STABILITY OF A GENERAL QUADRATIC FUNCTIONAL EQUATION IN NON-ARCHIMEDEAN NORMED SPACES

CHANG-JU LEE* AND YANG-HI LEE**

Abstract. In this paper, we investigate the stability for the functional equation
\[ f(x+y+z)+f(x-y)+f(x-z)-f(x-y-z)-f(x+y)-f(x+z) = 0 \]
in non-Archimedean normed spaces.

1. Introduction

A classical question in the theory of functional equations is “when is it true that a mapping, which approximately satisfies a functional equation, must be somehow close to a solution of the equation?”. This problem, called a stability problem of the functional equation, was formulated by S. M. Ulam [7] in 1940. In the next year, D. H. Hyers [2] gave a partial solution of Ulam problem for the case of an approximate additive mapping. Subsequently, his result was generalized by T. Aoki [1] for an additive mapping and by Th. M. Rassias [6] for a linear mapping with unbounded Cauchy differences.

We introduce some terminologies and notations used in the theory of non-Archimedean spaces (see [3]).

Definition 1.1. A field \( \mathbb{K} \) equipped with a function (valuation) \( | \cdot | \) from \( \mathbb{K} \) into \([0, \infty)\) is called a non-Archimedean field if the function \( | \cdot | : \mathbb{K} \rightarrow [0, \infty) \) satisfies the following conditions:

(i) \( |r| = 0 \) if and only if \( r = 0 \);
(ii) \( |rs| = |r||s| \);

Received February 05, 2013; Accepted April 04, 2013.
2010 Mathematics Subject Classification: Primary 46S40, 39B52.
Key words and phrases: non-Archimedean normed spaces, general quadratic functional equation.
Correspondence should be addressed to Yang-Hi Lee, lyhmzi@gju.ac.kr.
This work was supported by Gongju National University of Education Grant.
(iii) $|r + s| \leq \max\{|r|, |s|\}$ for all $r, s \in \mathbb{K}$.

Clearly, $|1| = |-1|$ and $|n| \leq 1$ for all $n \in \mathbb{N}$.

**Definition 1.2.** Let $X$ be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : X \to \mathbb{R}$ is a non-Archimedean norm if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|rx\| = |r|\|x\|$ ($r \in \mathbb{K}, x \in X$);
(iii) the strong triangle inequality, namely,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}$$

for all $x, y \in X$ and $r \in \mathbb{K}$. The pair $(X, \|\cdot\|)$ is called a non-Archimedean space if $\|\cdot\| : X \to \mathbb{R}$ is a non-Archimedean norm on $X$.

Due to the fact that

$$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n-1\} (n > m),$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space, we mean one in which every Cauchy sequence is convergent.


$$f(x + y) = f(x) + f(y)$$

and the quadratic functional equation

$$f(x + y) + f(x - y) - 2f(x) - 2f(y) = 0$$

in non-Archimedean normed spaces.

Now we consider the general quadratic functional equation

$$f(x+y+z)+f(x-y)+f(x-z)-f(x-y-z)-f(x+y)-f(x+z) = 0,$$

which solution is called a general quadratic mapping. Recently, Kim [4] and Jun et al [3] obtained a stability of the functional equation (1.3) by taking and composing an additive mapping $A$ and a quadratic mapping $Q$ to prove the existence of a general quadratic mapping $F$ which is close to the given function $f$. In their processing, $A$ is approximate to the odd part $\frac{f(x)-f(-x)}{2}$ of $f$ and $Q$ is close to the even part $\frac{f(x)+f(-x)}{2} - f(0)$ of $f$, respectively.

In this paper, we get a general stability result of the general quadratic functional equation (1.3) in non-Archimedean normed spaces.
2. Stability of the general quadratic functional equation

Throughout this section, we assume that $X$ is a real linear space and $Y$ is a complete non-Archimedean space with $|2|<1$.

For a given mapping $f : X \to Y$, we use the abbreviation

$$Df(x, y, z) := f(x + y + z) + f(x - y) + f(x - z)$$

$$- f(x - y - z) - f(x + y) - f(x + z)$$

for all $x, y, z \in X$. Now, we will prove the stability of the general quadratic functional equation (1.3).

**Theorem 2.1.** Let $\varphi : X^3 \to [0, \infty)$ be a function such that

$$\lim_{n \to \infty} \frac{\varphi(2^n x, 2^n y, 2^n z)}{|4^n|} = 0 \quad (x, y, z \in X).$$

Suppose that $f : X \to Y$ is a mapping satisfying

$$\|Df(x, y, z)\| \leq \varphi(x, y, z) \quad (x, y, z \in X).$$

Then there exists a unique general quadratic mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \lim_{n \to \infty} \max\{\psi_j(x) : 0 \leq j < n\} \quad (x \in X),$$

where $\psi_j : X \to [0, \infty)$ is defined by

$$\psi_j(x) := \max \left\{ \frac{\varphi(2^{-j-1} x, 2^{-j-1} x, 2^j x)}{|2| \cdot |4|^{j+1}}, \varphi(2^{-j-1} x, 2^{-j-1} x, -2^j x) \right\},$$

$$\varphi(-2^{-j-1} x, -2^{-j-1} x, -2^j x), \varphi(2^{j+1} x, 2^j x, 2^j x) \right\},$$

$$\varphi(2^j x, 2^{j+1} x, 2^j x), \varphi(2^j x, 2^j x, 2^j x) \right\}$$

for all $j \geq 0$. In particular, $T$ is given by

$$T(x) = \lim_{n \to \infty} \frac{f(2^n x) + f(-2^n x) - 2f(0)}{2 \cdot 4^n} + \frac{f(2^n x) - f(-2^n x)}{2n+1} + f(0)$$

for all $x \in X$.

**Proof.** Let $J_n f : X \to Y$ be a function defined by

$$J_n f(x) = \frac{f(2^n x) + f(-2^n x) - 2f(0)}{2 \cdot 4^n} + \frac{f(2^n x) - f(-2^n x)}{2n+1} + f(0)$$
for all \( x \in X \) and \( n \in \mathbb{N} \). Notice that \( J_0 f(x) = f(x) \) and

\[
(2.4) \quad \|J_j f(x) - J_{j+1} f(x)\| = \left\| - \frac{Df(2^{j-1}x, 2^{j-1}x, 2^j x)}{2 \cdot 4^{j+1}} - \frac{Df(-2^{j-1}x, -2^{j-1}x, -2^j x)}{2 \cdot 4^{j+1}} - \frac{Df(-2^{j-1}x, -2^{j-1}x, -2^{j-1}x)}{2 \cdot 4^{j+1}} + \frac{Df(2^{j+1}x, 2^j x, 2^j x)}{2^{j+2}} \right\| 
\]

\[
\leq \max \left\{ \left\| \frac{Df(2^{j-1}x, 2^{j-1}x, 2^j x)}{2 \cdot 4^{j+1}}, \frac{Df(2^{j-1}x, 2^{j-1}x, 2^{j-1}x)}{2 \cdot 4^{j+1}}, \right\| \left\| \frac{Df(-2^{j-1}x, -2^{j-1}x, -2^j x)}{2 \cdot 4^{j+1}}, \frac{Df(-2^{j-1}x, -2^{j-1}x, -2^{j-1}x)}{2 \cdot 4^{j+1}}, \right\| \right\} 
\]

\[
\leq \psi_j(x) 
\]

for all \( x \in X \) and \( j \geq 0 \). It follows from (2.1) and (2.4) that the sequence \( \{J_n f(x)\} \) is Cauchy. Since \( Y \) is complete, we conclude that \( \{J_n f(x)\} \) is convergent. Set

\[
T(x) := \lim_{n \to \infty} J_n f(x). 
\]

Using induction one can show that

\[
(2.5) \quad \|J_n f(x) - f(x)\| \leq \max \{\psi_j(x) : 0 \leq j < n\} 
\]

for all \( n \in \mathbb{N} \) and all \( x \in X \). By taking \( n \) to approach infinity in (2.5) and using (2.1), one obtains (2.3). Replacing \( x, y, \) and \( z \) by \( 2^n x, 2^n y, \) and \( 2^n z, \) respectively, in (2.2) we get

\[
\|D J_n f(x, y, z)\| = \left\| \frac{Df(2^n x, 2^n y, 2^n z) - Df(-2^n x, -2^n y, -2^n z)}{2^{n+1}} + \frac{Df(2^n x, 2^n y, 2^n z) + Df(-2^n x, -2^n y, -2^n z)}{2 \cdot 4^n} \right\| 
\]
Let $T$ satisfies the condition of uniqueness of (2.3), then for all $x$ for any $K$ over $T \in x$, $y$, $z$.

Taking the limit as $n \to \infty$ and using (2.1) we get $DT'(x, y, z) = 0$. If $T'$ is another general quadratic mapping satisfying (2.3), then

$$
T'(x) = \sum_{j=0}^{k-1} \left( - \frac{DT'(2^{j-1}x, 2^{j-1}x, 2^{j}x)}{2 \cdot 4^{j+1}} - \frac{DT'(2^{j-1}x, 2^{j-1}x, 2^{j}x)}{2 \cdot 4^{j+1}} \right)
$$

$$
= J_k T'(x)
$$

for any $k \in N$ and so

$$
||T(x) - T'(x)|| = \lim_{k \to \infty} ||J_k T(x) - J_k T'(x)||
$$

$$
\leq \lim_{k \to \infty} \max \{ ||J_k T(x) - J_k f(x)||, ||J_k f(x) - J_k T'(x)|| \}
$$

$$
\leq \lim_{k \to \infty} \max \{ ||T(2^k x) - f(2^k x)||, ||T(-2^k x) - f(-2^k x)||, ||f(2^k x) - T'(2^k x)||, ||f(-2^k x) - T'(-2^k x)|| \}
$$

$$
\leq \lim_{k \to \infty} \max \{ |2|^{-1} \psi_j(x), |2|^{-1} \psi_j(-x) : k \leq j < n + k \}
$$

$$
= 0
$$

for all $x \in X$. Therefore $T = T'$. This completes the proof of the uniqueness of $T$.

**Corollary 2.2.** Let $X$ and $Y$ be non-Archimedean normed spaces over $K$ with $|2| < 1$. If $Y$ is complete and for some $2 < r$, $f : X \to Y$ satisfies the condition

$$
||Df(x, y, z)|| \leq \theta(||x||^r + ||y||^r + ||z||^r)
$$


for all \(x, y, z \in X\). Then there exists a unique general quadratic mapping \(T : X \to Y\) such that

\[
\|f(x) - T(x)\| \leq 3|2|^{3-r}\theta\|x\|^r.
\]

**Proof.** Let \(\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r).\) Since \(|2| < 1\) and \(r - 2 > 0\),

\[
\lim_{n \to \infty} |4|^{-n}\varphi(2^n x, 2^n y, 2^n z) = \lim_{n \to \infty} |2|^{n(r-2)}\varphi(x, y, z) = 0
\]

for all \(x, y, z \in Y\). Therefore the conditions of Theorem 2.1 are satisfied. It is easy to see that \(\psi_0(x) = 3|2|^{-r}\theta\|x\|^r\). By Theorem 2.1 there is a unique general quadratic mapping \(T : X \to Y\) such that (2.6) holds. \(\square\)

**Theorem 2.3.** Let \(\varphi : X^3 \to [0, \infty)\) be a function such that

\[
\lim_{n \to \infty} |2|^n\varphi(2^{-n} x, 2^{-n} y, 2^{-n} z) = 0 \quad (x, y, z \in X).
\]

Suppose that \(f : X \to Y\) is a mapping satisfying

\[
\|Df(x, y, z)\| \leq \varphi(x, y, z) \quad (x, y, z \in X).
\]

Then there exists a unique general quadratic mapping \(T : X \to Y\) such that

\[
\|f(x) - T(x)\| \leq \lim_{n \to \infty} \max\{\psi_j(x) : 0 \leq j < n\} \quad (x \in X),
\]

where \(\psi_j : X \to [0, \infty)\) is defined by

\[
\psi_j(x) := \max\{|2|^{2j-1}\varphi\left(\frac{x}{2^{j+2}}, \frac{x}{2^{j+2}}, \frac{x}{2^{j+1}}\right), |2|^{2j-1}\varphi\left(\frac{x}{2^{j+2}}, \frac{x}{2^{j+2}}, \frac{x}{2^{j+2}}\right),
\]

\[
|2|^{2j-1}\varphi\left(\frac{-x}{2^{j+2}}, \frac{-x}{2^{j+2}}, \frac{-x}{2^{j+1}}\right), |2|^{2j-1}\varphi\left(\frac{-x}{2^{j+2}}, \frac{-x}{2^{j+2}}, \frac{-x}{2^{j+2}}\right),
\]

\[
|2|^{j-1}\varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right), |2|^{j-1}\varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right),
\]

\[
|2|^{j-1}\varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right)\}
\]

for all \(j \geq 0\). In particular, \(T\) is given by

\[
T(x) = \lim_{n \to \infty} \frac{4^n}{2} \left(f(2^{-n} x) + f(-2^{-n} x) - 2f(0)\right)
+ 2^{n-1} \left(f\left(\frac{x}{2^n}\right) - f\left(-\frac{x}{2^n}\right)\right) + f(0)
\]

for all \(x \in X\).
Proof. Let $J_n f : X \rightarrow Y$ be a function defined by

$$J_n f(x) = \lim_{n \rightarrow \infty} \frac{4^n}{2} \left( f(2^{-n}x) + f(-2^{-n}x) - 2f(0) \right)$$

$$+ 2^{n-1} \left( f \left( \frac{x}{2^n} \right) - f \left( -\frac{x}{2^n} \right) \right) + f(0)$$

for all $x \in X$ and $n \in \mathbb{N}$. Notice that $J_0 f(x) = f(x)$ and

$$\| J_j f(x) - J_{j+1} f(x) \|$$

$$= \left\| \frac{4^j}{2} \left( Df \left( \frac{x}{2^j+2}, \frac{x}{2^j+2}, \frac{x}{2^j+1} \right) + Df \left( \frac{x}{2^j+2}, \frac{x}{2^j+2}, \frac{x}{2^j+2} \right) \right) \right. $$

$$+ \left. Df \left( \frac{-x}{2^j+2}, \frac{-x}{2^j+2}, \frac{-x}{2^j+2} \right) + Df \left( \frac{x}{2^j+2}, \frac{x}{2^j+2}, \frac{x}{2^j+2} \right) \right\|$$

$$- 2^{j-1} \left( Df \left( \frac{x}{2^j}, \frac{x}{2^j}, \frac{x}{2^j} \right) - Df \left( \frac{x}{2^j+1}, \frac{x}{2^j+1}, \frac{x}{2^j+1} \right) \right)$$

$$\leq \psi_j(x)$$

for all $x \in X$ and $j \geq 0$. It follows from (2.7) and (2.10) that the sequence $\{ J_n f(x) \}$ is Cauchy. Since $Y$ is complete, we conclude that $\{ J_n f(x) \}$ is convergent. Set

$$T(x) := \lim_{n \rightarrow \infty} J_n f(x).$$

Using induction one can show that

$$\| J_n f(x) - f(x) \| \leq \max \left\{ \psi_j(x) : 0 \leq j < n \right\}$$

for all $n \in \mathbb{N}$ and all $x \in X$. By taking $n$ to approach infinity in (2.11) and using (2.7) one obtains (2.9). Replacing $x, y,$ and $z$ by $2^{-n}x, 2^{-n}y,$ and $2^{-n}z,$ respectively, in (2.8), we get

$$\| DJ_n f(x, y, z) \|$$

$$= \left\| 2^{n-1} Df \left( \frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n} \right) - 2^{n-1} Df \left( \frac{-x}{2^n}, \frac{-y}{2^n}, \frac{-z}{2^n} \right) \right\|$$

$$+ 2^{2n-1} Df \left( \frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n} \right) + 2^{2n-1} Df \left( \frac{-x}{2^n}, \frac{-y}{2^n}, \frac{-z}{2^n} \right)$$

$$\leq \max \left\{ \left| 2^{n-1} \varphi \left( \frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n} \right) \right|, \left| 2^{n-1} \varphi \left( \frac{-x}{2^n}, \frac{-y}{2^n}, \frac{-z}{2^n} \right) \right| \right\}.$$
Taking the limit as \( n \to \infty \) and using (2.7) we get \( DT(x, y, z) = 0 \). If \( T' \) is another general quadratic mapping satisfying (2.9), then
\[
T'(x) - J_kT'(x) = \sum_{j=0}^{k-1} \left( \frac{4j}{2} DT' \left( \frac{x}{2^{j+2}}, \frac{x}{2^{j+1}} \right) + DT' \left( \frac{-x}{2^{j+2}}, \frac{-x}{2^{j+1}} \right) \right) = 0
\]
for any \( k \in \mathbb{N} \) and so
\[
\|T(x) - T'(x)\| = \lim_{k \to \infty} \|J_kT(x) - J_kT'(x)\| \\
\leq \lim_{k \to \infty} \max\{\|J_kT(x) - J_kf(x)\|, \|J_kf(x) - J_kT'(x)\|\} \\
\leq \lim_{k \to \infty} |2|^{k-1} \max\left\{ \left\| T \left( \frac{x}{2^k} \right) - f \left( \frac{x}{2^k} \right) \right\|, \left\| T \left( -\frac{x}{2^k} \right) - f \left( -\frac{x}{2^k} \right) \right\|, \left\| T' \left( \frac{x}{2^k} \right) - f \left( \frac{x}{2^k} \right) \right\|\right\} \\
\leq \lim_{k \to \infty} |2|^{k-1} \lim_{n \to \infty} \max\{\psi_j \left( \frac{x}{2^k} \right), \psi_j \left( -\frac{x}{2^k} \right) : 0 \leq j < n\} \\
= \lim_{k \to \infty} |2|^{-1} \lim_{n \to \infty} \max\{\psi_j (x), \psi_j (-x) : k \leq j < n + k\} \\
= 0 \quad (x \in X).
\]
Therefore \( T = T' \). This completes the proof of the uniqueness of \( T \). \( \square \)

**Corollary 2.4.** Let \( X \) and \( Y \) be non-Archimedean normed spaces over \( K \) with \( |2| < 1 \). If \( Y \) is complete and for some \( 0 \leq r < 1 \), \( f : X \to Y \) satisfies the condition
\[
\|Df(x, y, z)\| \leq \theta (\|x\|^r + \|y\|^r + \|z\|^r)
\]
for all \( x, y, z \in X \). Then there exists a unique general quadratic mapping \( T : X \to Y \) such that
\[
(2.12) \quad \|f(x) - T(x)\| \leq 3|2|^{-1-2r}\theta \|x\|^r.
\]
Proof. Let $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$. Since $|2| < 1$ and $1 - r > 0$,

$$
\lim_{n \to \infty} |2^n \varphi(2^{-n}x, 2^{-n}y, 2^{-n}z) = \lim_{n \to \infty} |2^n (1-r) \varphi(x, y, z) = 0
$$

for all $x, y, z \in X$. Therefore the conditions of Theorem 2.3 are satisfied. It is easy to see that $\psi_0(x) = 3|2|^{-1-2r} \theta |x|^r$. By Theorem 2.3, there is a unique general quadratic mapping $T : X \to Y$ satisfying (2.12). □

References


* Department of Mathematics Education
Gongju National University of Education
Gongju 314-060, Republic of Korea
E-mail: chjlee@gjue.ac.kr

** Department of Mathematics Education
Gongju National University of Education
Gongju 314-060, Republic of Korea
E-mail: lyhmzi@gjue.ac.kr