HILBERT 3-CLASS FIELD TOWERS OF REAL CUBIC FUNCTION FIELDS

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Abstract. In this paper we study the infiniteness of Hilbert 3-class field tower of real cubic function fields over $\mathbb{F}_q(T)$, where $q \equiv 1 \mod 3$. We also give various examples of real cubic function fields whose Hilbert 3-class field tower is infinite.

1. Introduction

Let $k = \mathbb{F}_q(T)$ be a rational function field over the finite field $\mathbb{F}_q$ of $q$ and $\mathbb{A} = \mathbb{F}_q[T]$. For a finite separable extension $F$ of $k$, write $O_F$ for the integral closure of $\mathbb{A}$ in $F$. Let $\infty$ be the infinite place of $k$ and $S_\infty(F)$ be the set of places of $F$ lying over $\infty$. Let $\ell$ be a prime number. The Hilbert $\ell$-class field of $F$ is the maximal unramified abelian $\ell$-extension of $F$ in which every places $\nu_\infty \in S_\infty(F)$ splits completely. Let $F_0^{(\ell)} = F$ and inductively, $F_n^{(\ell)}$ be the Hilbert $\ell$-class field of $F_n^{(\ell)}$ for $n \geq 0$ (cf. [4]). Then the sequence of fields

$$F_0^{(\ell)} = F \subset F_1^{(\ell)} \subset \cdots \subset F_n^{(\ell)} \subset \cdots$$

is called the Hilbert $\ell$-class field tower of $F$ and we say that the Hilbert $\ell$-class field tower of $F$ is infinite if $F_n^{(\ell)} \neq F_{n+1}^{(\ell)}$ for each $n \geq 0$. For a multiplicative abelian group $A$, let $r_\ell(A) = \dim_{\mathbb{F}_\ell}(A/A^\ell)$ be the $\ell$-rank of $A$. Let $\text{Cl}_F$ and $\text{O}_F^*$ be the ideal class group and the group of units of $O_F$, respectively. Schoof [6] proved that the Hilbert $\ell$-class field tower of $F$ is infinite if

$$r_\ell(\text{Cl}_F) \geq 2 + 2\sqrt{r_\ell(\text{O}_F^*)} + 1.$$
In [1], Ahn and Jung studied the infiniteness of the Hilbert 2-class field tower of imaginary quadratic function fields. Assume that $q$ is odd with $q \equiv 1 \mod 3$. In [3], we studied the infiniteness of the Hilbert 3-class field tower of imaginary cubic function fields. By a real cubic function field, we always mean a finite (geometric) cyclic extension $F$ over $k$ of degree 3 in which $\infty$ splits completely. The aim of this paper is to study the infiniteness of the Hilbert 3-class field tower of real cubic function fields and give a various examples of real cubic function fields which have infinite Hilbert 3-class field tower.

2. Preliminaries

2.1. Rédei matrix and the invariant $\lambda_2$

Assume that $q$ is odd with $q \equiv 1 \mod 3$. Fix a generator $\gamma$ of $\mathbb{F}_q^*$. Let $\mathcal{P}(\mathbb{A})$ be the set of all monic irreducible polynomials in $\mathbb{A}$. Assume that we have given a total ordering “$<$” on the set $\mathcal{P}(\mathbb{A})$ such that $P < Q$ for any $P, Q \in \mathcal{P}(\mathbb{A})$ with $\deg P < \deg Q$. Then any real cubic function field $F$ can be written as $F = k(\sqrt[3]{D})$, where $D = P_1^{r_1} \cdots P_t^{r_t}$ with $P_i \in \mathcal{P}(\mathbb{A})$, $r_i \in \{1, 2\}$ for $2 \leq i \leq t$ and $3 \mid \deg D$. If we assume that $P_1 < \cdots < P_t$ and $r_1 = 1$, then $D$ is uniquely determined by $F$, and we denote $D_F = D$. We say that $D_F$ is special if $3 \mid \deg P_i$ for all $1 \leq i \leq t$.

Let $\sigma$ be a generator of $G = \text{Gal}(F/k)$. We have

\[
\lambda_3(F) = \dim_{\mathbb{F}_3} \left( \frac{\text{Cl}_F^{(1-\sigma)}}{\text{Cl}_F^{(1-\sigma)^2}} \right).
\]

Put

\[
\lambda_i(F) = \dim_{\mathbb{F}_3} \left( \frac{\text{Cl}_F^{(1-\sigma)^i-1}}{\text{Cl}_F^{(1-\sigma)^i}} \right) \quad (i = 1, 2).
\]

For $0 \neq N \in \mathbb{A}$, write $\omega(N)$ for the number of distinct monic irreducible divisors of $N$. Then, by the Genus theory (see [2, Corollary 3.5]), we have

\[
\lambda_1(F) = \begin{cases} 
\omega(D_F) - 1, & \text{if } D_F \text{ is special,} \\
\omega(D_F) - 2, & \text{otherwise.}
\end{cases}
\]

Let $\eta = \gamma \frac{q - 1}{3}$. Let $M_F$ be the $t \times t$ matrix $(e_{ij})_{1 \leq i, j \leq t}$ over $\mathbb{F}_3$, where, for $1 \leq i \neq j \leq t$, $e_{ij} \in \mathbb{F}_3$ is defined by $\eta^{e_{ij}} = (\frac{D}{P_i})_3$, and the diagonal entries $e_{ii}$ are defined to satisfy the relation $\sum_{i=1}^t r_i e_{ii} = 0$. Let $d_i \in \mathbb{F}_3$ be defined by $d_i \equiv \deg P_i \mod 3$ for $1 \leq i \leq t$. We associate a $(t+1) \times t$ matrix $R_F$ over $\mathbb{F}_3$ to $F$ as follows:
Hilbert 3-class field towers of real cubic function fields

If \( D_F \) is non-special or \( D_F \) is special with \( \gamma \in \mathcal{N}_{F/k}(\mathcal{O}_F^*) \), then \( R_F \) is the \((t+1)\times t\) matrix obtained from \( M_F \) by adjoining \((d_1, \ldots, d_t)\) in last row.

- If \( D_F \) is special with \( \gamma \not\in \mathcal{N}_{F/k}(\mathcal{O}_F^*) \), then \( R_F \) is the \((t+1)\times t\) matrix obtained from \( M_F \) by adjoining \((e_{B1}, \ldots, e_{Bi})\) in last row, where \( B \in \mathbb{A} \) is a monic polynomial such that \( B = N(B) \), \( \mathfrak{B}^{\sigma-1} = x\mathcal{O}_F \), \( N_{F/k}(x) \in F_q^* \backslash F_q^{*3} \) and \( e_{Bi} \in F_3 \) is defined by \( \eta^{e_{Bi}} = (\frac{F}{\mathfrak{B}})_F^t \).

Then we have (see [2, Corollary 3.8])

\[
\lambda_2(F) = \omega(D_F) - 1 - \text{rank } R_F.
\]

Hence, by inserting (2.2) and (2.3) into (2.1), we have

\[
(2.3) \quad r_3(C_F) = \begin{cases} 2\omega(D_F) - 2 - \text{rank } R_F, & \text{if } D_F \text{ is special}, \\ 2\omega(D_F) - 3 - \text{rank } R_F, & \text{otherwise}. \end{cases}
\]

2.2. Some lemmas

Let \( E \) and \( K \) be finite (geometric) separable extensions of \( k \) such that \( E/K \) is a cyclic extension of degree \( \ell \), where \( \ell \) is a prime number not dividing \( q \). Let \( \gamma_{E/K} \) be the number of prime ideals of \( \mathcal{O}_K \) that ramify in \( E \) and \( \rho_{E/K} \) be the number of places \( \mathfrak{p}_\infty \in S_\infty(K) \) that ramify or inert in \( E \). If

\[
(2.5) \quad \gamma_{E/K} \geq |S_\infty(K)| - \rho_{E/K} + 3 + 2\sqrt{\ell |S_\infty(K)|} + (1 - \ell)\rho_{E/K} + 1,
\]

then \( E \) has infinite Hilbert \( \ell \)-class field tower (see [1, Proposition 2.1]). By using this criterion, we give some sufficient conditions for a real cubic function field to have infinite Hilbert 3-class field tower. For \( D \in \mathbb{A} \), write \( \pi(D) \) for the set of all monic irreducible divisors of \( D \).

**Lemma 2.1.** Assume that \( q \) is odd with \( q \equiv 1 \mod 3 \). Let \( F \) be a real cubic function field over \( k \). If there is a nonconstant monic polynomial \( D' \) such that \( 3 \mid \deg D' \), \( \pi(D') \subset \pi(D_F) \) and \( (\frac{D'}{\mathfrak{p}_i})_F^t = 1 \) for \( P_i \in \pi(D) \setminus \pi(D') \) \((1 \leq i \leq 5)\), then \( F \) has infinite Hilbert 3-class field tower.

**Proof.** Put \( K = k(\sqrt[3]{D'}) \) and \( E = FK \). Since \( \gamma_{E/K} \geq 15 \), \( |S_\infty(K)| = 3 \) and \( \rho_{E/K} = 0 \), the inequality (2.5) is satisfied, so \( E \) has infinite Hilbert 3-class field tower. By hypothesis, \( P_1, P_2, P_3, P_4, P_5 \) and \( \infty \) split completely in \( K \), so \( E \) is contained in \( F_1^{(3)} \). Hence \( F \) also has infinite Hilbert 3-class field tower. \( \square \)

**Lemma 2.2.** Assume that \( q \) is odd with \( q \equiv 1 \mod 3 \). Let \( F \) be a real cubic function field over \( k \). If there are two distinct nonconstant monic polynomials \( D_1, D_2 \) such that \( 3 \mid \deg D_i \), \( \pi(D_i) \subset \pi(D_F) \) for \( i = 1, 2 \) and
(D_1 P_j)_3 = (D_2 P_j)_3 = 1 for some P_j \in \pi(D_F) \setminus (\pi(D_1) \cup \pi(D_2)) (1 \leq j \leq 3), then F has infinite Hilbert 3-class field tower.

Proof. Put K = k(\sqrt[3]{D_1}, \sqrt[3]{D_2}) and E = FK. Since \gamma_{E/K} \geq 27, |S_\infty(K)| = 9 and \rho_{E/K} = 0, the inequality (2.5) is satisfied, so E has infinite Hilbert 3-class field tower. By hypothesis, P_1, P_2, P_3 and \infty split completely in K, so E is contained in F^{(3)}_1. Hence F also has infinite Hilbert 3-class field tower.

3. Hilbert 3-class field tower of real cubic function field

Assume that q is odd with q \equiv 1 \mod 3. In this section we give several sufficient conditions for real cubic function fields to have infinite Hilbert 3-class field tower and examples.

Let F be a real cubic function field. Since r_3(\mathcal{O}_F^*) = 2, by Schoof's theorem, the Hilbert 3-class field tower of F is infinite if r_3(\text{Cl}_F) \geq 6. By (2.2), F has infinite Hilbert 3-class field tower if \omega(D_F) \geq 7 or \omega(D_F) \geq 8 according as D_F is special or not. We will consider the cases that \omega(D_F) \leq 6 if D_F is special and \omega(D_F) \leq 7 if D_F is non-special.

Theorem 3.1. Assume that q is odd with q \equiv 1 \mod 3. Let F be a real cubic function field.

(i) Assume that D_F is special. If \omega(D_F) = 6 with rank R_F \leq 4, \omega(D_F) = 5 with rank R_F \leq 2 or \omega(D_F) = 4 with rank R_F = 0, then F has infinite Hilbert 3-class field tower.

(ii) Assume that D_F is non-special. If \omega(D_F) = 7 with rank R_F \leq 5, \omega(D_F) = 6 with rank R_F \leq 3 or \omega(D_F) = 5 with rank R_F \leq 1, then F has infinite Hilbert 3-class field tower.

Proof. By using the Schoof's theorem with (2.4), we see that F has infinite Hilbert 3-class field tower if

\[ \text{rank } R_F \leq \begin{cases} 2\omega(D_F) - 8, & \text{if } D_F \text{ is special,} \\ 2\omega(D_F) - 9, & \text{if } D_F \text{ is non-special.} \end{cases} \]

Hence the result follows immediately.

Theorem 3.2. Assume that q is odd with q \equiv 1 \mod 3. Let F be a real cubic function field with \omega(D_F) \geq 6. If there exists Q \in \pi(D_F) with 3|\deg Q and P_i \in \pi(D_F) \setminus \{Q\} such that \left(\frac{Q}{P_i}\right)_3 = 1 (1 \leq i \leq 5), then F has infinite Hilbert 3-class field tower.
By applying Lemma 2.1 with \( D' = Q \), we see that \( F \) has infinite Hilbert 3-class field tower.

**Proof.** By applying Lemma 2.1 with \( D' = Q \), we see that \( F \) has infinite Hilbert 3-class field tower.

**Example 3.3.** Let \( k = \mathbb{F}_7(T) \) and \( \mathbb{A} = \mathbb{F}_7[T] \). Take \( Q = T^3 + 2T^2 + 1 \), \( P_1 = T, P_2 = T + 2, P_3 = T + 3, P_4 = T^2 + 2 \) and \( P_5 = T^3 + T + 1 \). By simple computations, we see that \( (\frac{Q}{P_i})_3 = 1 \) for \( 1 \leq i \leq 5 \). Then, for any \( D = QeP_1^{e_1}P_2^{e_2}P_3^{e_3}P_4^{e_4}P_5^{e_5} \) with \( e, e_i \in \{1, 2\} \) and \( e_1 + e_2 + e_3 + 2e_4 \equiv 0 \mod 3 \), \( k(\sqrt[3]{D}) \) is a real cubic function field whose Hilbert 3-class field tower is infinite by Theorem 3.2.

Let \( N(n, q) \) be the number of monic irreducible polynomials of degree \( n \) in \( \mathbb{A} = \mathbb{F}_q[T] \). Then it satisfies the following one ([5, Corollary of Proposition 2.1]):

\[
N(n, q) = \frac{1}{n} \sum_{d | n} \mu(d)q^{\frac{n}{d}}.
\]

For \( \alpha \in \mathbb{F}_q^* \), let \( \mathcal{N}(n, \alpha, q) \) be the set of monic irreducible polynomials of degree \( n \) with constant term \( \alpha \) in \( \mathbb{A} = \mathbb{F}_q[T] \) and \( N(n, \alpha, q) = |\mathcal{N}(n, \alpha, q)| \). Define

\[
D_n = \{ r : r | (q^n - 1), r \nmid (q^m - 1) \text{ for } m < n \}.
\]

For each \( r \in D_n \), let \( r = m_r d_r \), where \( d_r = \gcd(r, \frac{q^n - 1}{q-1}) \). In [7], Yucas proved that \( N(n, \alpha, q) \) satisfies the following formula:

\[
(3.1) \quad N(n, \alpha, q) = \frac{1}{n \varphi(f)} \sum_{\substack{r \in D_n \cap \mathbb{N} \atop m_r = f}} \phi(r),
\]

where \( f \) is the order of \( \alpha \) in \( \mathbb{F}_q^* \).

**Example 3.4.** Let \( k = \mathbb{F}_7(T) \) and \( \mathbb{A} = \mathbb{F}_7[T] \). We will find the number \( \mathcal{N}(3, 1, 7) \). We have

\[
D_3 = \{ r : r | (T^3 - 1), r \nmid (T^2 - 1), r \nmid (7 - 1) \} = \{9, 18, 19, 38, 57, 114, 171, 342\}
\]

and \( m_9 = 1, m_{18} = 6, m_{19} = 1, m_{38} = 2, m_{57} = 1, m_{114} = 2, m_{171} = 3, m_{342} = 6 \). Since the order of 1 in \( \mathbb{F}_7^* \) is 1, by (3.1), we have

\[
\mathcal{N}(3, 1, 7) = \frac{1}{3 \varphi(1)} (\phi(9) + \phi(19) + \phi(57)) = 20.
\]

Hence, \( \mathcal{N}(3, 1, 7) \) consists of exactly 20 distinct monic irreducible polynomials. Take \( Q = T, P_1 = T^2 + 2T + 1 \) with \( \alpha \in \{0, 3, 4\} \) and \( P_2, P_3, P_4, P_5 \in \mathcal{N}(3, 1, 7) \). Then, for any \( D = QeP_1^{e_1}P_2^{e_2}P_3^{e_3}P_4^{e_4}P_5^{e_5} \) with \( e, e_i \in \{1, 2\} \) and \( e_1 + e_2 + e_3 + 2e_4 \equiv 0 \mod 3 \), we have \( (\frac{Q}{P_i})_3 = 1 \).
for $1 \leq i \leq 5$, so $k(\sqrt[D]{\Delta})$ is a real cubic function field whose Hilbert 3-class field tower is infinite by Theorem 3.2.

**Theorem 3.5.** Assume that $q$ is odd with $q \equiv 1 \mod 3$. Let $F$ be a real cubic function field with $\omega(D_F) \geq 5$. If there exist $Q_1, Q_2 \in \pi(D_F)$ whose degrees are divisible by 3 and $P_i \in \pi(D_F) \setminus \{Q_1, Q_2\}$ such that $(\frac{Q_1}{P_i})_3 = (\frac{Q_2}{P_i})_3 = 1$ $(1 \leq i \leq 3)$, then $F$ has infinite Hilbert 3-class field tower.

**Proof.** By applying Lemma 2.2 with $D_1 = Q_1$ and $D_2 = Q_2$, we see that $F$ has infinite Hilbert 3-class field tower.

**Example 3.6.** Let $k = \mathbb{F}_7(T)$ and $A = \mathbb{F}_7[T]$. Take $P_1 = T, P_2 = T + 1, P_3 = T^2 + 1, Q_1 = T^3 + T^2 + 1$ and $Q_2 = T^3 + T + 1$. By simple computations, we see that $(\frac{Q_1}{P_i})_3 = (\frac{Q_2}{P_i})_3 = 1$ for $1 \leq i \leq 3$. Then, for any $D = P_1^{e_1}P_2^{e_2}P_3^{e_3}Q_1^{f_1}Q_2^{f_2}$ with $e_i, f_j \in \{1, 2\}$ and $e_1 + e_2 + 2e_3 \equiv 0 \mod 3$, $k(\sqrt[D]{\Delta})$ is a real cubic function field whose Hilbert 3-class field tower is infinite by Theorem 3.5.

**Theorem 3.7.** Assume that $q$ is odd with $q \equiv 1 \mod 3$. Let $F$ be a real cubic function field with non-special $D_F$. Assume that $\omega(D_F) = 6$ or $\omega(D_F) = 7$. If there exist $Q_1, Q_2, Q_3 \in \pi(D_F)$ whose degrees are divisible by 3 and $P_i \in \pi(D_F) \setminus \{Q_1, Q_2, Q_3\}$ such that $(\frac{Q_1Q_2}{P_i})_3 = (\frac{Q_1Q_3}{P_i})_3 = 1$ $(1 \leq i \leq 3)$, then $F$ has infinite Hilbert 3-class field tower.

**Proof.** By applying Lemma 2.2 with $D_1 = Q_1Q_2$ and $D_2 = Q_1Q_3$, we see that $F$ has infinite Hilbert 3-class field tower.

**Example 3.8.** Let $k = \mathbb{F}_7(T)$ and $A = \mathbb{F}_7[T]$. Take $P_1 = T, P_2 = T + 2, P_3 = T^2 + 1, Q_1 = T^3 + T + 1, Q_2 = T^3 + T^2 + 1$ and $Q_3 = T^3 + T - 1$. By simple computations, we see that

\[
\left(\frac{Q_1}{P_i}\right)_3 = \left(\frac{Q_1}{P_3}\right)_3 = 1 \text{ (1} \leq i \leq 3), \quad \left(\frac{Q_1}{P_2}\right)_3 = \eta^2, \quad \left(\frac{Q_2}{P_2}\right)_3 = \left(\frac{Q_3}{P_2}\right)_3 = \eta.
\]

Hence, we have

\[
\left(\frac{Q_1Q_2}{P_i}\right)_3 = \left(\frac{Q_1Q_3}{P_i}\right)_3 = 1 \text{ (1} \leq i \leq 3).
\]

Then, for any $D = P_1^{e_1}P_2^{e_2}P_3^{e_3}Q_1^{f_1}Q_2^{f_2}$ with $e_i, f_j \in \{1, 2\}$ and $e_1 + e_2 + 2e_3 \equiv 0 \mod 3$, $k(\sqrt[D]{\Delta})$ is a real cubic function field whose Hilbert 3-class field tower is infinite by Theorem 3.7.
References


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