \end{pmatrix}\) and \(B_j = \begin{pmatrix} 0 & b_j \\ c_j & 0 \end{pmatrix}\) for \(j = 0, 1, \ldots\), that is,

\[
T = \begin{pmatrix}
0 & b_0 & 0 & 0 & \ldots \\
c_0 & 0 & 0 & 0 & \ldots \\
p_0 & 0 & 0 & b_1 & \ldots \\
0 & g_0 & c_1 & 0 & \ldots \\
0 & 0 & p_1 & 0 & \ldots \\
0 & 0 & 0 & q_1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]
Proof. Let

\[ T_n := \begin{pmatrix}
B_n & 0 & 0 & 0 & \cdots \\
A_n & B_{n+1} & 0 & 0 & \cdots \\
0 & A_{n+1} & B_{n+2} & 0 & \cdots \\
0 & 0 & A_{n+2} & B_{n+3} & \cdots
\end{pmatrix} \quad (n = 0, 1, \cdots). \]

By [8], we can see that \( T_n \) is the minimal partially normal extension of \( T_{n+1} \) for each \( n \geq 0 \). Thus, by Lemma 1.1 (a), we can assume that \( A_n \) is positive for each \( n \geq 0 \).

Since \( A_0 \) and \( A_1 \) are diagonalizable and \( A_0 \) and \( A_1 \) commute, we can see that \( A_0 \) and \( A_1 \) are simultaneously diagonalizable. So we can write

\[ A_n := \begin{pmatrix} p_n & 0 \\ 0 & q_n \end{pmatrix} \quad (n = 0, 1). \]

Also write

\[ B_n := \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \quad (n = 0, 1). \]

By the third equality of (2.2), we have

\[
\begin{cases}
a_0 = a_1 =: a; \\
d_0 = d_1 =: d; \\
p_0b_1 = b_0q_0; \\
c_0p_0 = q_0c_1.
\end{cases}
\]

If \( a = d \) then by a translation we have

\[ B_n = \begin{pmatrix} 0 & b_n \\ c_n & 0 \end{pmatrix} \quad (n = 0, 1). \]

So by the third equality of (2.2), \( B_2 \) is skew diagonal and in turn, by the second equality of (2.2), \( A_2 \) is diagonal. Repeating this argument with a telescoping method shows that \( B_n \) is skew diagonal and \( A_n \) is diagonal for each \( n = 0, 1, \cdots \). Thus \( T \) is of the form (2.3).

Now suppose \( a \neq d \). By a translation and a multiplication by an appropriate scalar, write

\[ B_n := \begin{pmatrix} a & b_n \\ c_n & 0 \end{pmatrix} \quad (a \in \mathbb{R}, a \neq 0, n = 0, 1). \]

By the second equality of (2.2),

\[ [B_1^*, B_1] = \begin{pmatrix} |c_1|^2 - |b_1|^2 & ab_1 - \alpha c_1 \\ ab_1 - \alpha c_1 & |b_1|^2 - |c_1|^2 \end{pmatrix} \]
is diagonal, and hence $b_1 = c_1$. Thus $B_1$ is normal. Therefore, by Theorem 2.2, $T$ is subnormal.

We now give general sufficient conditions for the subnormality of $T$ in (2.3).

**Theorem 2.5.** The operator $T$ in (2.3) is subnormal if one of the following holds:

(i) $p_n \geq q_n$ (or $q_n \geq p_n$) for all $n = m, m + 1, \cdots$;

(ii) $q_n \geq |c_n|$ and $p_n \geq |b_n|$ for some $n \geq 0$;

(iii) $|b_n| = |c_n|$ for some $n \geq 0$;

(iv) $p_n = q_n$ for some $n \geq 0$;

(v) $p_n = p_{n+1}$ (or $q_n = q_{n+1}$) for some $n \geq 0$;

(vi) $|b_n| = |b_{n+1}|$ (or $|c_n| = |c_{n+1}|$) for some $n \geq 0$.

**Proof.** First of all, observe that from the second and third equalities of (2.2),

\[
\begin{align*}
(p_{n+1})^2 &= p_n^2 + |b_{n+1}|^2 - |c_{n+1}|^2; \\
(q_{n+1})^2 &= q_n^2 - |b_{n+1}|^2 + |c_{n+1}|^2; \\
p_n b_{n+1} &= b_n q_n; \\
c_n p_n &= q_n c_{n+1}. \\
\end{align*}
\]

(2.4)

(i) Without loss of generality we may assume $p_n \geq q_n$ for all $n = 0, 1, \cdots$. Thus $\{||b_n||\}$ is decreasing and $\{||c_n||\}$ is increasing. By using the fourth recursive formula of (2.4) repeatedly, we have

\[
c_{n+1} = \left( \prod_{j=0}^{n} \frac{p_j}{q_j} \right) c_0.
\]

Since $\frac{p_j}{q_j} \geq 1$ for each $j \geq 0$, the sequence $\{||c_n||\}$ should converge, so that $\sum_{j=0}^{\infty} \log \left( \frac{p_j}{q_j} \right)$ converges, and hence the sequence $\{\frac{p_j}{q_j}\}$ converges to 1. Similarly, the sequence $\{||b_n||\}$ converges. Say $b := \lim ||b_n||$ and $c := \lim ||c_n||$. We now claim that $b = c$. Assume to the contrary that $c > b$ and let $\epsilon := c^2 - b^2 > 0$. Then there exists $N \in \mathbb{Z}_+$ such that $|c_{n+1}|^2 - |b_{n+1}|^2 \geq \frac{\epsilon}{2}$ for all $n \geq N$. Then by the second equality of (2.4), if $n \geq N$ then

\[
q_{n+m}^2 \geq q_n^2 + \frac{\epsilon}{2} m \to \infty \text{ as } m \to \infty,
\]

which implies that the sequence $\{q_n\}$ is unbounded, a contradiction. If instead $b > c$ then again the sequence $\{p_n\}$ is unbounded, a contradiction. This proves $b = c$. Since $\{||b_n||\}$ is decreasing and $\{||c_n||\}$ is
increasing, we can see that

\[
(2.5) \quad b_n \geq c_n \quad \text{for all } n \geq 0.
\]

Thus by (2.4) and (2.5) we can conclude that \( \{p_n\} \) is increasing and \( \{q_n\} \) is decreasing. So both \( \{p_n\} \) and \( \{q_n\} \) converge. But since \( \frac{b_n}{q_n} \to 1 \), we can say \( p_n \to p \) and \( q_n \to q \) for some \( p > 0 \). So

\[
p_0 \leq p_1 \leq p_2 \leq \cdots \leq p \leq q_2 \leq q_1 \leq q_0.
\]

But since \( p_0 \geq q_0 \) it follows that \( p_n = p = q_n \) for all \( n \geq 0 \). By (2.5), this also implies that \( |b_n| = |c_n| \) for all \( n \geq 0 \). Therefore all the \( B_n \) are normal. By Theorem 2.2, we can conclude that \( T \) is subnormal.

(ii) Without loss of generality, we may assume that \( q_0 \geq |c_0| \) and \( p_0 \geq |b_0| \). If we put

\[
\hat{T} := \begin{pmatrix} B_{-1} & 0 \\ A_{-1}^{-1} & T \end{pmatrix} = \text{m.p.n.e.}(T).
\]

So we need to show that

\[
[\hat{T}^*, \hat{T}] = ([B_{-1}^*, B_{-1}] + A_{-1}^2) \oplus \infty \geq 0.
\]

A straightforward calculation shows that

\[
A_{-1} = \begin{pmatrix} |c_0|^2 - |b_0|^2 + p_0^2 & 0 \\ 0 & |b_0|^2 - |c_0|^2 + q_0^2 \end{pmatrix} \quad \text{and} \quad B_{-1} = \begin{pmatrix} 0 & \frac{p-1}{q-1}|b_0| \\ \frac{q-1}{p-1}c_0 & 0 \end{pmatrix},
\]

where

\[
p_{-1} := (|c_0|^2 - |b_0|^2 + p_0^2)^{\frac{1}{2}} \quad \text{and} \quad q_{-1} := (|b_0|^2 - |c_0|^2 + q_0^2)^{\frac{1}{2}}.
\]

So

\[
[B_{-1}^*, B_{-1}] + A_{-1}^2 = \begin{pmatrix} \left(\frac{\sigma}{p_{-1}}\right)^2 |c_0|^2 - \left(\frac{\sigma}{q_{-1}}\right)^2 |b_0|^2 + p_{-1}^2 & 0 \\ 0 & \left(\frac{\sigma}{q_{-1}}\right)^2 |b_0|^2 - \left(\frac{\sigma}{p_{-1}}\right)^2 |c_0|^2 + q_{-1}^2 \end{pmatrix}.
\]

Observe that if \( q_0 \geq |c_0| \) then

\[
\left(\frac{q_{-1}}{p_{-1}}\right)^2 |c_0|^2 - \left(\frac{p_{-1}}{q_{-1}}\right)^2 |b_0|^2 + p_{-1}^2
\]

\[
= \frac{1}{p_{-1}^2 q_{-1}^2} \left( q_{-1}^4 |c_0|^2 - p_{-1}^4 |b_0|^2 + q_{-1}^2 p_{-1}^4 \right)
\]

\[
= \frac{1}{p_{-1}^2 q_{-1}^2} \left( q_{-1}^4 |c_0|^2 + p_{-1}^4 (q_0^2 - |c_0|^2) \right)
\]

\[
\geq 0.
\]
and similarly, if \( p_0 \geq |b_0| \) then

\[
\left( \frac{p-1}{q-1} \right)^2 |b_0|^2 - \left( \frac{q-1}{p-1} \right)^2 |c_0|^2 + q_{-1}^2 \geq 0,
\]

and therefore \( [B_{-1}^*, B_{-1}] + A_{-1}^2 \geq 0 \). So \( T \) is 2-hyponormal. But since

if we put

\[
b_{-1} := \frac{p-1}{q-1} b_0 \quad \text{and} \quad c_{-1} := \frac{q-1}{p-1} c_0
\]

then

\[
p_{-1}^2 - |b_{-1}|^2 = \frac{p_{-1}^2}{q_{-1}^2} (q_{-1}^2 - |b_0|^2) = \frac{p_{-1}^2}{q_{-1}^2} (q_0^2 - |c_0|^2) \geq 0
\]

and

\[
q_{-1}^2 - |c_{-1}|^2 = \frac{q_{-1}^2}{p_{-1}^2} (p_{-1}^2 - |c_0|^2) = \frac{q_{-1}^2}{p_{-1}^2} (p_0^2 - |b_0|^2) \geq 0,
\]

we can repeat the above argument. Thus \( T \) is \( k \)-hyponormal for every \( k \in \mathbb{Z}_+ \) and hence \( T \) is subnormal.

(iii) Since \( |b_n| = |c_n| \) for some \( n \geq 0 \), \( B_n \) is normal. Thus, it follows from Theorem 2.2.

(iv) Without loss of generality, we may assume \( p_0 = q_0 \). So we can write \( A_0 = \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \) and hence \( B_0 = B_1 \) by the third equality of (2.2). Now if we define

\[
A_{-1} := ([B_0^*, B_0] + A_0^2)^{\frac{1}{2}} \quad \text{and} \quad B_{-1} := A_{-1}B_0A_{-1}^{-1},
\]

then \( \hat{T} := \left( \begin{smallmatrix} B_{-1}^* & 0 \\ A_{-1}^{-1} & T \end{smallmatrix} \right) \) is m.p.n.e.(\( T \)). So we need to show that

\[
[\hat{T}^*, \hat{T}] = ([B_{-1}^*, B_{-1}] + A_{-1}^2) \oplus 0 \geq 0.
\]

A straightforward calculation shows that

\[
A_{-1} = P^{-1}A_1P \quad \text{and} \quad B_{-1} = B_2
\]

where \( P := \left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right) \). So

\[
[B_{-1}^*, B_{-1}] + A_{-1}^2 = P^{-1}A_2^2P \geq 0.
\]

Thus we see that \( T \) is 2-hyponormal. Similarly, we can repeat this backward extension. Therefore we can conclude that \( T \) is subnormal.

(v) If \( p_n = p_{n+1} \) (or \( q_n = q_{n+1} \)), then by the first (or second) equality of (2.4) we have \( |b_{n+1}| = |c_{n+1}| \). Therefore the result follows from (iv).

(vi) If \( |b_n| = |b_{n+1}| \neq 0 \) (or \( |c_n| = |c_{n+1}| \neq 0 \)), then by the third (or fourth) equality of (2.4) we have \( p_n = q_n \). Therefore the result follows from (iii). If instead \( |b_n| = |b_{n+1}| = 0 \) (or \( |c_n| = |c_{n+1}| = 0 \), then
by the second (or first) equality of (2.4), we have $q_{n+1} \geq |c_{n+1}|$ (or $p_{n+1} \geq |b_{n+1}|$). Therefore the result follows from (ii).

We thus have:

**Theorem 2.6.** Let $T$ be the Putinar’s matricial model operator of rank 2. If the matrix $B_j$ in (2.1) is of the form $B_j = \begin{pmatrix} a & 0 \\ 0 & b_j \end{pmatrix}$ for some $a \in \mathbb{C}$ and for some $j \geq 0$ then $T$ is subnormal.

**Proof.** Without loss of generality we can assume $j = 0$. By Lemma 1.1 (a), we can also write $A_0 = \begin{pmatrix} p_0 & 0 \\ 0 & q_0 \end{pmatrix}$. From the third equality of (2.2) we can see that $B_1$ is of the form $B_1 = \begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix}$ for some $b_1 \in \mathbb{C}$. Let $A_1 = \begin{pmatrix} p_1 & r_1 \\ q_1 & 0 \end{pmatrix}$ be positive. Then by the second equality of (2.2), we have $r_1 = 0$. It thus follows that $A_1$ is a diagonal matrix. Therefore $T$ is subnormal by Theorem 2.4 and Theorem 2.5 (vi).

**References**


*Department of Mathematics Education
Sangmyung University
Seoul 110-743, Republic of Korea
E-mail: jilee@smu.ac.kr