LINEAR DIFFEOMORPHISMS WITH LIMIT SHADOWING

KEONHEE LEE*, MANSEOB LEE**, AND JUNMI PARK***

ABSTRACT. In this paper, we show that for a linear dynamical system \( f(x) = Ax \) of \( \mathbb{C}^n \), \( f \) has the limit shadowing property if and only if the matrix \( A \) is hyperbolic.

1. Introduction

Let \((X, d)\) be a compact metric space with the metric \( d \), and let \( f : X \rightarrow X \) be a homeomorphism. For \( \delta > 0 \), a sequence of points \( \{x_i\}_{i \in \mathbb{Z}} \) is called a \( \delta \)-pseudo orbit of \( f \) if \( d(f(x_i), x_{i+1}) < \delta \) for all \( i \in \mathbb{Z} \). We say that \( f \) has the shadowing property if for every \( \epsilon > 0 \), there is \( \delta > 0 \) such that for any \( \delta \)-pseudo orbit \( \{x_i\}_{i \in \mathbb{Z}} \), there is \( y \in X \) such that \( d(f^n(y), x_n) < \epsilon \) for all \( n \in \mathbb{Z} \). We introduce the limit shadowing property which founded in [2]. We say that \( f \) has the limit shadowing property if there exists \( \delta > 0 \) with the following property: if a sequence \( \{x_i\}_{i \in \mathbb{Z}} \) is \( \delta \)-limit pseudo orbit of \( f \) for which relations \( d(f(x_i), x_{i+1}) \rightarrow 0 \) as \( i \rightarrow +\infty \), and \( d(f^{-1}(x_{i+1}), x_i) \rightarrow 0 \) as \( i \rightarrow -\infty \) hold, then there is a point \( y \in X \) such that \( d(f^n(y), x_n) \rightarrow 0 \) as \( n \rightarrow \pm \infty \). It is easy to see that \( f \) has the limit shadowing property on \( \Lambda \) if and only if \( f^n \) has the limit shadowing property on \( \Lambda \) for \( n \in \mathbb{Z} \setminus \{0\} \). Note that the limit shadowing property is not the shadowing property. In fact, in [2], this concept is called the weak limit shadowing property and different from the notion of Pilyugin [3](see, [2] Example 3, 4).
The notion of the pseudo orbits very often appears in several branches of the modern theory of dynamical system. For instance, the pseudo-orbit tracing property (shadowing property) usually plays an important role in the stability theory (see, [3]).

Let \( A \) be a nonsingular matrix on \( \mathbb{C}^n \). We consider the dynamical system \( f(x) = Ax \) of \( \mathbb{C}^n \). We say that the matrix \( A \) is called hyperbolic if the spectrum does not intersect the circle \( \{ \lambda : |\lambda| = 1 \} \) (for more detail, see [1]).

**Theorem 1.1.** For a linear dynamical system \( f(x) = Ax \) of \( \mathbb{C}^n \), the following conditions are mutually equivalent:

(a) \( f \) has the limit shadowing property,
(b) the matrix \( A \) is hyperbolic.

2. Proof of Theorem 1.1

For the proof of \((a) \Rightarrow (b)\), we need the following two lemmas.

**Lemma 2.1.** Let \((X, d)\) be a metric space. Assume that for two dynamical systems \( f \) and \( g \) on \( X \), there exists a homeomorphism \( h \) on \( X \) such that \( f \circ h = h \circ g \). Then \( f \) has the limit shadowing property if and only if \( g \) has the limit shadowing property.

**Proof.** Suppose that \( f \) has the limit shadowing property. For any \( \delta > 0 \), let \( \xi = \{ x_i \}_{i \in \mathbb{Z}} \) be a \( \delta \)-limit pseudo orbit of \( f \). Then \( d(f(x_i), x_{i+1}) < \delta \), for all \( i \in \mathbb{Z} \) and \( d(f(x_i), x_{i+1}) \to 0 \) as \( i \to \pm \infty \). Since \( f \circ h = h \circ g \), we know that

\[
d(g(h^{-1}(x_i)), h^{-1}(x_{i+1})) < \delta \quad \text{for all } i \in \mathbb{Z},
\]

and \( d(g(h^{-1}(x_i)), h^{-1}(x_{i+1})) \to 0 \) as \( i \to \pm \infty \). Thus \( \{ h^{-1}(x_i) \}_{i \in \mathbb{Z}} \) is a \( \delta \)-limit pseudo orbit of \( g \). Since \( f \) has the limit shadowing property, there is a point \( y \in X \) such that \( d(f^i(y), x_i) \to 0 \) as \( i \to \pm \infty \). Then \( d(f^i(y), x_i) = d(g^i(h^{-1}(y)), h^{-1}(x_i)) \to 0 \) as \( i \to \pm \infty \). Then the point \( h^{-1}(y) \in X \) is the limit shadowing point of \( g \). Thus \( g \) has the limit shadowing property. \( \square \)

**Lemma 2.2.** [3] Let \( A \) be a nonhyperbolic matrix and \( \lambda \) be an eigenvalue of \( A \) with \( |\lambda| = 1 \). Then there exists a nonsingular matrix \( T \) such that \( J = T^{-1}AT \) is a Jordan form of \( A \) and the matrix \( J \) has the form

\[
\begin{pmatrix}
B & O \\
O & D
\end{pmatrix}
\]
where $B$ is the nonsingular $m \times m$ complex matrix with the form
\[
\begin{pmatrix}
\lambda & 0 & \cdots & 0 & 0 \\
1 & \lambda & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & \lambda
\end{pmatrix}
\]
and $D$ is the hyperbolic matrix.

**Proof of** $(a) \Rightarrow (b)$. Suppose that $f$ has the limit shadowing property. To derive a contradiction, we may assume that the matrix $A$ is non-hyperbolic. Then the matrix $A$ has an eigenvalue $\lambda$ with $|\lambda| = 1$. By Lemma 2.2, there is a nonsingular matrix $T$ such that $J = T^{-1}AT$ is a Jordan form of $A$ and the Jordan form $A$ has an eigenvalue $\lambda = 1$, then $J$ is a nonsingular matrix $T$ such that $J = T^{-1}AT$ is a Jordan form of $A$ and the Jordan form $A$ is the nonsingular matrix $T$ where $B$ and $D$ are as in Lemma 2.2. Let $g(x) = J(x) = T^{-1}AT(x)$, and let $h(x) = T(x)$ for $x \in \mathbb{C}^n$. Then $f \circ h = h \circ g$. Since $f$ has the limit shadowing property, by Lemma 2.1, $g$ has the limit shadowing property. Let $\delta > 0$ be the number of the definition of the limit shadowing property of $g$. Denote by $x^{(i)}$ the $i$-th component of a vector $x \in \mathbb{C}^n$. Then we construct a $\delta$-limit pseudo orbit as follows:
\[
x^{(1)}_{i+1} = \lambda x^{(1)}_i \left(1 + \frac{\delta}{2|\lambda| |x^{(1)}_i|}\right),
\]
and $x^{(i+1)}_i = (x^{(2)}_{i+1}, x^{(3)}_{i+1}, \ldots, x^{(n)}_{i+1}) = ((Jx^{(2)}_i, (Jx^{(3)}_i, \ldots, (Jx^{(n)}_i, (Jx^{(n)}_i) = (\lambda x^{(1)}_i, x^{(1)}_{i+1}),$ we know that if $\lambda = 1$, then
\[
d(g(x_i), x_{i+1}) = |x^{(1)}_i - x^{(1)}_{i+1}| = \frac{\delta}{2|\lambda|} < \delta,
\]
for all $i \in \mathbb{Z}$ and if $i \to \pm \infty$, then $d(g(x_i), x_{i+1}) = \delta/2|\lambda| \to 0$. Thus \(\{x_i\}_{i \in \mathbb{Z}}\) is a $\delta$-limit pseudo orbit of $g$. Since $g$ has the limit shadowing property, there is a point $y \in X$ such that $d(g^n(y), x_i) \to 0$ as $i \to \pm \infty$. If $y = (0, 0, \ldots, 0)$ then
\[
d(g^{i+1}(y), x_{i+1}) = |x^{(1)}_i + x^{(1)}_i| \geq |x^{(1)}_i| > 0.
\]
This is a contradiction. If $y = (0, y^{(2)}, y^{(3)}, \ldots, y^{(n)})$, then
\[
g^{i+1}(y) = (0, (Jy)^{(2)}, (Jy)^{(3)}, \ldots, (Jy)^{(n)}).
Then, we see that if for all \( i \in \mathbb{Z} \),
\[
|((Jx_i)^{(2)}, (Jx_i)^{(3)}, \ldots, (Jx_i)^{(n)}) - ((J^i y)^{(2)}, (J^i y)^{(3)}, \ldots, (J^i y)^{(n)})| = 0,
\]
then as in the proof of the above, for \( (J^i y)^{(1)} = 0 \), we get a contradiction. Thus we see that for the point \( y \in X \), the first component of \( y \), say \( y^{(1)} \), is not equal to 0. Then we consider the case \( g(y) = g(y^{(1)}, y^{(2)}, \ldots, y^{(n)}) = (y^{(1)}, (Jy)^{(2)}, (Jy)^{(3)}, \ldots, (Jy)^{(n)}) \). Thus, for all \( i \in \mathbb{Z} \),
\[
\left| x^{(1)}_i + \frac{x^{(1)}_i \delta}{2^i |x^{(1)}_i|} - y^{(1)} \right| \geq |x^{(1)}_i - y^{(1)}|.
\]

Take \( \eta > 0 \), let \( |x^{(1)}_0| = \eta \). For all \( i \in \mathbb{Z} \), we see that
\[
|x^{(1)}_i| = \eta + \delta + \frac{\delta}{2} + \frac{\delta}{2^2} + \cdots + \frac{\delta}{2^{i-1}} = \eta + 2\delta \left( 1 - \frac{1}{2^i} \right).
\]
If \( x_0 = y \) then by (2.1),
\[
|x^{(1)}_i - y^{(1)}| \geq |\eta + 2\delta \left( 1 - \frac{1}{2^i} \right)| - |\eta| \geq |\eta| - |2\delta \left( 1 - \frac{1}{2^i} \right)| - |\eta|,
\]
for all \( i \in \mathbb{Z} \). Then by (2.2), if \( i \to \infty \), then \( |x^{(1)}_i - y^{(1)}| \to -|2\delta| \neq 0 \). This is a contradiction. Finally, we consider \( x^{(1)}_0 \neq y^{(1)} \). Since \( |x^{(1)}_0 - y^{(1)}| \neq 0 \), we can take \( \gamma > 0 \) such that \( |x^{(1)}_0 - y^{(1)}| = \gamma \). Let \( |x^{(1)}_0| = \eta > 0 \). Then by (2.2),
\[
|x^{(1)}_i - y^{(1)}| \geq |\eta + 2\delta \left( 1 - \frac{1}{2^i} \right)| - |\eta| - |\gamma| \geq -|2\delta \left( 1 - \frac{1}{2^i} \right)| - |\gamma|,
\]
for all \( i \in \mathbb{Z} \). Then by (2.3), if \( i \to \infty \), then \( |x^{(1)}_i - y^{(1)}| \to -|2\delta| - |\gamma| \neq 0 \). This is a contradiction. Thus if \( f \) has the limit shadowing property, then the matrix \( A \) is hyperbolic. \( \square \)

Finally, we show that \( (b) \Rightarrow (a) \), that is proved by Lee [2] as follow.

**Lemma 2.3.** Let \( f(x) = Ax \) of \( \mathbb{C}^n \). If \( A \) is the hyperbolic matrix, then \( f \) has the limit shadowing property.

**Proof.** Denote by \( E_p \) the invariant subspace of \( T_p \mathbb{C}^n \) corresponding to the eigenvalues \( \lambda_i \) of \( A \) such that \( |\lambda_i| < 1 \), and by \( F_p \) the invariant subspace of \( T_p \mathbb{C}^n \) corresponding to the eigenvalues \( \lambda_i \) of \( A \) such that \( |\lambda_i| > 1 \). By [3], there exist \( C > 0 \), \( m \in \mathbb{N} \), \( 0 < \lambda < 1 \), and invariant linear subspaces \( E_p \) and \( F_p \) of \( T_p \mathbb{C}^n \) for \( p \in \mathbb{C}^n \) such that

1. \( T_p \mathbb{C}^n = E_p \oplus F_p \),
2. \( |A^mk(v)| < C\lambda^k |v| \), \( v \in E_p \), \( k \geq 0 \),
(3) $|A^{-mk}(v)| < C\lambda^{-k}|v|$, $v \in F_p$, $k < 0$.

This means that the dynamical system $f^m(x) = A^m(x)$ is hyperbolic. Then by [2], $f^m$ has the limit shadowing property, therefore, $f$ has the limit shadowing property.

References


* Department of Mathematics
Chungnam University
Daejeon 305-764, Republic of Korea
E-mail: khlee@cnu.ac.kr

** Department of Mathematics
Mokwon University
Daejeon 302-729, Republic of Korea
E-mail: lmsds@mokwon.ac.kr

*** Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea
E-mail: pjmds@cnu.ac.kr