BIPRODUCT BIALGEBRAS WITH A PROJECTION ONTO A HOPF ALGEBRA

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Abstract. Let \((D, B)\) be an admissible pair. Then recall that \(B \times_D D \iff \pi D \circ i_D = I\). We have solved a converse in case \(D\) is a Hopf algebra. Let \(D\) be a Hopf algebra with antipode \(s_D\) and be a left \(H\)-comodule algebra and a left \(H\)-module coalgebra over a field \(k\). Let \(A\) be a bialgebra over \(k\). Suppose \(A \iff j\) is an algebra map. We show that \((D, B)\) is an admissible pair and \(B \iff \pi j\) is an admissible mapping system and that the generalized biproduct bialgebra \(B \times_D D\) is isomorphic to \(A\) as bialgebras.

Given algebras \(A\) and \(B\), we put an algebra structure on the tensor product \(A \otimes B\) by

\[(a \otimes b)(a' \otimes b') = aa' \otimes bb' \quad (0)\]

where \(a, a' \in A\) and \(b, b' \in B\). We call \(A \otimes B\) the tensor product of the algebras \(A\) and \(B\). Its unit is \(1 \otimes 1\). Defining \(i_A(a) = a \otimes 1\) and \(i_B(b) = 1 \otimes b\), we get algebra morphisms \(i_A : A \rightarrow A \otimes B\) and \(i_B : B \rightarrow A \otimes B\). The following relation holds in view of \((0)\):

\[i_A(a)i_B(b) = i_B(b)i_A(a) = a \otimes b\]

for all \(a \in A\) and \(b \in B\).

Molnar constructed a smash coproduct \(C^#H\) of an \(H\)-comodule coalgebra \(C\) and a Hopf algebra \(H\) in [4] and usual smash product \(A\#H\) of an \(H\)-module algebra \(A\) and a Hopf algebra \(H\) has been defined in [8] or [9].
Definition 1 [1]. Let $H$ be a bialgebra over a field $k$ and $A$ be a left $H$-module algebra. Let $D$ be a left $H$-comodule algebra. The generalized smash product $A \#^L_H D$ is defined to be $A \otimes_k D$ as a vector space, with multiplication given by
\[(a \#^L_H d)(b \#^L_H e) = \Sigma a(d_{-1} \cdot b) \#^L_H d_0 e\]
and unit $1_A \otimes 1_D$ for all $a, b \in A$ and $d, e \in D$.

It is straightforward to show that $i_A : A \to A \#^L_H D$, $a \mapsto a \#^L_H 1_D$ and $i_D : D \to A \#^L_H D$, $d \mapsto 1_A \#^L_H d$ are algebra maps since $A$ is a left $H$-module algebra and $D$ is a left $H$-comodule algebra.

Definition 2 [2]. Let $H$ be a bialgebra over a field $k$ and $C$ be a left $H$-comodule coalgebra. Let $E$ be a left $H$-module coalgebra. The generalized smash coproduct $C \#^L_H E$ is defined to be $C \otimes_k E$ as a vector space with comultiplication given by
\[\Delta(c \#^L_H e) = \Sigma (c_1 \#^L_H c_2_{-1} \cdot e_1) \otimes (c_2_{0} \#^L_H e_2)\]
and counit
\[\varepsilon(c \#^L_H e) = \varepsilon_C(c) \varepsilon_E(e)\]
for all $c \in C$, $e \in E$.

It is straightforward to show that $\pi_C : C \# E \to C$, $c \# e \mapsto c \varepsilon_E(e)$ and $\pi_E : C \#^L_H E \to E$, $c \#^L_H e \mapsto \varepsilon_C(c) e$ are coalgebra surjections since $C$ is a left $H$-comodule coalgebra and $E$ is a left $H$-module coalgebra.

Definition 3 [5]. Let $H$ be a bialgebra over a field $k$. Let $B$ be a left $H$-module algebra and a left $H$-comodule coalgebra. Let $D$ be a left $H$-comodule algebra and a left $H$-module coalgebra. The generalized biproduct $B \times^L_H D$ of $B$ and $D$ is defined to be $B \#^L_H D$ as an algebra and $B \#^L_H D$ as a coalgebra.
Example 4. A bialgebra $H$ is a left $H$-comodule algebra via $\Delta_H$ because $\Delta_H$ is an algebra map. $H$ is a left $H$-module coalgebra via $m_H$ because $m_H$ is a coalgebra map. The generalized biproduct $B \times H^L_H$ is a biproduct $B \star H$ in [3]. We consider the case when $H = kG$, for $G$ an abelian group. Then $B \times H^L_H = B \star H$ is a bialgebra. As an algebra $B \times H^L_H = B \# H = B \star G$, the skew group ring.

Definition 5. Let $H$ be a bialgebra. Suppose that $B$ is a left $H$-module algebra and a left $H$-comodule coalgebra and $D$ is a left $H$-comodule algebra and a left $H$-module coalgebra. In case $(B \times H^L_H, m_{B \# H^L_H}, \eta_{B \# H^L_H}, \Delta_{B \# H^L_H}, \epsilon_{B \# H^L_H})$ is a bialgebra, we say the pair $(D, B)$ is admissible.

Throughout we let $H$ be a bialgebra over $k$. Suppose $B$ is a left $H$-module algebra and a left $H$-comodule coalgebra and $D$ is a left $H$-comodule algebra and a left $H$-module coalgebra.

Definition 6. Let $(D, B)$ be an admissible pair and suppose that $A$ be a bialgebra over $k$. Then

$$B \overset{\Pi}{\leftarrow} A \overset{\pi}{\rightarrow} D$$

is an admissible mapping system if the following conditions hold:

(a) $\Pi \circ j = I_B, \quad \pi \circ i = I_D$,

(b) $i$ and $\pi$ are algebra maps and coalgebra maps, $j$ is an algebra map, and $\Pi$ is a coalgebra map,

(c) $\Pi$ is a $D$-bimodule map ($A$ is given the $D$-bimodule structure via pullback along $i$ and $B$ is given the trivial right $D$-module structure),

(d) $j(B)$ is a sub-$D$-bicomodule of $A$ and $\Pi|_{j(B)}$ is a $D$-bicomodule map ($A$ is given the $D$-bicomodule structure via pushout along $\pi$, $B$ is given the trivial right $D$-comodule structure), and

(e) $(j \circ \Pi) \star (i \circ \pi) = I_A$.

Proposition 7 [5]. Let $(D, B)$ be an admissible pair. Then

$$B \overset{\Pi}{\leftarrow} A \overset{\pi}{\rightarrow} D$$

is an admissible mapping system where $i_D : D \rightarrow B \times H^L_H D, d \mapsto 1_B \times H^L_H d, \quad j_B : B \rightarrow B \times H^L_H D, b \mapsto b \times H^L_H 1_D, \quad \Pi_B : B \times H^L_H D \rightarrow B, b \times H^L_H d \mapsto \epsilon_D(d)b$
and $\pi_D : B \times^L_H D \longrightarrow D, b \times^L_H d \mapsto \varepsilon_B(b)d$.

Next result gives two mapping description of $B \leftrightarrow^\Pi B \leftrightarrow^j B \times^L_H D \leftrightarrow^\pi D$.

**Proposition 8** [5]. Let $(D, B)$ be an admissible pair and let $A$ be a bialgebra over $k$. Suppose that $B \leftrightarrow^\Pi B \leftrightarrow^j A \leftrightarrow^\pi D$ is an admissible mapping system.

(1) There exists a unique algebra map $f : B \times^L_H D \longrightarrow A$ such that the diagram

```
  B \times^L_H D  
  |          |    
  |   j_B    |  i_D  
  |          |    
  B          D  
  |          |    
  |   f     |  i   
  |          |    
  A          D
```

commutes. Furthermore the diagram

```
  B \times^L_H D  
  |          |    
  |   II_B    |  \pi_D  
  |          |    
  B          D  
  |          |    
  |   f     |  \pi   
  |          |    
  A          D
```

commutes and $f$ is a bialgebra isomorphism.

(2) There exists a unique coalgebra map $g : A \longrightarrow B \times^L_H D$ such that the diagram
commutes. Furthermore the diagram

\[
\begin{array}{ccc}
B \times^L_H D & \xrightarrow{\Pi_B} & B \\
\downarrow \scriptstyle{\pi_D} & & \downarrow \scriptstyle{\pi} \\
B & \xrightarrow{g} & \downarrow \scriptstyle{\Pi} \\
& & \downarrow \scriptstyle{\pi} \\
& & \downarrow \scriptstyle{\Pi} \\
A & \xleftarrow{j_B} & \downarrow \scriptstyle{j} \\
\end{array}
\]

commutes and \( g \) is a bialgebra isomorphism.

Let \((D, B)\) be an admissible pair. Then recall that \( B \times^L_H D \xleftarrow{\pi_D} D \) are bialgebra maps satisfying \( \pi_D \circ i_D = I_D \) by Proposition 7. We will solve a converse in case \( D \) is a Hopf algebra.

**Theorem 9.** Let \( D \) be a Hopf algebra with antipode \( s_D \) and be a left \( H \)-comodule algebra and a left \( H \)-module coalgebra over a field \( k \). Let \( A \) be a bialgebra over \( k \). Suppose \( A \xleftarrow{} D \) are bialgebra maps satisfying \( \pi \circ i = I_D \). Set \( \Pi = I_A \ast (i \circ s_D \circ \pi) \) and let \( B = \Pi(A) \). Let \( j : B \to A \) be the inclusion. Then

(i) \( B \) is a subalgebra of \( A \) and \( B \) has a coalgebra structure such that \( \Pi \) is a coalgebra map.

(ii) \( B \) is a left \( D \)-comodule coalgebra, a left \( D \)-module algebra, a left \( D \)-comodule algebra and \( B \) is a left \( D \)-module coalgebra.

Suppose that \( \Pi \) is an algebra map. Then

(iii) \( B \) is a left \( H \)-module algebra and \( B \) is a left \( H \)-comodule coalgebra,
(iv) \((D, B)\) is an admissible pair and \(B \ni_\pi^\Pi_j A \ni_\pi^\pi_i D\) is an admissible mapping system,

(v) The map \(f : B \times^L_H D \to A, \ b \times d \mapsto bi(d)\) is an isomorphism of bialgebras.

**Proof.** In the convolution algebra \(End_k(A)\), for all \(a \in A,\)
\[(i \circ s_D \circ \pi) \ast (i \circ \pi)(a) = \Sigma(i \circ s_D \circ \pi)(a_1)(i \circ \pi)(a_2)\]
\[= \Sigma i(s_D(\pi(a_1)))i(\pi(a_2)) = \Sigma i(s_D(\pi(a_1))\pi(a_2))\]
\[= \Sigma i(s_D(\pi(a_1))\pi(a_2)) = i(\varepsilon_D(\pi(a))1_D) = \varepsilon_D(\pi(a))1_A\]
\[= \varepsilon_A(a)1_A = u_A\varepsilon_A(a).\]
Therefore we have \(i \circ s_D \circ \pi = (i \circ \pi)^{-1}.\)

Hence
\[(j \circ \Pi) \ast (i \circ \pi) = [j \circ (I_A \ast (i \circ s_D \circ \pi))] \ast (i \circ \pi)\]
\[= [j \circ (I_A \ast (i \circ \pi)^{-1})] \ast (i \circ \pi) = I_A.\]
Then we have \((j \circ \Pi) \ast (i \circ \pi) = I_A.\)  

For all \(a, a' \in A,\)
\[\Pi(aa') = (I_A \ast (i \circ s_D \circ \pi))(aa')\]
\[= (I_A \ast (i \circ s_D \circ \pi))(\Sigma a_1a_1' \otimes a_2a_2')\]
\[= \Sigma I_A(a_1a_1')(i \circ s_D \circ \pi)(a_2a_2')\]
\[= \Sigma a_1a_1'(i \circ s_D)(\pi(a_2)\pi(a_2'))\]
\[= \Sigma a_1a_1'(i \circ s_D \circ \pi)(a_2')(i \circ s_D \circ \pi)(a_2)\]
\[= a_1\Pi(a')(i \circ s_D \circ \pi)(a_2).\]
Hence we have \(\Pi(aa') = a_1\Pi(a')(i \circ s_D \circ \pi)(a_2).\)

and
\[\Delta(\Pi(a)) = \Delta(I_A \ast (i \circ s_D \circ \pi))(a)\]
\[= \Delta(\Sigma a_1(i \circ s_D \circ \pi)(a_2))\]
\[= \Sigma a_{11}[i \circ s_D \circ \pi](a_2) \otimes a_{12}[i \circ s_D \circ \pi](a_2)]_2\]
\[= \Sigma a_{11}[i((s_D \circ \pi)(a_2)) \otimes a_{12}[i((s_D \circ \pi)(a_2)))]_2\]
\[= \Sigma a_{11}[i((s_D \circ \pi)(a_2)) \otimes a_{12}[i((s_D \circ \pi)(a_2)))]_2\]
\[= \Sigma a_{11}[i(s_D(\pi(a_2))) \otimes a_{12}[i(s_D(\pi(a_2)))]]_2\]
\[= \Sigma a_{11}[i(s_D(\pi(a_2))) \otimes a_{12}[i(s_D(\pi(a_2)])]_1\]
\[= \Sigma a_{11}[i(s_D(\pi(a_2))) \otimes a_{12}[i(s_D(\pi(a_2)))]_1\]
\[= \Sigma a_{11}i(s_D(\pi(a_2))) \otimes a_{12}[i(s_D(\pi(a_2)))]\]
\[= \Sigma a_{11}i(s_D(\pi(a_2))) \otimes a_{12}[i(s_D(\pi(a_2)))]\]
\[ \Sigma a_1(i \circ s_D \circ \pi)(a_4) \otimes a_2(i \circ s_D \circ \pi)(a_3) \]
\[ = \Sigma a_1(i \circ s_D \circ \pi)(a_3) \otimes \Pi(a_2) \]

Therefore we have \( \Delta(\Pi(a)) = \Sigma a_1(i \circ s_D \circ \pi)(a_3) \otimes \Pi(a_2) \). \hspace{1cm} \text{............... (3)}

For all \( d \in D \),
\[ \Pi(i(d)) = (I_A \ast (i \circ s_D \circ \pi))(i(d)) = \Sigma i(d_1)(i \circ s_D \circ \pi)(i(d_2)) \]
\[ = \Sigma i(d_1)(i \circ s_D \circ \pi)(i(d_2)) = \Sigma i((\pi \circ i)(d_2)) = \Sigma i(d_1 s_D(d_2)) \]
\[ = \Sigma i(d_1 s_D(d_2)) = \varepsilon(d) 1_D = \varepsilon(d) 1_A. \]

Therefore we have \( \Pi(i(d)) = \varepsilon(d) 1_A. \) \hspace{1cm} \text{......................... (4)}

For all \( a \in A \),
\[ (\pi \circ \Pi)(a) = \pi \circ (I_A \ast (i \circ s_D \circ \pi))(a) = \pi(\Sigma a_1(i \circ s_D \circ \pi)(a_2)) \]
\[ = \Sigma \pi(a_1)((\pi \circ i)((s_D \circ \pi)(a_2))) = \Sigma \pi(a_1)(s_D \circ \pi)(a_2) \]
\[ = \Sigma \pi(a_1 s_D(\pi(a_2))) = \Sigma \pi(a_1 s_D(\pi(a_2))) = \varepsilon(d) \Pi(a). \]

Therefore we have \( (\pi \circ \Pi)(a) = \varepsilon(d) \Pi(a) \). \hspace{1cm} \text{......................... (5)}

By (2) and (4),
\[ \Pi(ai(d)) = \Sigma a_1 \Pi(i(d))(i \circ s_D \circ \pi)(a_2) = \Sigma a_1 \varepsilon(d)(i \circ s_D \circ \pi)(a_2) \]
\[ = (\Sigma a_1(i \circ s_D \circ \pi)(a_2)) \varepsilon(d) = \varepsilon(d) \Pi(a). \]

Therefore we have \( \Pi(ai(d)) = \varepsilon(d) \Pi(a). \) \hspace{1cm} \text{......................... (6)}

For all \( b \in B = \Pi(A) \),
\[ \Delta(b) = \Delta(\Pi(a)) = \Sigma a_1(i \circ s_D \circ \pi)(a_3) \otimes \Pi(a_2). \]

Therefore
\[ \Sigma b_1 \otimes \pi(b_2) = \Sigma a_1(i \circ s_D \circ \pi)(a_3) \otimes \pi(\Pi(a_2)) \]
\[ = \Sigma a_1(i \circ s_D \circ \pi)(a_3) \otimes \varepsilon_A(a_2) 1_D = \Sigma(i \circ s_D \circ \pi)(\varepsilon_A(a_2)a_3) \otimes 1_D \]
\[ = \Sigma a_1(i \circ s_D \circ \pi)(a_2) \otimes 1_D = \Pi(a) \otimes 1_D = b \otimes 1. \]

Therefore we have \( \Sigma b_1 \otimes \pi(b_2) = b \otimes 1. \) \hspace{1cm} \text{......................... (7)}

Let \( A \otimes D \longrightarrow A, \ a \otimes d \mapsto a \cdot d = ai(d) \) be a right \( D \)-module structure map of \( A \) and \( B \otimes D \longrightarrow B, \ \Pi(e) \otimes d \mapsto \Pi(a) \cdot d = \varepsilon(d) \Pi(a) \) be a right \( D \)-module structure map of \( A \). Let \( \rho_A : A \longrightarrow A \otimes D, \ a \mapsto \Sigma a_1 \otimes \pi(a_2) \) be a right \( D \)-comodule structure map of \( A \) and \( \rho_B : B \longrightarrow B \otimes D, \ b \mapsto b \otimes 1_D \) be a right \( D \)-comodule structure map of \( B \). By (6) and (7)
\[ \Pi(a \cdot h) = \Pi(ai(h)) = \Pi(a) \varepsilon(h) = \Pi(a) \cdot h, \]
\[ \Sigma(\Pi(b))_0 \otimes (\Pi(b))_1 = \Pi(b) \otimes 1 = \Sigma \Pi(b_0) \otimes b_1. \]

Therefore we have \( \Pi \) is a right \( D \)-module map and \( B \) is a right \( D \)-subcomodule and \( \Pi|_B \) is a right \( D \)-comodule map. \hfill(8)

By (1) and (7), we have
\[
\begin{align*}
\Sigma(j(\Pi(b_1))) & \pi(\pi(b_2)) = \Sigma(\Pi(b_1)) \pi(\pi(b_2)) = \Pi(b).
\end{align*}
\]
Therefore we have \( \Pi(b) = b \). \( b \in B \). \hfill(9)

Hence
\[
\Pi \circ j = I_B \quad \hfill(10)
\]

By (3) and (7),
\[
\rho_A(bb') = \Sigma(bb')_1 \otimes \pi((bb')_2) = \Sigma b_1 b'_1 \otimes \pi(b_2 b'_2) = \Sigma b_1 b'_1 \otimes \pi(b_2 \pi(b'_2)) = (\Sigma b_1 \otimes \pi(b_2))(\Sigma b'_1 \otimes \pi(b'_2)) = (b \otimes 1)(b' \otimes 1) = bb' \otimes 1.
\]
Therefore we have \( B \) is a subalgebra with \( \Delta(B) \subseteq A \otimes B \). \hfill(11)

and
\[
\text{the inclusion map } j : B \rightarrow A \text{ is an algebra map}. \hfill(12)
\]

Let \( d \cdot a = ad_i(d \otimes a) = \Sigma i(d_1) a_i(s_D(d_2)) \) be the adjoint action of \( D \) on \( A \). By (2),
\[
\Pi(i(d)a') = \Sigma i(d_1) \Pi(a')(i \circ s_D \circ \pi)(i(d_2)) = \Sigma i(d_1) \Pi(a')(i \circ s_D)(d_2) = \Sigma i(d_1) \Pi(a')(s_D(d_2)) = d \cdot \Pi(a').
\]
So \( d \cdot \Pi(a') = \Pi(i(d)a') \in B \).

Therefore we have \( B \) is a left \( D \)-module under \( ad_i \) and \( \Pi \) is a left \( D \)-module map. \hfill(13)

Define the comultiplication on \( B \) as \( \Delta_B : B \rightarrow B \otimes B \) by
\[
\Delta_B(\Pi(a)) = \Sigma \Pi(a_1) \otimes \Pi(a_2).
\]

Let \( D^+ = D \cap ker(\varepsilon_D) \). By (6), \( A^i (D^+) \subseteq ker(\Pi) \) since \( \Pi(ai(d)) = \varepsilon_D(d)\Pi(a) = 0 \). If \( \Pi(a) = 0 \), then
\[
a = I(a) = \Sigma \Pi(a_1) (i \circ \pi)(a_2) = \Sigma \Pi(a_1)(i \circ \pi)(a_2) - \Pi(a) = \Sigma \Pi(a_1)(i \circ \pi)(a_2) - \Pi(\Sigma a_1 \varepsilon_A(a_2)) = \Sigma \Pi(a_1)[i(\pi(a_2)) - \varepsilon_A(a_2)1_A] = \Sigma \Pi(a_1)[i(\pi(a_2)) - \varepsilon_A(a_2)1_A].
\]
Therefore $\varepsilon(a_2)1_A)\in A$ since $\varepsilon_D(\pi(a_2) - \varepsilon(a_2)1_A) = \varepsilon_A(a_2)1_A - \varepsilon_A(a_2)1_A = 0$. Therefore $\ker(\Pi) = A\ i(D^+)$. So $\ker(\Pi)$ is a coideal of $A$ and $\Delta_B$ is well-defined. Since $\varepsilon_B \circ \Pi = \varepsilon_A$,

$$B$$ is a coalgebra and $\Pi : A \longrightarrow B$ is a coalgebra map. \hspace{1cm} (14)

Since $\Pi$ is a coalgebra map,

$$\pi(b) = \pi(\Pi(a)) = \varepsilon_A(a)1_D = \varepsilon_B(\Pi(a))1_D = \varepsilon_B(b)1_D$$

by (5). So

$$\Sigma\pi(b_1) \otimes b_2 = \Sigma\pi(b_1)s_D(1) \otimes \varepsilon(b_2)b_2 = \Sigma\pi(b_1)s_D(\varepsilon(b_3))1_D \otimes b_2$$

$$= \Sigma\pi(b_1)s_D(\pi(b_3)) \otimes b_2 = \Sigma\pi(b_1)((s_D \circ \pi)(b_3) \otimes b_2.$$ Therefore we have $\Sigma\pi(b_1) \otimes b_2 = \Sigma\pi(b_1)((s_D \circ \pi)(b_3) \otimes b_2.$ \hspace{1cm} (15)

If we define the left $D$-comodule structure map of $B$ as $\rho'_B(b) = \Sigma\pi(b_1) \otimes b_2$ then $\rho'_B$ is well-defined since $\Delta_B \subseteq A \otimes B$. Then

$$B$$ is a left $D$-comodule under $\rho'_B$. \hspace{1cm} (16)

By (8),

$$(\rho_A \circ \Pi|_B)(b) = \rho'_A(\Pi(b)) = \Sigma\pi(b_1) \otimes b_2 = \Sigma\pi(b_1) \otimes \Pi(b_2) = (I_D \otimes \Pi|_B)\rho_B(b).$$

Therefore we have $\Pi|_B$ is a left $D$-comodule map. \hspace{1cm} (17)

Since $A$ is a left $D$-module algebra under $ad_i$ and $B$ is a submodule of $A$,

$$B$$ is a left $D$-module algebra. \hspace{1cm} (18)

For all $d \in D$ and $a \in A$,

$$\Delta_D(d \cdot \Pi(a)) = \Delta_D(\Pi(i(da))) = \Sigma\Pi(i(d_1)a_1) \otimes \Pi(i(d_2)a_2) = \Sigma d_1 \cdot \Pi(a_1) \otimes d_2 \cdot \Pi(a_2)$$

and

$$\varepsilon(d \cdot a) = \varepsilon(\Sigma i(d_1)ai(s_D(d_2))) = \Sigma\varepsilon(i(d_1))\varepsilon(a)\varepsilon(i(s_D(d_2)))$$

$$= \Sigma\varepsilon(a)\varepsilon(i(d_1)i(d_2))) = \Sigma\varepsilon(a)\varepsilon(i(d_1)s_D(d_2)) = \varepsilon(a)\varepsilon(i(\varepsilon(d)1)) = \varepsilon(a)\varepsilon(d)1.$$ Therefore

$$B$$ is a left $D$-module coalgebra. \hspace{1cm} (19)

Since $\rho_B'(b) = \Sigma\pi(b_1) \otimes b_2$ it follows that

$$B$$ is a left $D$-comodule algebra. \hspace{1cm} (20)

$$(I \otimes \Pi) \circ \rho'_A(a) = (I \otimes \Pi)(\Sigma\pi(a_1) \otimes a_2) = \Sigma\pi(a_1) \otimes \Pi(a_2) = \Sigma\pi(a_1)s_D \circ \Pi(a_2)$$
\[ \pi(a_3) \otimes \Pi(a_2) = \Sigma \pi(a_1)(\pi \circ i \circ s_D \circ \pi)(a_3) \otimes \Pi(a_2) = \Sigma (a_1 \circ s_D \circ \pi)(a_3) \otimes \Pi(a_2) = (\pi \otimes I)(\Sigma a_1(i \circ s_D \circ \pi)(a_3)) \otimes \Pi(a_2) = (\pi \otimes I)\Delta(\Pi(a)) = \rho_B^p(\Pi(a)). \]

Thus \( \Pi : A \rightarrow B \) is a surjective coalgebra map such that \((I \otimes \Pi) \circ \rho_A^p = \rho_B^p \circ \Pi \) where \( \rho_B^p = \rho_A^p |_B \). Since \( A \) is a left \( D \)-comodule under \( \rho_A^p \), \( B \) is a left \( D \)-comodule coalgebra under \( \rho_B^p \).

\begin{enumerate}
  \item From (11) and (14).
  \item From (18), (19), (20) and (21).
  \item We have defined the multiplication and the unit of \( B \) as \( m_B : B \otimes B \rightarrow B, \quad \Pi(a) \otimes \Pi(a') \mapsto \Pi(a)\Pi(a') = \Pi(aa') \) and \( u_B : k \rightarrow B, \quad 1_k \mapsto u_B(1_k) = 1_B = 1_A. \)

We have defined the comultiplication and the counit of \( B \) as \( \Delta_B : B \rightarrow B \otimes B, \quad \Pi(a) \mapsto \Delta_B(\Pi(a)) = \Sigma \Pi(a_1) \otimes \Pi(a_2) \) and \( \varepsilon_B : B \rightarrow k, \quad \Pi(a) \mapsto \varepsilon_B(\Pi(a)) = \varepsilon_A(a). \)

Then \((B, m_B, u_B)\) is a algebra and \((B, \Delta_B, \varepsilon_B)\) is a coalgebra. We define \( H \otimes B \rightarrow B, \quad h \otimes \Pi(a) \mapsto h \cdot \Pi(a) = \varepsilon_H(h)\varepsilon_A(a)1_B = \varepsilon_H(h)\varepsilon_B(b)1_B \) and \( \rho_B'' : B \rightarrow H \otimes B, \quad \Pi(a) \mapsto \varepsilon_A(a)(1_H \otimes (\Pi \circ i)(1_D)) = \varepsilon_A(a)(1_H \otimes 1_B). \)

Then \( B = \Pi(A) \) is a left \( H \)-module and \( B \) is a left \( H \)-comodule. For all \( \Pi(a) \in B \), we compute \( m_B(h \cdot (\Pi(a) \otimes \Pi(a'))) = h \cdot m_B(\Pi(a) \otimes \Pi(a')) \) and \( u_B(h \cdot 1_k) = h \cdot u(1_k) \), since \( \Pi \) is an algebra map.

Therefore we have \( B \) is a left \( H \)-module algebra.

For all \( \Pi(a) \in B \), we compute \( (\rho_{B \otimes B} \circ \Delta_B)(\Pi(a)) = (I \otimes \Delta_B)\rho_B''(\Pi(a)) \) and \( ((I \otimes \varepsilon) \circ \rho_B'')(\Pi(a)) = (\rho_k \circ \varepsilon_B)(\Pi(a)). \)

Therefore we have \( B \) is a left \( H \)-comodule coalgebra.
(iv) : We will show that \((D, B)\) is a admissible pair. Let
\[(b \times d)(b' \times d') = \Sigma b(d_{-1} \cdot b') \times d_0 d' = \Sigma b \varepsilon_H(d_{-1}) \varepsilon_A(a') \times d_0 d' = \Sigma \varepsilon_A(a') b \times dd',\]
where \(b' = \Pi(a')\).

Then \(B#^H D\) is an associative algebra with identity \(1_B#1_D\) by [6, Proposition 1]. Let
\[
\Delta (b \times d) = \Sigma (b_1 \times b_{2,-1} \cdot d_1) \otimes (b_{2,0} \times d_2), \quad \varepsilon (b \times d) = \varepsilon_B(b) \varepsilon_D(d).
\]

Then \(B#^H D\) is a coassociative coalgebra by [6, Proposition 2].

Define \(\rho'_A : \longrightarrow H \otimes A, \quad a \mapsto \varepsilon_A(a)(1_H \otimes 1_A)\). Then \(A\) is a left \(H\)-comodule and \(\Pi\) is a left \(H\)-comodule map. Since \(\Pi\) is a coalgebra map and \(\Pi\) is a left \(H\)-comodule map, since \(\Pi\) is a coalgebra map and \(\Pi\) is a left \(H\)-comodule map,
\[
\Delta (b \times d) \Delta (b' \times d')
\]
where \(b = \Pi(a)\) and \(b' = \Pi(a')\). We have
\[
\Delta (1_B \times 1_D) = \Sigma \Pi(1_A) \times \Pi(1_A)_1 \cdot 1_D \otimes [1_B]_0 \times [1_D]_2
\]
by (22).
Therefore we have $\Delta(b \times d)\Delta(b' \times d') = \Delta((b \times d)(b' \times d'))$ and $\Delta(1_B \times 1_D) = (1_B \times 1_d) \otimes (1_B \times 1_D)$. So $\Delta$ is an algebra map. Since $A$ and $D$ are bialgebras, we compute

$$
\varepsilon((b \times d)(b' \times d')) = \varepsilon(b \times d)\varepsilon(b' \times d')
$$

$\varepsilon(1_B \times 1_D) = 1_k$.

So $\varepsilon$ is an algebra map. Therefore we have $B \times_\Pi D$ is a bialgebra so $(D, B)$ is an admissible pair. By (1),(12),(13),(16) and (17), $B \equiv_\Pi A \equiv_\pi D$ is an admissible mapping system.

(v) : From (iv) and Proposition 8. $\square$

**Remark 10.** If we assume that $h \cdot 1_D = \varepsilon_H(h)1_D$ then the $H$-module structure of $B$ in the proof of Theorem 1 is reduced from the $H$-module structure of $D$:

$$
h \cdot \Pi(a) = (\Pi \circ i)(h \cdot \pi(\Pi(a))) = (\Pi \circ i)(h \cdot \varepsilon_A(a)1_D)
$$

$$
= \varepsilon_A(a)(\Pi \circ i)(h \cdot 1_D) = \varepsilon_A(a)(\Pi \circ i)(\varepsilon_H(h)1_D)
$$

$$
= \varepsilon_H(h)\varepsilon_A(a)\Pi(1_A) = \varepsilon_H(h)\varepsilon_A(a)\Pi(1_A) = \varepsilon_H(h)\varepsilon_A(a)1_B
$$

$$
= \varepsilon_H(h)\varepsilon_B(\Pi(a))1_B = \varepsilon_H(h)\varepsilon_B(b)\circ(1_B).
$$

**Corollary 11.** Let $B$ be as in the Theorem above. Then the following are equivalent:

(1) $\Pi$ is an algebra map.

(2) $d \cdot b = \varepsilon(d)b, \; d \in D$ and $b \in B$.

**Proof.** (1) $\Rightarrow$ (2) : By (4), for all $d \in D$ and $b \in B$,

$$
d \cdot b = ad_1(d \otimes a) = \Sigma i(d_1)bi(s_D(d_2)) = \Sigma i(d_1)\Pi(b)(i \circ s_D)(d_2)
$$

$$
= \Sigma i(d_1)\Pi(b)(i \circ s_D \circ \pi)(i(d_2)) = \Sigma i(d_1)\Pi(b)(i \circ s_D \circ \pi)(i(d_2))
$$

$$
= \Pi(i(d)b) = \Pi(i(d))\Pi(b) = \varepsilon(d)b,
$$

since $\Pi$ is an algebra map, $\Pi(b) = b$ and $\pi \circ i = I$.

(2) $\Rightarrow$ (1) : By Theorem 1 (v), $a = bi(d)$. For $a' \in A$,

$$
\Pi(aa') = \Pi(bi(d)a') = \Sigma b_1\Pi(i(d)a')(i \circ s \circ \pi)(b_2)
$$

$$
= \Sigma b_1\Pi(i(d)a')(i \circ s)(\pi(b_2)) = b\Pi(i(d)a')(i \circ s)(i_D)
$$

$$
= b\Pi(i(d)a') = b(d \cdot \Pi(a')) = b(\varepsilon(d)\Pi(a')) = \varepsilon(d)b\Pi(a')
$$

$$
= \Pi(bi(d))\Pi(a') = \Pi(a)\Pi(a'),
$$

by the right $D$-module structure of $B$. Therefore $\Pi$ is an algebra map. $\square$
Biproduct Bialgebras with a projection onto a Hopf Algebra

References


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