MINIMAL QUASI-\( F \) COVERS OF SOME EXTENSION

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Abstract. Observing that every Tychonoff space \( X \) has an extension \( kX \) which is a weakly Lindelöf space and the minimal quasi-\( F \) cover \( QF(kX) \) of \( kX \) is a weakly Lindelöf, we show that \( \Phi_{kX} : QF(kX) \rightarrow kX \) is a \( z^\# \)-irreducible map and that \( QF(\beta X) = \beta QF(kX) \). Using these, we prove that \( QF(kX) = kQF(X) \) if and only if \( \Phi_{kX} : kQF(X) \rightarrow kX \) is an onto map and \( \beta QF(X) = QF(\beta X) \).

1. Introduction

All spaces in this paper are assumed to be Tychonoff and \( \beta X (vX, \text{resp.}) \) denotes the Stone-Čech compactification (Hewitt realcompactification, resp.) of \( X \).

Iliadis constructed the absolute of a Hausdorff space \( X \), which is the minimal extremally disconnected cover \( (E(X), \pi_X) \) of \( X \) and they turn out to be the perfect onto projective covers ([6]). To generalize extremally disconnected spaces, basically disconnected spaces, quasi-\( F \) spaces and cloz-spaces have been introduced and their minimal covers have been studied by various authors ([1], [4], [5], [8], [9]). In these ramifications, minimal covers of compact spaces can be nicely characterized.

In particular, Henriksen and Gillman introduced the concept of quasi-\( F \) spaces in which every dense cozero-set is \( C^* \)-embedded ([2]). Each space \( X \) has the minimal quasi-\( F \) cover \((QF(X), \Phi_X)\) ([5]). In [5], authors investigated when \( \beta QF(X) = QF(\beta X) \) and \( QF(X) = \Phi_{\beta X}^{-1}(X) \), where \((QF(\beta X), \Phi_{\beta X})\) is the minimal quasi-\( F \) cover of \( \beta X \).

It is well-known that each space has the minimal extremally disconnected cover \((E(X), k_X)\) and that \( \beta E(X) = E(\beta X) \) ([8]). Moreover,
internal characterizations of a space $X$ that is equivalent to $E(vX) = vE(X)$ is known ([8]). Similar results for the minimal basically disconnected cover $(\Lambda X, \Lambda X)$ are given by [7].

For any space $X$, there is an extension $(kX, kX)$ of $X$ such that

1. $kX$ is a weakly Lindelöf space, and
2. for any continuous map $f : X \to Y$, there is a continuous map $f^k : kX \to kY$ such that $f^k|_X = f([10])$.

The purpose to write this paper is to find the relation of the minimal quasi-$F$ cover $QF(kX)$ of $kX$ and $kQF(X)$. For any space $X$, we show that $QF(kX)$ is a weakly Lindelöf space and $\Phi^k_{kX} : QF(kX) \to kX$ is a $z^\#$-irreducible map and that $QF(\beta X) = \beta QF(kX)$. Moreover, we show that $kQF(X) = QF(kX)$ if and only if $\Phi^k_{kX} : kQF(X) \to kX$ is an onto map and $QF(\beta X) = \beta QF(X)$.

For the terminology, we refer to [2] and [9].

2. Quasi-$F$ covers

Let $X$ be a space. It is well-known that the collection $\mathcal{R}(X)$ of all regular closed sets in $X$, when partially ordered by inclusion, becomes a complete Boolean algebra, in which the join, meet, and complementation operations are defined as follows:

For any $A \in \mathcal{R}(X)$ and any $\mathcal{F} \subseteq \mathcal{R}(X)$,

\[
\bigvee \mathcal{F} = cl_X \left( \bigcup \{ F | F \in \mathcal{F} \} \right),
\]

\[
\bigwedge \mathcal{F} = cl_X \left( int_X \left( \bigcap \{ F | F \in \mathcal{F} \} \right) \right),
\]

\[
A^c = cl_X (X - A).
\]

A sublattic of $\mathcal{R}(X)$ is a subset of $\mathcal{R}(X)$ that contains $\emptyset$, $X$ and is closed under finite joins and finite meets ([8]).

A map $f : Y \to X$ is called a covering map if it is an onto continuous, perfect, and irreducible map ([8]).

**Lemma 2.1.** ([8])

1. Let $X$ be a dense subspace of $Y$. Then the map $\phi : R(Y) \to R(X)$, defined by $\phi(A) = A \cap X$, is a Boolean isomorphism.
2. Let $f : Y \to X$ be a covering map. Then the map $\psi : R(Y) \to R(X)$, defined by $\psi(A) = f(A)$, is a Boolean isomorphism.

In the above lemma, the inverse map $\phi^{-1} : R(X) \to R(Y)$ of $\phi$ is given by $\phi^{-1}(B) = cl_Y (B)$ ($B \in R(X)$) and the inverse map $\psi^{-1} : R(X) \to R(Y)$ of $\psi$ is given by $\psi^{-1}(B) = cl_Y (int_Y (f^{-1}(B))) = cl_Y (f^{-1}(int_X (B)))$ ($B \in R(X)$).
DEFINITION 2.2. A space $X$ is called a quasi-$F$ space if for any zero-sets $A, B$ in $X$, $cl_X(int_X(A \cap B)) = cl_X(int_X(A)) \cap cl_X(int_X(B))$, equivalently, every dense cozero-set in $X$ is $C^*$-embedded in $X$.

It is well-known that a space $X$ is a quasi-$F$ space if and only if $\beta X$ (or $vX$) is a quasi-$F$ space.

DEFINITION 2.3. Let $X$ be a space. Then a pair $(Y, f)$ is called

1. a cover of $X$ if $f : X \to Y$ is a covering map,
2. a quasi-$F$ cover of $X$ if $(Y, f)$ is a cover of $X$ and $Y$ is a quasi-$F$ space, and
3. a minimal quasi-$F$ cover of $X$ if $(Y, f)$ is a quasi-$F$ cover of $X$ and for any quasi-$F$ cover $(Z, g)$ of $X$, there is a covering map $h : Z \to Y$ such that $f \circ h = g$.

Let $X$ be a space, $Z(X) = \{Z \mid Z$ is a zero-set in $X\}$ and $Z(X)^\# = \{cl_X(int_X(A)) \mid A \in Z(X)\}$. Then $Z(X)^\#$ is a sublattice of $R(X)$.

Suppose that $X$ is a compact space. Let $QF(X) = \{\alpha \mid \alpha$ is a $Z(X)^\#$-ultrafilter\} and for any $A \in Z(X)^\#$, let $\sum_A Z(X)^\# = \{\alpha \in QF(X) \mid A \in \alpha\}$. Then the space $QF(X)$, equipped with the topology for which \{\(QF(X) - \sum_A Z(X)^\# \mid A \in Z(X)^\#\}\} is a base, is a quasi-$F$ space. Define the map $\Phi_X : QF(X) \to X$ by $\Phi_X(\alpha) = \cap A \mid A \in \alpha\}$. Then $(QF(X), \Phi_X) \text{is the minimal quasi-$F$ cover of } X \text{ and for any } A \in Z(X)^\#, \Phi_X(\sum_A Z(X)^\#) = A([4])$.

Let $X, Y$ be spaces and $f : Y \to X$ a map. For any $U \subseteq X$, let $f_U : f^{-1}(U) \to U$ denote the restriction and co-restriction of $f$ with respect to $f^{-1}(U)$ and $U$, respectively. For any space $X$, let $(QF(\beta X), \Phi_\beta)$ denote the minimal quasi-$F$ cover of $\beta X$.

We recall that a covering map $f : Y \to X$ is called $z^\# - \text{irreducible}$ if $f(Z(Y)^\#) = Z(X)^\#$. Let $f : Y \to X$ be a covering map and $Z$ a zero-set in $X$. By Lemma 2.1, $f(cl_Y(int_Y f^{-1}(Z))) = cl_X(int_X(Z))$ and $cl_Y(int_Y f^{-1}(Z)) = Z(Y)^\#$. Hence $Z(X)^\# \subseteq f(Z(Y)^\#)$ and so $f : Y \to X$ is $z^\#$-irreducible if and only if $f(Z(Y)^\#) \subseteq Z(X)^\#$. Using these we have the following:

PROPOSITION 2.4. Let $f : Y \to X$ and $g : W \to Y$ be covering maps. Then $f \circ g : W \to X$ is $z^\#$-irreducible if and only if $f : Y \to X$ and $g : W \to Y$ are $z^\#$-irreducible.

It is well-known that $\Phi_\beta$ is $z^\#$-irreducible ([5]).
3. Minimal quasi-$F$ covers of $kX$

A $z$-filter $F$ on a space $X$ is called real if $F$ is closed under the countable intersection.

For any space $X$, let $kX = vX \cup \{ p \in \beta X - vX \mid p \text{ is a real } z\text{-filter } F \text{ on } X \text{ such that } \cap \{cl_{\kappa X}(F) \mid F \in F\} = \emptyset \text{ and } p \in \cap \{cl_{\beta X}(F) \mid F \in F\} \}$. Then $kX$ is an extension of a space $X$ such that $vX \subseteq kX \subseteq \beta X$ ([10]).

We recall that a space $X$ is called a weakly Lindel"of space if for any open cover $U$ of $X$, there is a countable subfamily $V$ of $U$ such that $\cup \{V \mid V \in V\}$ is a dense subset of $X$.

**Lemma 3.1.** ([10]) For any space $X$, $kX$ is a weakly Lindel"of space.

It is well known that a space $X$ is weakly Lindel"of if and only if for any $Z(X)$-filter $\mathcal{A}$ with the countable meet property, $\cap \{A \mid A \in \mathcal{A}\} \neq \emptyset$.

Let $X$ be a space. For any $A \in Z(\beta X)$, let $\sum_A Z(\beta X) = \sum_A$ and $\sum_A \cap QF(kX) = \lambda_A$. Then for any $A \in Z(\beta X)$, $\Phi_\beta(\sum_A) = A$, and $\Phi_{kX}(\lambda_A) = A \cap kX$, because $QF(kX) = \Phi_\beta^{-1}(kX)$ and $\Phi_{kX} = \Phi_{kX}([7])$.

**Theorem 3.2.** Let $X$ be a space. Then we have the following:

1. $QF(kX)$ is a weakly Lindel"of space, and
2. $\Phi_{kX} : QF(kX) \rightarrow kX$ is a $z^\#$-irreducible map.

**Proof.** (1) Let $\mathcal{A}$ be a $z$-filter on $QF(kX)$ with the countable meet property and $\cap \{A \mid A \in \mathcal{A}\} = \emptyset$. Suppose that $\cap \{\Phi_{kX}(A) \mid A \in \mathcal{A}\} \neq \emptyset$. Pick $x \in \cap \{\Phi_{kX}(A) \mid A \in \mathcal{A}\}$. Since $\mathcal{A}$ is a $z$-filter on $QF(kX)$, $\mathcal{A}$ has the finite intersection property. Hence $\cap \{\Phi_{kX}(A) \mid A \in \mathcal{A}\}$ is a family of closed set in $\Phi_{kX}^{-1}(x)$ with the finite intersection property. Since $\Phi_{kX}^{-1}(x)$ is a compact subset in $QF(kX)$, $\cap \{A \cap \Phi_{kX}^{-1}(x) \mid A \in \mathcal{A}\} = \emptyset$ and so $\cap \{A \mid A \in \mathcal{A}\} \neq \emptyset$. This is a contradiction. Thus $\cap \{\Phi_{kX}(A) \mid A \in \mathcal{A}\} = \emptyset$.

(k) $kX$ is a weakly Lindel"of space, there is a sequence $(A_n)$ in $\mathcal{A}$ such that $cl_{\kappa X}(\cup \{kX - \Phi_{kX}(A_n) \mid n \in N\}) = kX$. Let $A \in \mathcal{A}$. Then $\Phi_{kX}(QF(kX) - A) \supseteq kX - \Phi_{kX}(A)$ and hence $\Phi_{kX}(A) \supseteq \Phi_{kX}(QF(kX) - A) \supseteq kX - \Phi_{kX}(A)$. Thus $cl_{\kappa X}(\cup \{\Phi_{kX}(A_n) \mid n \in N\}) = kX$. Note that

$kX = cl_{\kappa X}(\cup \{\Phi_{kX}(A_n) \mid n \in N\})$

$= cl_{\kappa X}(\Phi_{kX}(\cup \{A_n^\prime \mid n \in N\}))$

$= \Phi_{kX}(cl_{\kappa X}(\cup \{A_n^\prime \mid n \in N\}))$

$= \Phi_{kX}(\cup \{A_n^\prime \mid n \in N\})$.  

Since $\Phi_{kX}$ is a covering map, $\bigvee \{ A_n \mid n \in N \} = QF(kX)$ and so $\bigwedge (\bigvee \{ A_n \mid n \in N \})' = \{ A_n \mid n \in N \} = \emptyset$. Since $A$ has the countable meet property, it is a contradiction. Hence $\bigcap \{ A \mid A \in \mathcal{A} \} = \emptyset$ and so $QF(kX)$ is a weakly Lindelöf space.

(2) Take any zero-set $Z$ in $QF(kX)$. Since $QF(kX)$ is a weakly Lindelöf space, $QF(kX) - Z$ is an open weakly Lindelöf subspace of $QF(kX)$. Hence there is a sequence $(Z_n)$ in $Z(\beta X)^\#$ such that for any $n \in N$, $QF(kX) - (\Sigma Z_n \cap QF(kX)) \subseteq QF(kX) - Z$ and

$$
c\beta QF(kX)(\bigvee \{ QF(kX) - (\Sigma Z_n \cap QF(kX)) \mid n \in N \}) \cap (QF(kX) - Z)
= c\beta QF(kX)(\bigvee \{ QF(kX) - \lambda Z_n \mid n \in N \}) \cap (QF(kX) - Z)
= QF(kX) - Z.
$$

Hence $\bigvee \{ \lambda Z_n \mid n \in N \} \supseteq QF(kX) - Z \supseteq \bigvee \{ \lambda Z_n \mid n \in N \}$. Thus $\bigwedge \{ \lambda Z_n \mid n \in N \} = c\beta QF(kX)(\text{int}_{QF(kX)}(Z))$. Note that for any $A \in Z(\beta X)^\#$, $\Phi_{QF(kX)}(\lambda A) = A \cap kX$. By Lemma 2.1,

$$
\Phi_{QF(kX)}(c\beta QF(kX)(\text{int}_{QF(kX)}(Z)))
= \Phi_{QF(kX)}(\bigwedge \{ \lambda Z_n \mid n \in N \})
= \bigwedge \{ \Phi_{QF(kX)}(\lambda Z_n) \mid n \in N \}
= \bigwedge \{ Z_n \cap kX \mid n \in N \}.
$$

and hence $\Phi_{QF(kX)}(c\beta QF(kX)(\text{int}_{QF(kX)}(Z))) \in Z(kX)^\#$. Thus $\Phi_{QF(kX)}$ is a $z^\#$-irreducible map. □

Let $X$ be a space. Then $\beta QF(X) = QF(\beta X)$ if and only if $\Phi_X$ is $z^\#$-irreducible ([5]). Using this, we have the following:

**Corollary 3.3.** For any space, $QF(\beta X) = \beta QF(kX)$.

**Lemma 3.4.** ([10]) For any continuous map $f : X \to Y$, there is a unique continuous map $f^k : kX \to kY$ such that $f^k \circ k_X = k_f \circ f$.

Let $X$ be a space. Then there is a covering map $h : \beta QF(X) \to QF(\beta X)$ such that $\Phi_{\beta} \circ h \circ \beta QF(kX) = \beta_X \circ \Phi_X$. By Lemma 3.4, there is a continuous map $\Phi_X^k : kQF(X) \to kX$ such that $\Phi_X^k \circ k_{QF(X)} = k_X \circ \Phi_X$. Since $\Phi_{\beta}^{-1}(kX) = QF(kX)$, there is a continuous map $t_X : QF(kX) \to QF(kX)$ such that $j \circ t_X = h \circ \beta QF(kX)$ and $\Phi_{QF(kX)} \circ t_X = \Phi_X^k$, where $j : QF(kX) \to QF(\beta X)$ is a dense embedding. If $t_X$ is a homeomorphism, then we write $kQF(X) = QF(kX)$.

**Corollary 3.5.** Let $X$ be a space. If $kQF(X) = QF(kX)$, then $\beta QF(X) = QF(\beta X)$. 


Proof. Since \( t_X : kQF(X) \to QF(kX) \) is a homeomorphism and 
\( \Phi_{kX} : QF(kX) \to kX \) is \( z^\# \)-irreducible, \( \Phi_X^k : kQF(X) \to kX \) is \( z^\# \)-irreducible. Take any zero-set \( Z \) in \( \beta QF(X) \). Then, by Lemma 2.1, \( cl_{\beta QF(X)}(int_{\beta QF(X)}(Z)) \cap kQF(X) \in Z(kQF(X))^\# \) and 
\[
\Phi_X^k(cl_{\beta QF(X)}(int_{\beta QF(X)}(Z)) \cap kX) = \Phi_{\beta}(h(cl_{\beta QF(X)}(int_{\beta QF(X)}(Z)))) \cap kX \in Z(kX)^\#.
\]
By Lemma 2.1, \( \Phi_{\beta}(h(cl_{\beta QF(X)}(int_{\beta QF(X)}(Z)))) \in Z(\beta X)^\# \) and so \( \Phi_{\beta} \circ h \) is a \( z^\# \)-irreducible map. Proposition 2.4, \( h : \beta QF(X) \to QF(\beta X) \) is a \( z^\# \)-irreducible map. Since \( \beta QF(X) \) and \( QF(\beta X) \) are quasi-\( F \) spaces, \( h \) is a homeomorphism.

Let \( X \) be a space such that \( \beta QF(X) = QF(\beta X) \). By Corollary 3.3, there is a homeomorphism \( m_X : \beta QF(X) \to \beta QF(kX) \) such that \( \beta_{QF(kX)} \circ t_X = m_X \circ \beta_{QF(X)} \). Since \( m_X \circ \beta_{QF(X)} \) is an embedding, \( t_X \) is an embedding.

A subspace \( X \) of a space \( Y \) is called \( C^* \)-embedded in \( Y \) if for any real-valued continuous map \( f : X \to R \), there is a continuous map \( g : Y \to R \) such that \( g|_X = f \). For any space \( X \), \( X \) is \( C^* \)-embedded in \( \beta X \) and if \( X \supseteq Y \supseteq W \supseteq \beta X \), then \( Y \) is \( C^* \)-embedded in \( W ([2]) \). Hence we have the following

**Corollary 3.6.** Let \( X \) be a space such that \( \beta QF(X) = QF(\beta X) \). Then \( kQF(X) \) is a \( C^* \)-embedded subspace of \( QF(kX) \).

**Theorem 3.7.** Let \( X \) be a space. Then the following are equivalent:

1. \( kQF(X) = QF(kX) \),
2. \( t_X \) is an onto map and \( \beta QF(X) = QF(\beta X) \), and
3. \( \Phi_X^k \) is an onto map and \( \beta QF(X) = QF(\beta X) \).

**Proof.** (1) \( \Rightarrow \) (2) By Corollary 3.5, it is trivial.
(2) \( \Rightarrow \) (3) Since \( \Phi_X \) and \( t_X \) are onto maps, \( \Phi_X^k \) is an onto map.
(3) \( \Rightarrow \) (1) Let \( f = \Phi_X^k \). Take any \( x \in kX \). Since \( f \) is an onto map and \( \Phi_X \) is a covering map, \( f(kQF(X) - QF(X)) = kX - X([8]) \). Since \( \beta_{kX} \circ f = \Phi_{\beta} \circ h \circ \beta_{QF(X)} \), \( f^{-1}(x) = (\Phi_{\beta} \circ h)^{-1}(X) = \phi_{\beta}^{-1}(X) \subseteq kQF(X) - QF(X) \). Since \( \Phi_{\beta} \circ h \) is a covering map, \( f^{-1}(x) \) is a compact subset of \( kQF(X) \) and hence \( f \) is a compact map. By Corollary 3.6, \( f^{-1}(x) = \Phi_{\beta}^{-1}(x) \subseteq QF(kX) \).

Let \( F \) be a closed set in \( kQF(X) \) and \( x \in kX - f(F) \). Then \( f^{-1}(x) \cap F = \emptyset \). Since \( f^{-1}(x) \) is compact, there are \( A, B \in Z(\beta X)^\# \) such that \( f^{-1}(x) \subseteq \Sigma_A, F \subseteq \Sigma_B \) and \( A \cap B = \emptyset \). Since \( \Phi_{\beta}(\Sigma_B) = B \) and
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\[ \Phi_\beta^{-1}(x) \cap \Sigma_B = f^{-1}(x) \cap \Sigma_B = \emptyset, \; x \notin B. \]  Since \( \text{cl}_{kX}(f(F)) \subseteq B, \)
x \notin \text{cl}_{kX}(f(F)). Thus \( f \) is a closed map and so \( f \) is a perfect map.

Since \( m_X \circ \Phi_\beta \circ \beta_{kQF}(X) = \beta_{kX} \circ \Phi_X^k \) and \( m_X \circ \Phi_\beta \) is a covering map, 
\( \Phi_X^k \) is a covering map. Since \( kQF(X) \) is a quasi-$F$ space, there is a covering map \( l : kQF(X) \to QF(kX) \) such that \( \Phi_{QF(kX)} \circ l = \Phi_X^k \). Since \( QF(X) = \Phi_\beta^{-1}(X) \) and \( QF(kX) = \Phi_X^{-1}(kX) \), \( l \circ kQF(X) = t_X \circ kQF(X) \). Since \( kQF(X) \) is a dense embedding, \( l = t_X \) is a homeomorphism.

References


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