SOME EXAMPLES OF THE UNION OF TWO LINEAR
STAR-CONFIGURATIONS IN $\mathbb{P}^2$ HAVING GENERIC
HILBERT FUNCTION

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Abstract. In [20] and [22], the author proved that the union of
two linear star-configurations in $\mathbb{P}^2$ of type $t \times s$ with $3 \leq t \leq 10$
and $t \leq s$ has generic Hilbert function. In this paper, we prove that
the union of two linear star-configurations in $\mathbb{P}^2$ of type $t \times s$ with
$3 \leq t$ and $\binom{t}{2} - 1 \leq s$ has also generic Hilbert function.

1. Introduction

Let $R = \mathbb{k}[x_0, x_1, \ldots, x_n]$ be an $(n + 1)$-variable polynomial ring and
$A = R/I$ where $I$ is a homogeneous ideal in $R$. Then $A = \bigoplus_{i=0}^{\infty} A_i$
is also a graded ring. In this situation the Hilbert function of $A$ is the
function
\[ H(A, i) := \dim_{\mathbb{k}} A_i = \dim_{\mathbb{k}} R_i - \dim_{\mathbb{k}} I_i = \binom{i+n}{n} - \dim_{\mathbb{k}} I_i. \]
If $I := I_X$ is the ideal of a subscheme $X$ in $\mathbb{P}^n$, then we denote the Hilbert
function of $X$ by $H_X(t) = H(R/I_X, t)$
(see [1, 2, 3, 6, 7, 8, 9, 10, 11, 12, 13]). In particular, If $X$ is a subscheme
in $\mathbb{P}^2$ and
\[ H_X(d) = \min \left\{ \binom{d+2}{2}, \deg(X) \right\} \]
for every $d \geq 0$, then we say that $X$ has generic Hilbert function.

In this paper, we study the union of two star-configurations in $\mathbb{P}^2$
defined by general forms (see also [2, 20, 21, 22]). In [21], the author
found conditions for a star-configuration in $\mathbb{P}^2$ to have generic Hilbert
function based on the degrees of these general forms. In [2, 21], the

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authors also found conditions when a graded Artinian ring \( R/(I_X + I_Y) \) has the Weak Lefschetz property for two star-configurations \( X \) and \( Y \) in \( \mathbb{P}^2 \) (see also [14, 15, 16, 17, 18, 19]).

The following proposition in [3] is about the ideal of general forms in \( R \), which leads to the definition of a star-configuration and a linear star-configuration in \( \mathbb{P}^n \).

**Proposition 1.1.** [3, Proposition 2.1] Let \( F_1, F_2, \ldots, F_s \) be general forms in \( R = k[x_0, x_1, \ldots, x_n] \) with \( s \geq 3 \). Then
\[
\cap_{1 \leq i < j \leq s} (F_i, F_j) = (\tilde{F}_1, \ldots, \tilde{F}_s), \quad \text{where} \quad \tilde{F}_i = \prod_{j=1}^{s} F_j / F_i \quad \text{for} \quad i = 1, \ldots, s.
\]

The variety \( X \) in \( \mathbb{P}^n \) of the ideal \( \cap_{1 \leq i < j \leq s} (F_i, F_j) = (\tilde{F}_1, \ldots, \tilde{F}_s) \) in Proposition 1.1 is called a star-configuration in \( \mathbb{P}^n \). Furthermore, if the \( F_i \) are all general linear forms in \( R \), the star-configuration \( X \) is called a linear star-configuration in \( \mathbb{P}^n \).

In this paper, if \( X := X(t,s) \) is the union of two linear star-configurations \( X_1 \) and \( X_2 \) in \( \mathbb{P}^2 \) of types \( t \) and \( s \) (type \( t \times s \) for short), then \( X \) has generic Hilbert function for \( 3 \leq t \) and \( (t^2 - 1) \leq s \). Moreover, we also show that \( \sigma(X) = s \) for such \( t \) and \( s \), where \( \sigma(X) := \min\{d \mid H_X(d-1) = H_X(d)\} \).

In Section 3, we propose some questions for further study.

### 2. The union of two linear star-configurations in \( \mathbb{P}^2 \)

Before we start to prove the main theorem, we introduce some notations for convenience. Let \( L_1, \ldots, L_{s-1}, L_s \), and \( M_1, \ldots, M_t \) be general linear forms for \( s \geq 3 \) and \( t \geq 3 \), respectively. Define

\( X_1 = Y_1 \) is a linear star-configuration in \( \mathbb{P}^2 \) defined by \( M_1, \ldots, M_t \),

\( X_2 \) is a linear star-configuration in \( \mathbb{P}^2 \) defined by \( L_1, \ldots, L_{s-1}, L_s \),

\( Y_2 \subseteq X_2 \) is a linear star-configuration in \( \mathbb{P}^2 \) defined by \( L_1, \ldots, L_s \),

\( Y := X^{(t,s-1)} := Y_1 \cup Y_2 \), \( X := X^{(t,s)} := X_1 \cup X_2 \), and

\( G_{s-1} := L_1 \cdots L_{s-1} \), respectively.

The first idea is that if \( \mathcal{X}' \) is the union of two finite sets of points defined by linear forms \( M_1, \ldots, M_t \) and \( L_1, L_2, \ldots, L_s \) in \( R \) (not necessarily general), respectively, then the points in \( \mathcal{X} \) are more general than the points in \( \mathcal{X}' \). This implies for every \( i \geq 0 \) we get
\[
H_{\mathcal{X}}(i) \leq H_{\mathcal{X}'}(i).
\]

The second idea is using Bezout’s Theorem in \( \mathbb{P}^2 \) to find the union \( \mathcal{X}' \) of two sets of points defined by linear forms \( M_1, \ldots, M_t \) and \( L_1, L_2, \ldots, L_s \).
in \(R\), respectively, such that

\[
H_X(i) = H_X'(i) = \min \{ |X|, (i+2\over 2) \} \quad \text{for some} \quad i \geq 0.
\]

In other words, if a form \(F\) of degree \(d\) in \(R\) vanishes on \((d+1)\)-points on a line defined by a linear form \(M\) in \(R\), then \(F\) is divided by a linear form \(M\). Throughout this section, we shall not distinguish \(X\) from \(X'\) for convenience.

**Proposition 2.1.** With notation as above, \(X := X(t,s)\) has generic Hilbert function and \(\sigma(X) = s\) for \(s \geq \binom{t}{2}\) and \(t \geq 3\).

**Proof.** We shall prove this by induction on \(s\). First, let \(s = \binom{t}{2}\), and assume that \(X := X_1 \cup X_2\) where \(X_1\) and \(X_2\) are linear star-configurations in \(\mathbb{P}^2\) defined by general linear forms \(M_1, \ldots, M_t\) and \(L_1, \ldots, L_s\), respectively. Let \(X_i := \{ Q_1, \ldots, Q_s \}\). Without loss of generality, we may assume that \(L_i\) vanishes on a point \(Q_i\) for \(i = 1, \ldots, s - 1\). If \(F \in (I_X)_{s-1}\) then, by Bezout’s Theorem,

\[
F = \alpha L_1 \cdots L_{s-1}
\]

for some \(\alpha \in k\). Moreover, since \(F\) also vanishes on the point \(Q_s\), which none of \(L_1, \ldots, L_{s-1}\) vanishes, we get that \(F = 0\), that is \(, (I_X)_{s-1} = 0\).

Hence

\[
H(R/I_X, s - 1) = \binom{s+1}{2} = \binom{s}{2} + s = \binom{s}{2} + \binom{t}{2} = \deg(X),
\]

and so \(X\) has generic Hilbert function as

\[
H_X : 1 \quad \binom{3}{2} \quad \cdots \quad \binom{(s-3)+2}{2} \quad \binom{(s-2)+2}{2} \quad \binom{(s-1)+2}{2} \quad \cdots \quad \binom{(s-1)+2}{\deg(X)} \quad \rightarrow,
\]

and \(\sigma(X) = s\), as we wished.

Now suppose \(s > \binom{t}{2}\). Let \(Y := X(t,s-1)\) be the union of two linear star-configurations in \(\mathbb{P}^2\) defined by linear forms \(M_1, \ldots, M_t\) and \(L_1, \ldots, L_s\), respectively. Now we consider the following equations:

\[
H(R/I_X, -) : 1 \quad \binom{1+2}{2} \quad \cdots \quad \binom{(s-2)-nd}{2} \quad \binom{t}{2} \quad \rightarrow,
\]

\[
H(R/I_Y, -) : 1 \quad \binom{1+2}{2} \quad \cdots \quad \binom{(s-1)}{2} \quad \binom{t}{2} \quad \rightarrow,
\]

\[
H(R/(L_s, G_{s-1}), -) : 1 \quad 2 \quad \cdots \quad s - 1 \quad s - 1 \quad \rightarrow,
\]

\[
H(R/(I_Y, L_s, G_{s-1}), -) : 1 \quad 2 \quad \cdots \quad 0 \quad 0 \quad \rightarrow,
\]

\[
H(R/(I_Y, L_s), -) : 1 \quad 2 \quad \cdots \quad \binom{t}{2} \quad 0 \quad \rightarrow.
\]

Since \(\deg G_{s-1} = s - 1\), we have

\[
H(R/(I_Y, L_s, G_{s-1}), s - 2) = H(R/(I_Y, L_s), s - 2) = \binom{t}{2},
\]
With notation as above, note that, by Proposition 2.1, we assume that Hilbert function and
and thus
\[ H(R/I_X, s - 2) = H(R/I_Y, s - 2) + H(R/(L_s, G_{s-1}), s - 2) - H(R/(I_Y, L_s, G_{s-1}), s - 2) = \binom{(s-3)+2}{2} + \binom{1}{2} + (s - 1) - \binom{1}{2} = \binom{(s-3)+2}{2} + (s - 1) = \binom{(s-2)+2}{2}. \]

This means that \( X \) has generic Hilbert function as
\[ H_X : 1 \ 1+2 \ \cdots \ \frac{(s-3)+2}{2} \ \frac{(s-2)+2}{2} \ \frac{s}{2} + \frac{1}{2} \ \frac{1}{2} \to, \]
and \( \sigma(X) = s \), which completes the proof.

**Corollary 2.2.** With notation as above, \( X := X^{(t, s-1)} \) has generic Hilbert function and \( \sigma(X) = s \) for \( s = \binom{t}{2} \) and \( t \geq 3 \).

**Proof.** Note that, by Proposition 2.1, \( Z := X^{(t, s)} \) has generic Hilbert function, and so we get the following equation.

\[
\begin{align*}
H(R/I_Z, -) & : 1 \ \frac{1+2}{2} \ \cdots \ \frac{(s-3)+2}{2} \ \frac{(s-2)+2}{2} \ \frac{s}{2} + \frac{1}{2} \ \frac{1}{2} \to, \\
H(R/I_X, -) & : 1 \ \frac{1+2}{2} \ \cdots \ \frac{(s-4)+1}{2} + \frac{s}{2} \ \frac{(s-2)+1}{2} + \frac{1}{2} \to, \\
H(R/(L_s, G_{s-1}), -) & : 1 \ 2 \ \cdots \ s-1 \ s-1 \to, \\
H(R/(I_X, L_s, G_{s-1}), -) & : 1 \ 2 \ \cdots \ 0 \ 0 \to, \\
H(R/(I_Z, L_s), -) & : 1 \ 2 \ \cdots \ - \ 0 \to.
\end{align*}
\]

Let \( F \in (I_X)_{s-2} \) and let \( X_1 := \{Q_1, \ldots, Q_s\} \). Without loss of generality, we assume that
\[
\begin{align*}
L_1 & \text{ vanishes on } (s-1)\text{-points} \quad P_{1,2}, \ldots, P_{1,s-1}, Q_1, \\
L_2 & \text{ vanishes on } (s-2)\text{-points} \quad P_{2,3}, \ldots, P_{2,s-1}, Q_2, \\
& \vdots \\
L_{s-1} & \text{ vanishes on } (s-t+1)\text{-points} \quad P_{t-1,t}, \ldots, P_{t-1,s-1}, Q_{t-1}, \\
& \vdots \\
L_{s-3} & \text{ vanishes on } 3\text{-points} \quad P_{s-3,s-2}, P_{s-3,s-1}, Q_{s-3}, \\
L_{s-2} & \text{ vanishes on } 2\text{-points} \quad P_{s-2,s-1}, Q_{s-2},
\end{align*}
\]

where \( P_{i,j} \) is the point defined by two linear forms \( L_i \) and \( L_j \) for \( i < j \). Then, by Bezóut's theorem, \( F = \alpha L_1 \cdots L_{s-2} \). Moreover, since \( F \) has to vanish on two more points \( Q_{s-1} \) and \( Q_s \), we see that \( F = 0 \), that is, \( (I_X)_{s-2} = 0 \). It follows that \( X \) has generic Hilbert function
\[ H(R/I_X, -) : 1 \ 3 \ \cdots \ \frac{(s-2)+2}{2} \ \frac{(s-1)}{2} + \frac{1}{2} \ \frac{s-1}{2} + \frac{1}{2} \to, \]
and \( \sigma(X) = s \), as we wished. \( \square \)
3. Additional comments and questions

In [4], the authors proved that the secant variety Sec\(_{s-1}(\text{Split}_d(P^n))\) to the variety \(\text{Split}_d(P^n)\) of split forms in \(R = \mathbb{k}[x_0, x_1, \ldots, x_n]\) is not defective for \(3(s-1) \leq n\) and \(2 < d\) (see also [5]). Moreover, in [20], the author proved that the secant variety Sec\(_1(\text{Split}_d(P^2))\) to the variety \(\text{Split}_d(P^2)\) of split forms in \(R = \mathbb{k}[x_0, x_1, x_2]\) is not defective for \(2 < d\), which is not covered by the result of [4], calculating the Hilbert function of two linear star-configurations in \(P^2\) of type \(d \times d\) with \(d > 2\).

In particular, in [20, 22], the author found that the union of two linear star-configurations in \(P^2\) of type \(t \times s\) has generic Hilbert function for \(3 \leq t \leq 10\) and \(t \leq s\), and we also found that some different type of the union of two linear star-configurations in \(P^2\) has also generic Hilbert function (see Proposition 2.1 and Corollary 2.2). Hence it is natural to ask the following question.

**Question 3.1.** Let \(X_1\) and \(X_2\) be star-configurations in \(P^2\) defined by \(s\)-general forms of degrees \(1 \leq d_1 \leq \cdots \leq d_s\) with \(3 \leq s\), respectively, and let \(X := X_1 \cup X_2\).

(a) Does \(X\) have generic Hilbert function in general?

(b) Does \(X\) have generic Hilbert function if \(1 = d_1 = \cdots = d_s\)?

(c) Does \(X\) have generic Hilbert function if \(1 = d_1 = \cdots = d_s\)?

In fact, Question 3.1 (a) is not true in general. Here is an example.

**Example 3.2.** Let \(L_i, M_j \in R_1\) for \(i, j = 1, \ldots, 5\) and \(F, G \in R_5\). Assume \(X\) is the union of two star-configurations in \(P^2\) defined by 6-forms \(L_1, \ldots, L_5, F\) and \(M_1, \ldots, M_5, G\), respectively. Then there exists one generator \(L_1 \cdots L_5 M_1 \cdots M_5 \in (I_X)_{10}\), and hence, by Proposition 1.1, the Hilbert function of \(X\) is of the form

\[
H_X : 1 \left(\begin{array}{c} 1+2 \\ 2 \end{array}\right) \cdots \left(\begin{array}{c} 9+2 \\ 2 \end{array}\right) \left(\begin{array}{c} 10+2 \\ 2 \end{array}\right) - 1 \cdots,
\]

which indicates \(H_X(10) = 65 \neq 70 = \deg(X)\). Thus, \(X\) does not have generic Hilbert function.

Indeed, we can generalize Example 3.2 as follows:

**Remark 3.3.** Let \(L_1, \ldots, L_{s-1}, M_1, \ldots, M_{s-1} \in R_1\) and \(F, G \in R_c\) with \(s \geq 6\) and \(c \geq s - 1\). Assume \(X\) is the union of two star-configurations \(X_1\) and \(X_2\) in \(P^2\) defined by \(s\)-forms \(L_1, \ldots, L_{s-1}, F\) and \(M_1, \ldots, M_{s-1}, G\), respectively. Since the ideal \(I_X\) has one generator \(L_1 \cdots L_{s-1} M_1 \cdots M_{s-1}\) in degree \(d = 2(s-1)\), the Hilbert function of \(X\)
is of the form
\[ H_X : 1 \ (1+2) \ \ldots \ (2s-3)+2 \ (2(s-1)+2) - 1 \ \ldots , \]
and hence \( H_X(d) < (d+2) \). Moreover, since \( s \geq 6 \), we also have that
\[ H_X(d) < (d+2) < \deg(X), \]
which follows that \( X \) does not have generic Hilbert function.

Note that if \( X \) is the union of two star-configurations in \( P^2 \) defined by forms of degrees 1, 1, 1, 1, 4, then \( X \) has generic Hilbert function as
\[ H_X : 1 \ 3 \ 6 \ 10 \ 21 \ 28 \ 36 \ 44 \rightarrow . \]

However, we don’t have any counter example to Question 3.1 (b) and (c) yet.

References

Some examples of the union of two linear star-configurations in \( \mathbb{P}^2 \)


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