BOUNDLESSNESS AND COMPACTNESS OF SOME TOEPLITZ OPERATORS

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Abstract. We consider the problem to determine when a Toeplitz operator is bounded on weighted Bergman spaces. We introduce some set $CG$ of symbols and we prove that Toeplitz operators induced by elements of $CG$ are bounded and characterize when Toeplitz operators are compact and show that each element of $CG$ is related with a Carleson measure.

1. Introduction

Let $dA$ denote normalized Lebesgue area measure on the unit disk $\mathbb{D}$. For $\alpha > -1$, the weighted Bergman space $A^p_\alpha$ consists of the analytic functions in $L^p(\mathbb{D},dA_\alpha)$, where $dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^{\alpha} dA(z)$. Since $A^2_\alpha$ is a closed subspace of $L^2(\mathbb{D},dA_\alpha)$, for any $z \in \mathbb{D}$, there is a unique function $K^\alpha_z$ in $A^2_\alpha$ such that $f(z) = \langle f, K^\alpha_z \rangle$ for all $f \in A^2_\alpha$, in fact, $K^\alpha_z(w) = \frac{1}{(1 - \frac{z}{w})^{2+\alpha}}$ and the normalized reproducing kernel $k^\alpha_z$ is the function $\frac{K^\alpha_z(w)}{||K^\alpha_z||_{2,\alpha}} = \frac{(1 - |z|^{1+\frac{\alpha}{2}}}{(1 - \frac{z}{w})^{2+\alpha}}$, where the norm $|| \cdot ||_{p,\alpha}$ and the inner product are taken in the space $L^p(\mathbb{D},dA_\alpha)$ and $L^2(\mathbb{D},dA_\alpha)$, respectively.

For a linear operator $S$ on $A^2_\alpha$, $S$ induces a functions $\tilde{S}$ on $\mathbb{D}$ given by $\tilde{S}(z) = \langle Sk^\alpha_z, k^\alpha_z \rangle$, $z \in \mathbb{D}$. The function $\tilde{S}$ is called the Berezin transform of $S$.

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For \( u \in L^1(D, dA_\alpha) \), the Toeplitz operator \( T_u^\alpha \) with symbol \( u \) is the operator on \( A^2_\alpha \) defined by \( T_u^\alpha(f) = P_\alpha(uf) \), \( f \in A^2_\alpha \), where \( P_\alpha \) is the orthogonal projection from \( L^2(D, dA_\alpha) \) onto \( A^2_\alpha \) and let \( \tilde{u} \) denote \( T_u^\alpha \). Many mathematicians working in operator theory are interested in the boundedness and compactness of Toeplitz operators on the Bergman spaces. It is well-known that the Toeplitz operator \( T_u^\alpha \) induced by any element of \( L^\infty(D, dA_\alpha) \) is bounded. Since \( L^\infty(D, dA_\alpha) \) is dense in \( L^1(D, dA_\alpha) \), for any \( u \in L^1(D, dA_\alpha) \), \( T_u^\alpha \) is densely defined on \( A^2_\alpha \) but in general, \( T_u^\alpha \) is not bounded. We note that Berezin transforms and Carleson measures are useful tools in the study of Toeplitz operators ([2], [4], [5]). Using those tools, many mathematicians working in the operator theory characterized the boundedness and compactness of Toeplitz operators.

In this paper, we introduce some set \( CG \) and prove that Toeplitz operators induced by elements of \( CG \) are bounded and \( ||u||_G \) having vanishing property implies the compactness of Toeplitz operators \( T_u^\alpha \) and \( T_{\tilde{u}}^\alpha \).

Sections 3 contains some upper bounds of Toeplitz operators induced by elements of \( CG \) and relationship between elements of \( CG \) and Carleson measures and we deal with the compactness of appropriate products of Toeplitz operators and Hankel operators.

Throughout this paper, we use the symbol \( A \preceq B \) for nonnegative constants \( A \) and \( B \) to indicate that \( A \) is dominated by \( B \) time some positive constant and \( p' \) to denote the conjugate of \( p \), that is, \( \frac{1}{p} + \frac{1}{p'} = 1 \).

2. Some linear operators

A nice survey of previously known results connecting Toeplitz operators with bounded symbol can be found in [2].

For \( z \in D \), let \( \varphi_z(w) = \frac{z - w}{1 - \overline{z}w}. \) Then \( \varphi_z \) is an element of \( Aut(D) \) which is the set of all bianalytic map of \( D \) onto \( D \). Moreover, \( \varphi_z \circ \varphi_z \) is the identity map on \( D \) and \( Aut(D) \) is the Möbius group under composition.

For \( \alpha > -1 \) and \( z \in D \), let \( U_z^\alpha : L^2(D, dA_\alpha) \to L^2(D, dA_\alpha) \) be an isometry operator defined by

\[
U_z^\alpha f(w) = f \circ \varphi_z(w) \frac{(1 - |z|^2)^{1+\frac{\alpha}{2}}}{(1 - \overline{z}w)^{2+\alpha}},
\]

\( f \in L^2(D, dA_\alpha) \) and \( w \in D \).
Since \((1 - z\varphi_z(w))^{2+\alpha} = \left(\frac{1 - |z|^2}{1 - \overline{z}w}\right)^{2+\alpha}\), \((U^\alpha_z)^{-1} = U^\alpha_z\) and hence \(U^\alpha_z\) is a self-adjoint unitary operator on \(A^2_\alpha\) and \(U^\alpha_z 1 = k^\alpha_z(w)\).

For a linear operator \(S\) on \(A^2_\alpha\), define \(S_z\) by \(U^\alpha_z SU^\alpha_z\). Since \(U^\alpha_z\) is a self-inverse operator, \(S_z\) is the operator given by conjugation with \(U^\alpha_z\).

Now we are ready to state useful properties.

**Lemma 2.1.** For \(u \in L^1(\mathbb{D}, dA_\alpha)\) and \(z \in \mathbb{D}\), \((T^\alpha_u)_z = T^\alpha_{u \circ \varphi_z}\).

**Proof.** Take any \(f\) in \(A^2_\alpha\) and any \(w\) in \(\mathbb{D}\). Since \(U^\alpha_z\) is self-adjoint,

\[
U^\alpha_z T^\alpha_u(f)(w) = <U^\alpha_z T^\alpha_u(f), K^\alpha_w> = <U^\alpha_z uf, K^\alpha_w> = <(u \circ \varphi_z)(f \circ \varphi_z), \frac{(1 - |z|^2)^{1/2}}{(1 - \overline{z}w)^{2+\alpha}} K^\alpha_w> = <T^\alpha_{u \circ \varphi_z}(U^\alpha_z f), K^\alpha_w> = T^\alpha_{u \circ \varphi_z}(U^\alpha_z f)(w).
\]

Thus \((T^\alpha_u)_z = T^\alpha_{u \circ \varphi_z}\).

**Corollary 2.2.** For \(u_1, u_2, \ldots, u_n \in L^1(\mathbb{D}, dA_\alpha)\) and \(z \in \mathbb{D}\),

\[
U^\alpha_z T^\alpha_{u_1} T^\alpha_{u_2} \cdots T^\alpha_{u_n} U^\alpha_z = T^\alpha_{u_1 \circ \varphi_z} \cdots T^\alpha_{u_n \circ \varphi_z}.
\]

**Proof.** If follows immediately from the fact that \((U^\alpha_z)^{-1} = U^\alpha_z\) and Lemma 2.1.

**Proposition 2.3.** For \(u \in L^1(\mathbb{D}, dA_\alpha)\) and \(z \in \mathbb{D}\), \(\widetilde{T^\alpha_u} \circ \varphi_z = T^\alpha_u\) and hence \((\widetilde{T^\alpha_u})_z = T^\alpha_{u \circ \varphi_z} = T^\alpha_u \circ \varphi_z\).

**Proof.** Take any \(w\) in \(\mathbb{D}\). Since \(<u \circ \varphi_z k^\alpha_w, k^\alpha_w> = <uk^\alpha_{\varphi_z(w)}, k^\alpha_{\varphi_z(w)}>\),

\[
\widetilde{T^\alpha_u}(w) = <T^\alpha_{u \circ \varphi_z} k^\alpha_w, k^\alpha_w> = <u \circ \varphi_z k^\alpha_w, k^\alpha_w> = <uk^\alpha_{\varphi_z(w)}, k^\alpha_{\varphi_z(w)}> = <P^\alpha(u k^\alpha_{\varphi_z(w)}), k^\alpha_{\varphi_z(w)}> = \widetilde{T^\alpha_u}(\varphi_z(w)) = T^\alpha_u \circ \varphi_z(w).
\]

This completes the proof.

**Proposition 2.4.** If \(S : A^2_\alpha \to A^2_\alpha\) is a bounded linear operator then \(\tilde{S}\) and \(S_11\) are in \(L^2(\mathbb{D}, dA_\alpha)\).
Proof. Since \( \|S_1\|_{2,\alpha} = \|SU_{z}^1\|_{2,\alpha} \leq \|S\| \) and
\[
\|\tilde{S}\|_{2,\alpha} = \int_{\mathbb{D}} |\tilde{S}(z)|^2 dA_{\alpha}(z) \leq \int_{\mathbb{D}} \|S\|^2 dA_{\alpha}(z) = \|S\|^2,
\]  
\( \tilde{S} \) and \( S_{z} \) are in \( L^2(\mathbb{D}, dA_{\alpha}) \).

We notice that \( P_{\alpha} : L^2(\mathbb{D}, dA_{\alpha}) \rightarrow \mathcal{A}_{\alpha}^2 \) is bounded linear operator and hence for any \( u \in L^\infty(\mathbb{D}, dA_{\alpha}) \), \( \|P_{\alpha}(uf)\|_{2,\alpha} \leq \|u\|_{\infty} \|f\|_{2,\alpha} \). Thus \( T_{u}^\alpha \) is a bounded linear operator. Moreover, we extend the domain of \( C \) for any \( \sup_{z} \) imply a bounded Toeplitz operator. Let \( \bar{f} = \sum_{k=1}^{\infty} k\chi_{\left(\frac{1}{2k+1},\frac{1}{2k+2}\right)}(|z|) \) for all \( z \in \mathbb{D} \). Then \( f \) is a radial function and \( f \notin L^\infty(\mathbb{D}, dA_{\alpha}) \). Since
\[
\|f\|_{1,\alpha} = \int_{\mathbb{D}} |f(z)|(1 - |z|^2)^\alpha dA(z)
\leq \begin{cases}
(1 - \frac{1}{4}) \sum_{k=1}^{\infty} k^{2k+1}, & \alpha < 0 \\
\sum_{k=1}^{\infty} k^{2k+1}, & \alpha \geq 0
\end{cases}, f \in L^1(\mathbb{D}, dA_{\alpha}).
\]

For \( p > 2 \),
\[
\|\bar{T}_{f}^\alpha z_1\|_{p,\alpha} = \|U_{z}^\alpha T_{f}^\alpha U_{z}^1\|_{p,\alpha} \leq \|f k_{z}^\alpha\|_{p,\alpha} < \infty
\]
because \( \sup\{k_{z}^\alpha(w) : w \leq \frac{1}{2}\} \leq 2^{2+\alpha} \). Since for each \( z \in \mathbb{D} \),
\[
|f(z)| = \int_{\mathbb{D}} k_{z}^\alpha(w)|f(w)|dA_{\alpha}(w) \leq 2^{1+2\alpha} c \sum_{k=1}^{\infty} k^{2k+1}
\]
for some constant \( c \), \( |f|dA_{\alpha} \) is a Carleson measure and hence \( T_{f}^\alpha \) is a bounded linear operator. But every element of \( L^1(\mathbb{D}, dA_{\alpha}) \) does not imply a bounded Toeplitz operator. Let \( CG = \{u \in L^1(\mathbb{D}, dA_{\alpha}) : \sup_{z} \|\bar{T}_{u}^\alpha z_1\|_{p,\alpha} < \infty \text{ and } \sup_{z} \|\bar{T}_{u}^{\alpha*} z_{1}\|_{p,\alpha} < \infty \text{ for some } p \in (2, \infty)\} \).

Suppose \( f, g \in A_{\alpha}^2 \). Since \( \bar{T}_{u}^\alpha f, g \geq \bar{u} f, g \geq \bar{f}, T_{u}^\alpha g \geq \bar{T}_{u}^{\alpha*} \bar{T}_{u}^\alpha g \). If \( \|\bar{T}_{u}^\alpha z_1\|_{p,\alpha} < \infty \) then \( \|\bar{T}_{u}^{\alpha*} z_{1}\|_{p,\alpha} = \|\bar{T}_{u}^{\alpha*} z_{1}\|_{p,\alpha} < \infty \) and clearly \( CG \) is closed under the formation of conjugation and hence \( \{T_{u}^\alpha : u \in CG\} \) is self-adjoint in \( \mathcal{L}(A_{\alpha}^2) \) which is the set of all bounded linear operators on \( A_{\alpha}^2 \). Moreover, \( CG \) is a vector space over \( \mathbb{C} \) and we define \( \|u\|_{G} = \max_{z} \|\bar{T}_{u}^\alpha z_1\|_{p,\alpha}, \|\bar{T}_{u}^\alpha z_1\|_{p,\alpha} \).
By the above observation, \( L^\infty(\mathbb{D}, dA_\alpha) \) is a proper subset of \( CG \).

Since \( f(z) = 0 \) for all \(|z| > \frac{1}{2}\), \( \lim_{z \to \partial \mathbb{D}} T_\alpha^\varphi(z) = 0 = \lim_{z \to \partial \mathbb{D}} \| (T_\alpha^\varphi)_z 1 \|_{p,\alpha} \). Since \( T_\alpha^\varphi(z^n) \neq 0 \) for all \( n \in \mathbb{N} \), \( T_\alpha^\varphi \) has an infinite-dimensional range and hence it is not compact, that is, the vanishing property does not imply the compactness of Toeplitz operators.

3. Some operators

This section contains the boundedness of some operators. We begin by starting well-known lemma (see Lemma 3.10 in [5]) which is some integral estimates.

**Lemma 3.1.** Suppose \( a - 1 < \alpha \). If \( a + b < 2 + \alpha \) then
\[
\int_{\mathbb{D}} \frac{dA_\alpha(w)}{(1 - |w|^2)^a |1 - \overline{w}w|^b} \text{ is bounded on } \mathbb{D}.
\]

Note that \((T_u^\varphi)^* = T_u^\varphi\). Thus for \( z \in \mathbb{D} \),
\[
(T_u^\varphi)^* K_w^\alpha(z) = \langle (T_u^\varphi)^* K_w^\alpha, K_z^\alpha \rangle = \langle K_w^\alpha, T_u^\varphi K_z^\alpha \rangle = T_u^\varphi K_w^\alpha(z).
\]

Moreover, \( \| (T_u^\varphi)_z 1 \|_{t,\alpha} \) in the right side of the next lemma may not be finite but it will be infinite, making the corresponding inequality true.

**Lemma 3.2.** Suppose \( u \in L^1(\mathbb{D}, dA_\alpha) \) and \( 0 < a < 1 \). If \( 2 < \frac{2 + a}{a} < t \) then there is a constant \( c \) such that
\[
\int_{\mathbb{D}} \frac{|(T_u^\varphi K_w^\alpha)(w)|}{(1 - |w|^2)^a} dA_\alpha(w) \leq \frac{c \| (T_u^\varphi)_z 1 \|_{t,\alpha}}{(1 - |z|^2)^a}
\]
for all \( z \in \mathbb{D} \) and
\[
\int_{\mathbb{D}} \frac{|(T_u^\varphi K_z^\alpha)(w)|}{(1 - |w|^2)^a} dA_\alpha(z) \leq \frac{c \| (T_u^\varphi)_w 1 \|_{t,\alpha}}{(1 - |w|^2)^a}
\]
for all \( w \in \mathbb{D} \).

**Proof.** Take any \( z \) in \( \mathbb{D} \). Since \( U_z^\alpha 1 = k_z^\alpha \), \( T_u^\varphi K_z^\alpha = \frac{(T_u^\varphi)_z 1 \circ \varphi_z(\varphi_z)^{1 + \alpha}}{(1 - |z|^2)^{1 + \frac{\alpha}{2}}} \)

and hence put \( w = \varphi_z(\lambda) \) to obtain the following:
\[
\int_{D} \frac{|T_{u}^{\alpha} K_{w}^{\alpha}(w)|}{(1 - |w|^{2})^{\alpha}} dA_{\alpha}(w)
= \int_{D} \frac{|(T_{u}^{\alpha})_{z} 1(\lambda)||\varphi_{z}(\varphi_{z}(\lambda))|^{1+\frac{\alpha}{2}}}{(1 - |z|^{2})^{1+\frac{\alpha}{2}}(1 - |\varphi_{z}(\lambda)|^{2})^{\alpha}} |\varphi_{z}(\lambda)|^{2} (1 - |\varphi_{z}(\lambda)|^{2})^{\alpha} dA(\lambda)
= \frac{1}{(1 - |z|^{2})^{\alpha}} \int_{D} \frac{|(T_{u}^{\alpha})_{z} 1(\lambda)|}{|1 - \pi \lambda|^{2-2\alpha+\alpha} (1 - |\lambda|^{2})^{\alpha-\alpha} dA(\lambda)}
\leq \frac{|| (T_{u}^{\alpha})_{z} ||_{t, \alpha}}{(1 - |z|^{2})^{\alpha}} \left( \int_{D} \frac{dA(\lambda)}{|1 - \pi \lambda|^{(2-2\alpha+\alpha)t'}} \right)^{\frac{1}{t'}}.
\]

Here, the inequality comes from Hölder’s inequality.

If \((2 - a + \alpha)t' - \alpha < 2\) then the final integral is finite. Since \(\frac{2a+\alpha}{a} < t\), \(t' < \frac{2a+\alpha}{a}\). This makes the corresponding inequality true. The second inequality follows from the above observation.

**Corollary 3.3.** Suppose \(0 < a < 1\) and \(||u||_{G}\) is finite with respect to \(||\cdot||_{p, \alpha}\) for some \(p \in (2, \infty)\), that is, \(u \in CG\). If \(2 < \frac{2a+\alpha}{a} < p\) then there is a constant \(c\) such that

\[
\int_{D} \frac{|(T_{u}^{\alpha} K_{w}^{\alpha}(w))|}{(1 - |w|^{2})^{\alpha}} dA_{\alpha}(w) \leq \frac{c|| (T_{u}^{\alpha})_{z} ||_{p, \alpha}}{(1 - |z|^{2})^{\alpha}} \leq \frac{||u||_{G}}{(1 - |z|^{2})^{\alpha}}
\]

for all \(z \in D\) and

\[
\int_{D} \frac{|(T_{u}^{\alpha} K_{w}^{\alpha}(w))|}{(1 - |w|^{2})^{\alpha}} dA_{\alpha}(z) \leq \frac{c|| (T_{u}^{\alpha})_{w} ||_{p, \alpha}}{(1 - |w|^{2})^{\alpha}} \leq \frac{||u||_{G}}{(1 - |w|^{2})^{\alpha}}
\]

for all \(w \in D\).

**Proof.** If follows immediately from the definition of \(||u||_{G}\) and Lemma 3.2.

**Proposition 3.4.** If \(u \in CG\) and \(||u||_{G}\) is finite with respect to \(||\cdot||_{t, \alpha}\) then \(T_{u}^{\alpha}(h)(w) \leq \frac{1}{(1 - |w|^{2})^{1+\frac{\alpha}{2}}} ||h||_{2, \alpha} ||u||_{t, \alpha}\) for every \(h \in A_{\alpha}^{2}\) and every \(w \in D\).

**Proof.** Suppose \(h \in A_{\alpha}^{2}\) and \(w \in D\). Then

\[
(T_{u}^{\alpha}h)(w) = < T_{u}^{\alpha}h, K_{w}^{\alpha} > = \frac{1}{(1 - |w|^{2})^{1+\frac{\alpha}{2}}} \times < h, \overline{\pi k_{w}^{\alpha}} >.
\]

By Hölder’s inequality, we get

\[
| < h, \overline{\pi k_{w}^{\alpha}} > | \leq ||h||_{t', \alpha} ||\overline{\pi k_{w}^{\alpha}}||_{t, \alpha}.
\]

Since \(1 < t' < 2\) and \(A_{\alpha}(\mathbb{D}) = 1\), \(||h||_{t', \alpha} \leq ||h||_{t, \alpha}\) and hence one has the result.

\[\square\]
Suppose $f \in A^2_\alpha$ and $z \in \mathbb{D}$. Then
\[
(T^\alpha_u f)(z) = \langle T^\alpha_u f, K^\alpha_w \rangle = \int_\mathbb{D} f(w)((T^\alpha_u)^* K^\alpha_w)(w) dA_\alpha(w)
\]
Thus $T^\alpha_u$ is the integral operator with kernel $T^\alpha_u K^\alpha_w(z)$ and hence we find some upper bound of $\|T^\alpha_u\|_p$ to use the Schur test (see page 126 of [3]), where $\|T^\alpha_u\|_p$ is the operator norm on $A^p_\alpha$.

**Theorem 3.5.** Suppose $u \in CG$ and $\|u\|_G$ is finite with respect to $\| \cdot \|_{p,\alpha}$. If $pp'(2 + \alpha) < t$ then $T^\alpha_u$ is a bounded linear operator on $A^p_\alpha$ and $A^{p'}_\alpha$ and $\|T^\alpha_u\|_p \leq \|u\|_G$.

**Proof.** Since $0 < \frac{1}{pp'} < 1$, let $h(\lambda) = \frac{1}{(1-|\lambda|^2)^{pp'}}$. Then $h$ is a positive measurable function. Since $\|(T^\alpha_u)_1\|_{t,\alpha}$ and $\|(T^\alpha_w)_1\|_{t,\alpha}$ are less than or equal to $\|u\|_G$, the results follow from Lemma 3.2 and the Schur test.

Using the concept of a Carleson measure, we get the boundedness and compactness of Toeplitz operators.

**Proposition 3.6.** Suppose $u \in CG$ and $\|u\|_G$ is finite with respect to $\| \cdot \|_{t,\alpha}$.

1. Then $|u|dA_\alpha$ is a Carleson measure on $A^p_\alpha$ and hence $T^\alpha_u$ is a bounded linear operator.
2. If $\|(T^\alpha_u)_1\|_{t,\alpha} \to 0$ as $z \to \partial \mathbb{D}$ then $T^\alpha_u$ is compact.

**Proof.** (1) For $z \in \mathbb{D}$, $|\tilde{u}(z)| = |\langle T^\alpha_u k^\alpha_z, k^\alpha_z \rangle|$
\[
= (1 - |z|^2)^{1+\frac{\alpha}{2}} |\langle T^\alpha_u K^\alpha_z, k^\alpha_z \rangle|
\]
\[
\leq (1 - |z|^2)^{1+\frac{\alpha}{2}} \|T^\alpha_u K^\alpha_z\|_{2,\alpha}
\]
\[
= (1 - |z|^2)^{1+\frac{\alpha}{2}} \|(T^\alpha_u)_1\|_{2,\alpha}
\]
\[
\leq (1 - |z|^2)^{1+\frac{\alpha}{2}} \|(T^\alpha_u)_1\|_{t,\alpha},
\]
where the last inequality follows from $A_\alpha(\mathbb{D}) = 1$. Since $\tilde{u}$ is bounded, $|u|dA_\alpha$ is a Carleson measure on $A^p_\alpha$.

(2) In the proof of (1), for $z \in \mathbb{D}$, $|\tilde{u}(z)| \leq (1 - |z|^2)^{1+\frac{\alpha}{2}} \|(T^\alpha_u)_1\|_{t,\alpha}$ and hence $|u|dA_\alpha$ is a vanishing Carleson measure. Thus $T^\alpha_u$ is a compact linear operator.\qed
Corollary 3.7. Suppose \( u \in CG \) and \( \|u\|_G \) is finite with respect to \( || \cdot ||_{p,\alpha} \). If \( \|u\|_G \) vanishes on \( \partial \mathbb{D} \) then \( T^G_u \) and \( T^\alpha_u \) are compact operators.

Proof. It follows immediately from the fact that \( \|T^G_u\|_{1,\alpha} \) and \( \|T^\alpha_u\|_{1,\alpha} \) are less than or equal to \( \|u\|_G \).

Proposition 3.8. Suppose \( u \in CG \) and \( \|u\|_G \) is finite with respect to \( || \cdot ||_{t,\alpha} \). If \( T^G_u \) is a compact operator then \( \|T^\alpha_u\|_{1,\alpha} \to 0 \) as \( z \to \partial \mathbb{D} \) and hence \( u \) has the vanishing property on \( \partial \mathbb{D} \). Moreover, \( \|T^\alpha_u\|_{1,\alpha} \to 0 \) as \( z \to \partial \mathbb{D} \).

Proof. We note that \( H^\infty \) is dense in \( A^2_\alpha \). Take any \( f \) in \( A^2_\alpha \). Then \( (f, k^\alpha_z) = (1 - |z|^2)^{1+\frac{\alpha}{2}} f(z) \) and hence \( k^\alpha_z \to 0 \) weakly in \( A^2_\alpha \) as \( z \to \partial \mathbb{D} \). Since \( \|T^\alpha_u\|_{1,\alpha} = \|T^\alpha_u k^\alpha_z\|_{1,\alpha} \) and \( T^\alpha_u \) is compact, \( \|T^\alpha_u\|_{1,\alpha} \to 0 \) as \( z \to \partial \mathbb{D} \).

For \( u \in L^1(\mathbb{D}, dA_\alpha) \), we define an operator \( H^\alpha_u : A^2_\alpha \to (A^2_\alpha)^\perp \) by \( H^\alpha_u(g) = (I - P_\alpha)(ug) \), \( g \in A^2_\alpha \). Then \( H^\alpha_u \) is called the Hankel operator on the weighted Bergman space with symbol \( u \). Since \( L^\infty(\mathbb{D}, dA_\alpha) \) is dense in \( L^1(\mathbb{D}, dA_\alpha) \), \( H^\alpha_u \) is densely defined and if \( u \in L^\infty(\mathbb{D}, dA_\alpha) \) then \( \|H^\alpha_u\| \leq \|u\|_\infty \) and hence \( H^\alpha_u \) is bounded. By Lemma 2.1, \( (T^\alpha_u)_z = T^\alpha_{w\varphi_z} \) and hence \( \|(T^\alpha_u)_z\|_{1,\alpha} = \|H^\alpha_u k^\alpha_z\|_{1,\alpha} \leq \|H^\alpha_u\| \) and \( (H^\alpha_u)_z = (I - T^\alpha_u)_z = I - T^\alpha_{w\varphi_z} = H^\alpha_{w\varphi_z} \). Thus one has the following properties:

Proposition 3.9. Suppose \( u_1, u_2 \in L^1(\mathbb{D}, dA_\alpha) \) and \( u_1 = u_2 \circ \varphi_z \) for some \( z \in \mathbb{D} \). Then the following pairs are unitary equivalent:

1. \( T^\alpha_{u_1} \) and \( T^\alpha_{u_2} \)
2. \( H^\alpha_{u_1} \) and \( H^\alpha_{u_2} \).

Proposition 3.10. Suppose \( H^\alpha_u \) is bounded, where \( u \in L^1(\mathbb{D}, dA_\alpha) \). Then \( (H^\alpha_u)_z \) and \( H^\alpha_{w\varphi_z} k^\alpha_z \) are in \( L^2(\mathbb{D}, dA_\alpha) \) and \( H^\alpha_{w\varphi_z} \) is bounded.

Proof. By the above observation, \( (H^\alpha_u)_z \) and \( H^\alpha_{w\varphi_z} k^\alpha_z \) are in \( L^2(\mathbb{D}, dA_\alpha) \). Take any \( f \) in \( A^2_\alpha \). Since \( (H^\alpha_u)_z = H^\alpha_{w\varphi_z} k^\alpha_z \), \( \|H^\alpha_{w\varphi_z} f\|_{2,\alpha} = \|(H^\alpha_u)_z f\|_{2,\alpha} = \|H^\alpha_u U^\beta_z (f)\|_{2,\alpha} \leq \|H^\alpha_u\| \|f\|_{2,\alpha} \). This completes the proof.

Proposition 3.11. If \( u^2 \in CG \) then \( H^\alpha_u \) is bounded and hence we get the results of Proposition 3.10.

Proof. Take any \( f \) in \( A^2_\alpha \). By Proposition 3.6, \( \|u\|^2 dA_\alpha \) is a Carleson measure on \( A^2_\alpha \) and hence there is a constant \( c \) such that

\[
\int_{\mathbb{D}} |f(z)|^2 |u(z)|^2 dA_\alpha(z) \leq c \|f\|_{2,\alpha}^2.
\]
Then $\|H^\alpha_u(f)\|_{2,\alpha}^2 = \|(I - P_u)(uf)\|_{2,\alpha}^2 \leq \|uf\|_{2,\alpha}^2 \leq c\|f\|_{2,\alpha}^2$. Thus $H^\alpha_u$ is bounded.

Consider some products of Toeplitz operators and Hankel operators. Suppose $u, v, u^\alpha, v^\alpha \in A^2$. Since $\langle vf, P_u(ug) \rangle = \langle \overline{v}T^\alpha_v(f), g \rangle = \langle \overline{T^\alpha_wT^\alpha_v}(f), g \rangle = \langle \overline{T^\alpha_wT^\alpha_v}T^\alpha_u(f), g \rangle$ and $(H^\alpha_u)^*H^\alpha_v = T^\alpha_vT^\alpha_u$. In particular, if $u = v$ then $(H^\alpha_u)^*H^\alpha_u = T^\alpha_u - T^\alpha_uT^\alpha_u$. If $H^\alpha_u$ is compact then $(H^\alpha_u)^*H^\alpha_u$ is compact. Proposition 3.8 implies that $((H^\alpha_u)^*H^\alpha_u)^\sim(z) \to 0$ as $z \to \partial \mathbb{D}$ and hence $\|H^\alpha_u\|_{2,\alpha} \to 0$ as $z \to \partial \mathbb{D}$ because $\|H^\alpha_u\|_{2,\alpha}^2 = \langle H^\alpha_u, H^\alpha_u \rangle = \langle (H^\alpha_u)^*H^\alpha_u \rangle^\sim(z)$.

Suppose $u, v, u^\alpha, v^\alpha$ are in CG and $T^\alpha_u$ and $H^\alpha_u$ are compact. Then $(T^\alpha_u)^*$ and $(H^\alpha_u)^*$ are also compact. Since $U^\alpha_z$ is a bounded linear operator and $(T^\alpha_u)_z = U^\alpha_zT^\alpha_uU^\alpha_z$, the above equality implies that the following are compact:

1. $T^\alpha_uT^\alpha_v$
2. $T^\alpha_uT^\alpha_v$
3. $T^\alpha_uT^\alpha_v$
4. $(H^\alpha_u)^*H^\alpha_v$
5. $H^\alpha_u(H^\alpha_u)^*$
6. $T^\alpha_uT^\alpha_v$
7. $T^\alpha_uT^\alpha_v$
8. $T^\alpha_uT^\alpha_v$
9. $H^\alpha_uT^\alpha_u$
10. $H^\alpha_uT^\alpha_u$
11. $H^\alpha_uT^\alpha_u$
12. $T^\alpha_uT^\alpha_v$
13. $T^\alpha_vT^\alpha_v$
14. $(H^\alpha_u)^*H^\alpha_v$
15. $(H^\alpha_u)^*H^\alpha_v$

References


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