WEAK SOLUTIONS OF GRADIENT FLOW OF LANDAU-DE GENNES ENERGY

Jinhae Park*

ABSTRACT. Taking into account the flexoelectric effects, we consider a gradient flow of Landau-de Gennes energy which generalizes the Oseen-Frank energy. In this article, we discuss existence of weak solutions of the gradient flow in an appropriate function space.

1. Introduction

Molecules in Nematic Liquid Crystals are described by a traceless symmetric second order tensor

\[ Q = \int_{S^2} \ell \otimes \ell f(\ell) \, d\ell \ - \frac{1}{3} I, \]

where \( f \) is a probability distribution function satisfying \( f(\ell) = f(-\ell) \) for all \( \ell \in S^2 \). Shapes of molecules are characterized by three eigenvalues of \( Q \) and the direction of a molecule is defined by the unit eigenvector whose corresponding eigenvalue has the largest magnitude. The order tensor \( Q \) is a measure of the local degree of orientational order in liquid crystals. The liquid crystal is said to be uniaxial if two eigenvalues of \( Q \) are equal, and it is biaxial when \( Q \) has three distinct eigenvalues. The tensor \( Q \) is zero in the isotropic phase. Since \( Q \) is a symmetric matrix, all eigenvalues of \( Q \) are real and expressed in term of \( Q \) as [4]

Received July 11, 2013; Accepted July 30, 2013.
2010 Mathematics Subject Classification: Primary 46T99, 34A34; Secondary 34K18, 49J99.

Key words and phrases: weak solution, Landau-de Gennes, Q-tensor.

*This study was financially supported by research fund of Chungnam National University in 2011.
\[
\begin{aligned}
\lambda_1 &= 2\sqrt{\frac{\text{tr} Q^2}{6}} \cos \alpha, \\
\lambda_2 &= 2\sqrt{\frac{\text{tr} Q^2}{6}} \left( -\frac{1}{2} \cos \alpha - \frac{\sqrt{3}}{2} \sin \alpha \right), \\
\lambda_3 &= 2\sqrt{\frac{\text{tr} Q^2}{6}} \left( -\frac{1}{2} \cos \alpha + \frac{\sqrt{3}}{2} \sin \alpha \right),
\end{aligned}
\]

where
\[
\cos(3\alpha) = -\frac{\sqrt{6\text{tr} Q^3}}{\text{tr} Q^2\sqrt{\text{tr} Q^2}}, \quad \sin(3\alpha) = \sqrt{1 - \frac{6(\text{tr} Q^3)^2}{(\text{tr} Q^2)^3}}, \quad \alpha \in \left[0, \frac{\pi}{3}\right].
\]

It follows from \(\text{tr} Q = 0\) and \(Q = Q^T\) that \(6(\text{tr} Q^3)^2 \leq (\text{tr} Q^2)^3\). Moreover, \(Q\) has two distinct eigenvalues if and only if \(6(\text{tr} Q^3)^2 = (\text{tr} Q^2)^3\). From (1.1), it can be easily seen that \(-\frac{1}{3} \leq \lambda_i \leq \frac{2}{3}\) for \(i = 1, 2, 3\). It then follows that \(\text{tr} Q^2 \leq \frac{1}{6}\).

If \(Q\) is expressed by
\[
Q = S_1 \left( m \otimes m - \frac{1}{3} I \right) + S_2 \left( n \otimes n - \frac{1}{3} I \right),
\]
where \(\{m, n, m \times n\}\) is an orthonormal basis for \(\mathbb{R}^3\) consisting of unit eigenvectors of \(Q\), then the eigenvalues are
\[
\frac{1}{3}(2S_1 - S_2), \quad -\frac{1}{3}(S_1 + S_2), \quad \frac{1}{3}(2S_2 - S_1).
\]

In the Landau-de Gennes theory, neglecting the higher derivatives and powers of \(Q\) the free energy density \(F\) for nematic liquid crystals is given by
\[
F(Q, \nabla Q) = \frac{1}{2} \left( L_1 Q_{\alpha \beta, \gamma} Q_{\alpha \beta, \gamma} + L_2 Q_{\alpha \beta, \gamma} Q_{\alpha \gamma, \gamma} + L_3 Q_{\alpha \beta, \gamma} Q_{\alpha \gamma, \beta} \right) + f_{\text{bulk}}(Q),
\]
where
\[
f_{\text{bulk}}(Q) = \frac{A}{2} \text{tr} Q^2 - \frac{B}{3} \text{tr} Q^3 + \frac{C}{4} \left( \text{tr} Q^2 \right)^2.
\]

The bulk energy \(f_{\text{bulk}}\) is a potential function for uniaxial nematic liquid crystals, meaning that \(f_{\text{bulk}}\) favors molecules to be uniaxial. In order to study biaxial liquid crystals, we need to add higher powers of \(Q\) to \(f_{\text{bulk}}\). In liquid crystals, there exists a polarization induced by a splay and bending distortion [2, 1]. Such a polarization is called flexoelectric polarization which is analogous to piezoelectric polarization in solids. The flexoelectric polarization can be written in terms of \(Q\) as
\[ \mathbf{P}^f = (P_1, P_2, P_3), \]

\[ P_i = \epsilon_3 Q_{ij,j} + \epsilon_4 Q_{jk,k} + \epsilon_5 Q_{ijj,k,k} + \text{higher order}. \]

Due to the appearance of the flexoelectric polarization, the following electrostatic equations (Maxwell’s equations) will be taken into account in the system

\[ \nabla \cdot (\epsilon(Q) \mathbf{E}) = - \nabla \cdot \mathbf{P}^f, \quad \nabla \times \mathbf{E} = 0, \tag{1.2} \]

where \( \epsilon(Q) \) is the dielectric permittivity tensor given by

\[ \epsilon(Q) = \epsilon_0 \mathbf{I} + \epsilon_1 Q + \epsilon_2 Q^2. \]

Hence the electrostatic energy is

\[ f_{\text{elec}} = - \frac{1}{2} (\epsilon(Q) \mathbf{E}) \cdot \mathbf{E} - \mathbf{P}^f \cdot \mathbf{E}. \]

If we let

\[ Q = \frac{3}{2} S (\mathbf{n} \otimes \mathbf{n} - \frac{1}{3} \mathbf{I}), \]

then

\[ \epsilon(Q) \mathbf{E} \cdot \mathbf{E} = \left( \epsilon_0 - \frac{\epsilon_1}{2} S + \frac{\epsilon_2}{4} S^2 \right) |\mathbf{E}|^2 + \frac{3}{2} S \left( \epsilon_1 + \frac{\epsilon_2}{2} S \right) (\mathbf{n} \cdot \mathbf{E})^2, \]

\[ \mathbf{P}^f = \epsilon_{11} (\nabla \cdot \mathbf{n}) \mathbf{n} + \epsilon_{33} \mathbf{n} \times \nabla \times \mathbf{n}, \]

\[ \epsilon_{11} = \frac{3}{2} \epsilon_3 S + \frac{3}{4} (2 \epsilon_5 - \epsilon_4) S^2, \quad \epsilon_{33} = \frac{3}{2} \epsilon_3 S + \frac{3}{4} (2 \epsilon_4 - \epsilon_5) S^2. \]

Then the permittivity \( \epsilon_\perp \) and dielectric anisotropic constant \( \epsilon_a \) are defined by

\[ \epsilon_\perp = \epsilon_0 - \frac{\epsilon_1}{2} S + \frac{\epsilon_2}{4} S^2, \quad \epsilon_a = \frac{3}{2} S \left( \epsilon_1 + \frac{\epsilon_2}{2} S \right). \]

Now, since eigenvalues of \( Q \) are in between \(-\frac{1}{3}\) and \(\frac{2}{3}\), we impose the following condition for strong ellipticity of (1.2)

\[ 3\epsilon_0 > \epsilon_1 \quad \text{if} \quad \epsilon_1 > 0, \quad \text{and} \quad 3\epsilon_0 > -2\epsilon_1 \quad \text{if} \quad \epsilon_1 \leq 0. \]

Since some material can have \( \epsilon_1 > 0 \) and \( S > 0 \), we have to include \( \epsilon_2 \)-term in order to satisfy solvability condition \( \epsilon_\perp > |\epsilon_a| \). For a sake of simplicity, we take \( \epsilon_4 = \epsilon_5 = 0 \) so that equations (1.2) become

\[ \nabla \cdot \left[ (\epsilon_0 \mathbf{I} + \epsilon_1 Q + \epsilon_2 Q^2) \nabla \varphi \right] = - \epsilon_3 \nabla \cdot (\nabla \cdot \mathbf{Q}), \tag{1.3} \]

where \( \nabla \cdot \mathbf{Q} = Q_{1ij} \mathbf{e}_x + Q_{2ij} \mathbf{e}_y + Q_{3ij} \mathbf{e}_z, \ {\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z} \) is a set of unit vectors in \( x, y, z \) axes respectively, and \( \varphi \) is an electric potential function, i.e. \( \mathbf{E} = \nabla \varphi \).
By Maxwell’s equation, the electrostatic energy functional can be written as
\[
\int_{\Omega} f_{\text{elec}} \, dx = -\frac{1}{2} \int_{\Omega} (\nabla \cdot Q) \cdot \nabla \varphi \, dx.
\]
The total energy functional \(E\) is
\[
E(Q, \varphi) = \int_{\Omega} \left\{ \frac{1}{2} L|\nabla Q|^2 + \frac{A}{2} \text{tr} \, Q^2 - \frac{B}{3} \text{tr} \, Q^3 + \frac{C}{4} (\text{tr} \, Q^2)^2 - \frac{1}{2} \varepsilon_3 (\nabla \cdot Q) \cdot \nabla \varphi \right\} \, dx.
\]
In the absence of a flow, equations for dynamic problems are
\[
\frac{\partial Q}{\partial t} = L\Delta Q - A Q + B \left( Q^2 - \frac{\text{tr} \, Q^2}{3} I \right) - C(\text{tr} \, Q^2) Q
\]
subject to
\[
\nabla \cdot \left( \varepsilon(Q) \nabla \varphi \right) = -\varepsilon_3 \nabla \cdot (\nabla \cdot Q) \quad \text{in } \Omega,
\]
and boundary conditions
\[
\frac{\partial Q(x,t)}{\partial \nu} = 0 \quad \text{on } \Gamma, \quad Q(x,t) = Q_1(x) \quad \text{on } \partial \Omega \setminus \Gamma,
\]
where \(Q_1\) is fixed.

From now on, we study existence of weak solutions of the system (1.4)-(1.5) with the boundary conditions (1.6).

2. A priori estimates

In this section, we study a priori estimates for solutions which will be used in the next section. Let us introduce
\[
W^{1,2}(\Omega, S_0) = \{ Q : \|Q\|_{L^2(\Omega)} + \|\nabla Q\|_{L^2(\Omega)} < \infty, Q : \Omega \to S_0 \},
\]
\[
H^1(\Omega) = \left\{ \psi \in H^1(\Omega) : \psi = 0 \text{ on } \Gamma, \frac{\partial \psi}{\partial \nu} = 0 \text{ on } \partial \Omega \setminus \Gamma \right\}.
\]
For any \(p > 0\), and \(t > 0\), we denote by \(L^p(0, t; V)\) the space of all functions \(Q : (0, t) \to V\) such that
\[
\int_0^t \|Q\|_V \, dt < \infty,
\]
\[
\frac{\partial Q(x,t)}{\partial \nu} = 0 \quad \text{on } \Gamma, \quad Q(x,t) = Q_1(x) \quad \text{on } \partial \Omega \setminus \Gamma,
\]
where \(Q_1\) is fixed.

From now on, we study existence of weak solutions of the system (1.4)-(1.5) with the boundary conditions (1.6).
where $V$ is a function space equipped with its norm $|| \cdot ||_V$. We look for a weak solution of the system (1.4), (1.5), and (1.6). In other words, the problem is to find $Q \in L^2(0, T; W^{1,2}(\Omega; S_0))$ and $\varphi \in L^2(0, T; H^1(\Omega))$ satisfying

\[
\begin{aligned}
&\begin{cases}
\frac{d}{dt} \int_{\Omega} \left( \frac{\partial Q}{\partial t} + L \nabla Q + A Q - B Q^2 + C(\text{tr} Q^2) Q \right) \cdot T \right) dx = \frac{1}{2} \epsilon_3 \int_{\Omega} \nabla \varphi \cdot (\nabla \cdot T) dx, \\
\int_{\Omega} (\epsilon(Q) \nabla \varphi) \cdot \nabla \psi dx = \int_{\Gamma} (\epsilon(Q_1) \nabla \varphi_0 \cdot \nu) \psi dA - \int_{\Omega} (\nabla \cdot Q) \cdot \nabla \psi dx, \\
\varphi(x, t) = \varphi_0(x) \quad \text{on} \quad \Gamma, \\
\frac{\partial \varphi(x, t)}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega \\
\end{cases}
\end{aligned}
\]

for all $T \in W^{1,2}(\Omega; S_0)$ and $\psi \in H^1_\Gamma(\Omega)$.

**Lemma 2.1.** Let $(Q, \varphi)$ be a solution pair of functions to (1.4), (1.5), and (1.6). Then

$Q \in L^2(0, T; W^{1,2}(\Omega; S_0)) \cap L^4(0, T; L^4(\Omega; S_0)), \quad \varphi \in L^2(0, T; H^1(\Omega))$.

**Proof.** Let $(Q, \varphi)$ be a solution pair of functions to (1.4), (1.5), and (1.6). Multiplying each equation in (1.4) by $Q_{ij}$ and integrating by parts followed by summing up, we obtain

\[
\frac{d}{dt} \int_{\Omega} |Q|^2 dx + \int_{\Omega} (L |\nabla Q|^2 + A \text{tr} Q^2 - B \text{tr} Q^3 + C(\text{tr} Q^2)^2) dx = \frac{1}{2} \epsilon_1 \int_{\Omega} \nabla \varphi \cdot (\nabla \cdot Q) dx.
\]

(2.2)

Similarly, multiplying (1.5) by $\varphi$ and integrating by parts yield

\[
\int_{\Omega} (\epsilon(Q) \nabla \varphi) \cdot \nabla \varphi dx = -\epsilon_3 \int_{\Omega} (\nabla \cdot Q) \cdot \nabla \varphi dx.
\]

(2.3)

Combining (2.2) with (2.3) we obtain

\[
\frac{d}{dt} \int_{\Omega} |Q|^2 dx + \int_{\Omega} \left( L |\nabla Q|^2 + C(\text{tr} Q^2)^2 + \frac{1}{2} (\epsilon(Q) \nabla \varphi) \cdot \nabla \varphi \right) dx = \int_{\Omega} (\text{tr} Q^3) dx.
\]

(2.4)

By Hölder inequality, choose $\eta > 0$ such that $C - B \eta^2 > 0$ and

\[
\int_{\Omega} \text{tr} Q^3 dx \leq \int_{\Omega} \left\{ \frac{1}{\eta^2} \text{tr} Q^2 + \eta^2(\text{tr} Q^2)^2 \right\} dx.
\]

(2.5)
It follows from (2.4) and (2.5) that
\[
\frac{d}{dt} \int_\Omega |Q|^2 \, dx + \int_\Omega \left( L\nabla Q \cdot \nabla \varphi \right) \, dx \leq \mathcal{M}||Q||_{L^2} + D, \tag{2.6}
\]
where \( \tilde{C} = C - \frac{1}{\eta^2} \), and \( \mathcal{M} = -A + \frac{B}{\eta^2} \). Hence we get
\[
\frac{d}{dt} ||Q||_{L^2}^2 \leq \mathcal{M}||Q||_{L^2}^2 + D,
\]
and Grownwall’s inequality leads us to have
\[
||Q(t)||_{L^2}^2 \leq ||Q(0)||_{L^2}^2 e^{\mathcal{M}T} + \frac{D}{\mathcal{M}} \left( e^{\mathcal{M}T} - 1 \right). \tag{2.7}
\]
This implies that
\[
\sup_{0 \leq t \leq T} ||Q(t)||_{L^2}^2 \leq ||Q(0)||_{L^2}^2 e^{\mathcal{M}T} + \frac{D}{\mathcal{M}} \left( e^{\mathcal{M}T} - 1 \right),
\]
and integrating (2.6) with respect to \( t \) yields
\[
\int_0^T \int_\Omega \left( L\nabla Q \cdot \nabla \varphi \right) \, dx \, dt < \infty.
\]
Since \( (\epsilon(Q)\nabla \varphi) \cdot \nabla \varphi \geq \lambda ||\nabla \varphi||^2 \) for some \( \lambda > 0 \), by Poincare inequality we have
\[
Q \in L^2(0, T; W^{1,2}(\Omega; S_0)) \cap L^4(0, T; L^4(\Omega; S_0)), \varphi \in L^2(0, T; H^1(\Omega)).
\]

\[\square\]

3. Existence of weak solution

**Theorem 3.1.** For any given \( T > 0, Q_0 \in L^2(\Omega; S_0) \), there exists a solution pair \( (Q, \varphi) \) to (2.1) such that \( Q \in L^2(0, T; W^{1,2}(\Omega; S_0)) \) and \( \varphi \in L^2(0, T; H^1(\Omega)) \). Moreover, if \( Q_0 \in W^{1,2}(\Omega; S_0) \), then
\[
Q \in C(0, T; W^{1,2}(\Omega; S_0)) \cap L^4(0, T; L^4(\Omega; S_0)), \frac{\partial Q}{\partial t} \in L^2(0, T; L^2(\Omega; S_0)).
\]

**Proof.** We use the Galerkin Method \([6]\) to obtain a weak solution \( (Q, \varphi) \) to (2.1). We first approximate \( W^{1,2}(\Omega; S_0) \) and \( H^1(\Omega) \) by increasing sequences of finite dimensional subspaces \( X^m \subset W^{1,2}(\Omega, S_0), \) and \( Y^m \subset H^1(\Omega) \) such that
\[
\cup_{m=1}^\infty X^m = W^{1,2}(\Omega, S_0), \cup_{m=1}^\infty Y^m = H^1(\Omega).
\]
For each $m \in \mathbb{N}$, let \( \{x_i\}_{i=1}^m \) and \( \{y_i\}_{i=1}^m \) be orthonormal bases for \( X^m \) and \( Y^m \), respectively. We first seek a solution pair \((Q^m, \varphi^m)\) in \( X^m \times Y^m \) in the form
\[
Q^m(x, t) = \sum_{i=1}^m p_i(t)x_i(x), \quad \varphi^m(x, t) = \sum_{i=1}^m q_i(t)y_i(x).
\]
Substituting \((Q^m, \varphi^m)\) for \((Q, \varphi)\) in (2.1), and taking \( T = x_j, \psi = y_k \), we obtain a system of nonlinear ordinary differential equations for \( \{p_i(t), q_i(t)\}_{i=1}^m \). It follows from the standard theory of ODEs that the new system has a unique solution on some interval \([0, t_m] \subset [0, T]\). By lemma 2.1, we know that
\[
\sup_{0 \leq t \leq T} \{||Q^m(t)||_{L^2}, ||\varphi^m(t)||_{L^2}\} < \infty.
\]

We extend \( Q^m, \varphi^m \) to the interval \([0, T]\) by the standard continuation method [3, 6]. Apply Lemma 2.1 again to show that \( \{Q^m\}_{m \in \mathbb{N}} \) is bounded in \( L^2(0, T; W^{1,2}(\Omega; S_0)) \cap L^4(0, T; L^4(\Omega; S_0)) \), and \( \{\varphi^m\}_{m \in \mathbb{N}} \) is bounded in \( L^2(0, T; H^1(\Omega)) \). Note that \( \{(\operatorname{tr}(Q^m)^2)Q^m\}_{m \in \mathbb{N}} \) is bounded in \( L^{\frac{4}{3}}((0, T) \times \Omega) \).

We can extract a subsequence (not relabeled) \( \{(Q^m, \varphi^m)\}_{m \in \mathbb{N}} \) such that
\[
Q^m \rightharpoonup \tilde{Q} \text{ weakly in } L^2(0, T; W^{1,2}(\Omega; S_0)),
\]
\[
Q^m \rightharpoonup \tilde{Q} \text{ weakly in } L^4(0, T; L^4(\Omega; S_0)),
\]
\[
(\operatorname{tr}(Q^m)^2)Q^m \rightharpoonup P \text{ weakly in } L^\frac{4}{3}((0, T) \times \Omega),
\]
\[
\varphi^m \rightharpoonup \tilde{\varphi} \text{ weakly in } L^2(0, T; H^1(\Omega)).
\]
Using the Sobolev imbedding \( W^{1,2} \subset L^4[5] \), we obtain imbeddings
\[
L^4(0, T; W^{1,2}(\Omega; S_0)) \hookrightarrow L^4((0, T) \times \Omega; S_0),
\]
\[
L^\frac{4}{3}((0, T) \times \Omega; S_0) \hookrightarrow L^\frac{4}{3}(0, T; [W^{1,2}(\Omega; S_0)]').
\]
It follows that \( \\{\frac{\partial Q^m}{\partial t}\}_{m \in \mathbb{N}} \) is bounded in \( L^\frac{4}{3}(0, T; [W^{1,2}(\Omega; S_0)]') \). Since \( \{Q^m\}_{m \in \mathbb{N}} \) is bounded in \( L^2(0, T; W^{1,2}(\Omega; S_0)) \), Aubin’s compactness shows that
\[
Q^m \rightarrow \tilde{Q} \text{ strongly in } L^2(0, T; L^2(\Omega; S_0)).
\]
This concludes that \( \operatorname{tr} \tilde{Q}^2 \tilde{Q} = P \), and therefore \( (\tilde{Q}, \tilde{\varphi}) \) is a weak solution pair.

**Corollary 1.** There exists a weak solution pair \((Q, \varphi)\) which belongs to \( L^2(0, \infty; W^{1,2}(\Omega; S_0)) \times L^2(0, \infty; H^1(\Omega)) \) to (1.4), (1.5), and (1.6).
Proof. As in the proof of lemma 2.1, multiplying (1.4),(1.5) by $Q$ and $\varphi$ followed by integration by parts we obtain

\begin{equation}
\int_{\Omega} \left( L|\nabla Q|^2 + A \text{tr} Q^2 - B \text{tr} Q^3 + C(\text{tr} Q^2)^2 + \frac{1}{2} (\epsilon(Q)\nabla \varphi) \cdot \nabla \varphi \right) \, dx \\
+ \frac{d}{dt} \int_{\Omega} |Q|^2 \, dx = 0
\end{equation}

Since $C > 0$, there exists $D$ such that $A \text{tr} Q^2 - B \text{tr} Q^3 + C(\text{tr} Q^2)^2 \geq -D$. It follows from (3.2) and the Poincaré inequality that

\begin{equation}
\frac{d}{dt} \int_{\Omega} |Q|^2 \, dx + M \int_{\Omega} |Q|^2 \, dx \leq D|\Omega|,
\end{equation}

where $M = \frac{L}{2} > 0$ with the Poincaré constant $K$. By Grownwall’s inequality we have

\[ ||Q(t)||_{L^2} \leq ||Q(0)||_{L^2} e^{-Mt} + \frac{D|\Omega|}{M} (1 - e^{-Mt}). \]

Therefore $\sup_{0 \leq t < \infty} ||Q(t)||_{L^2} \leq \frac{D|\Omega|}{M}$ and the proof is complete.

Next, we prove that such a weak solution is unique and it converges to an equilibrium solution of the energy functional $E$.

**THEOREM 3.2.** If $\epsilon_1 = \epsilon_2 = 0$ in (1.5), then there exists a unique weak solution to (1.4),(1.5), and (1.6).

**Proof.** Let $(Q_1, \varphi_1)$ and $(Q_2, \varphi_2)$ be two weak solutions. Then

\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |Q|^2 \, dx &+ \int_{\Omega} \left[ L|\nabla Q|^2 + (f'_{\text{bulk}}(Q_1) - f'_{\text{bulk}}(Q_2)) \cdot Q \right] \\
&= \frac{1}{2} \epsilon_3 \int_{\Omega} \nabla \varphi \cdot (\nabla \cdot Q) \, dx,
\end{align*}

\begin{align*}
\int_{\Omega} (\epsilon_0 \nabla \varphi) \cdot \nabla \varphi \, dx &= -\epsilon_3 \int_{\Omega} (\nabla \cdot Q) \cdot \nabla \varphi \, dx,
\end{align*}

where $Q = Q_1 - Q_2$, $\varphi = \varphi_1 - \varphi_2$. Plugging the second equation into the first one, we get

\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |Q|^2 \, dx &+ \int_{\Omega} \left( L|\nabla Q|^2 + \frac{1}{2} \epsilon_0 |\nabla \varphi|^2 \right) \, dx \\
&= \int_{\Omega} \left[ (f'_{\text{bulk}}(Q_1) - f'_{\text{bulk}}(Q_2)) \cdot Q \right] \, dx \\
&\leq M \int_{\Omega} |Q|^2 \, dx \text{ for some } M > 0.
\end{align*}
Hence \( ||Q||_{L^2} \leq ||Q(0)||_{L^2}e^t = 0 \) so that \( Q_1 = Q_2 \) and \( \varphi_1 = \varphi_2 \).

**Theorem 3.3.** If \( Q_0 \in W^{1,2}(\Omega, S_0) \), then there is a subsequence of solutions to (1.4) which converges to a solution of the steady state problem as \( t \to \infty \).

**Proof.** Multiplying individual equation by \( \frac{\partial Q_{ij}}{\partial t} \) and integrating by parts followed by summing up, we obtain

\[
\int_\Omega |Q_t|^2 = -\frac{d}{dt} \int_\Omega \left[ \frac{L}{2} |\nabla Q|^2 + A \text{tr} Q^2 - B \text{tr} Q^3 + C (\text{tr} Q^2)^2 \right] dx \\
+ \epsilon_3 \int_\Omega \nabla \cdot Q_t \cdot \nabla \varphi dx - \frac{d}{dt} \int_\Omega |\nabla \varphi|^2 dx \\
= \epsilon_3 \int_\Omega \nabla \cdot Q_t \cdot \nabla \varphi dx.
\]

Then

\[
\int_\Omega |Q_t|^2 = -\frac{d}{dt} \int_\Omega \left[ \frac{L}{2} |\nabla Q|^2 + A \text{tr} Q^2 - B \text{tr} Q^3 + C (\text{tr} Q^2)^2 + |\nabla \varphi|^2 \right] dx
\]

and

\[
\int_0^T \int_\Omega |Q_t|^2 dx \, dt = -\int_\Omega \left[ \frac{L}{2} |\nabla Q|^2 + A \text{tr} Q^2 - B \text{tr} Q^3 + C (\text{tr} Q^2)^2 + |\nabla \varphi|^2 \right]_{t=T} dx \\
+ \int_\Omega \left[ \frac{L}{2} |\nabla Q|^2 + A \text{tr} Q^2 - B \text{tr} Q^3 + C (\text{tr} Q^2)^2 + |\nabla \varphi|^2 \right]_{t=0} dx \\
\leq \int_\Omega \left[ \frac{L}{2} |\nabla Q|^2 + A \text{tr} Q^2 - B \text{tr} Q^3 + C (\text{tr} Q^2)^2 + |\nabla \varphi|^2 \right]_{t=0} dx + M|\Omega|,
\]

where \( M \) is the minimum value of \( A \text{tr} Q^2 - B \text{tr} Q^3 + C (\text{tr} Q^2)^2 \).

Hence we obtain \( Q_t \in L^2(0, \infty; L^2(\Omega; S_0)) \). This shows that

\[
\int_\Omega |Q_t(x, t_i)|^2 dx \to 0 \quad \text{as} \quad i \to \infty,
\]

for almost all sequence \( \{t_i\}_{i \in \mathbb{N}} \) satisfying \( t_i \to \infty \) as \( i \to \infty \). Furthermore, we also get

\[
(\nabla Q, \nabla \varphi) \in L^\infty(0, \infty; L^2(\Omega; S_0)) \times L^\infty(0, \infty; L^2(\Omega)).
\]
By Poincare inequality, we have
\((Q, \varphi) \in L^\infty(0, \infty; W^{1,2}(\Omega; S_0)) \times L^\infty(0, \infty; W^{1,2}(\Omega))\)
and there is a sequence \(\{t_i\} \in \mathbb{N}\) with \(t_i \to \infty\) as \(i \to \infty\) such that
\((Q(x, t_i), \varphi(x, t_i)) \to (\bar{Q}, \bar{\varphi})\) weakly in \(W^{1,2}\) as \(t_i \to \infty\).

Since \((Q, \varphi)\) is a weak solution pair,
\[
\begin{cases}
\frac{\partial \bar{Q}}{\partial t} + \langle \bar{L} \nabla \bar{Q} + A \bar{Q} - B \text{tr} \bar{Q}^2 + C(\text{tr} \bar{Q}^2) \bar{Q}, \nabla \tilde{Q} \rangle \\
+ \varepsilon \langle \nabla \varphi, \nabla \cdot \tilde{Q} \rangle = 0, \\
\int_{\Omega} (\nabla \varphi \cdot \nabla \psi + \nabla \cdot \bar{Q} \cdot \nabla \psi) \, dx = 0,
\end{cases}
\]
for all \(\tilde{Q} \in W^{1,2}(\Omega, S_0), \psi \in W^{1,2}(\Omega)\). Passing to the limit as \(t_i \to \infty\), we obtain
\[
\begin{cases}
\int_{\Omega} (\bar{L} \nabla \bar{Q} + A \bar{Q} - B \text{tr} \bar{Q}^2 + C(\text{tr} \bar{Q}^2) \bar{Q}) \cdot \nabla \tilde{Q} \\
+ \varepsilon \int_{\Omega} \nabla \varphi \cdot (\nabla \cdot \tilde{Q}) \, dx = 0, \\
\int_{\Omega} (\nabla \bar{\varphi} \cdot \nabla \psi + \nabla \cdot \bar{Q} \cdot \nabla \psi) \, dx = 0.
\end{cases}
\]
This completes the proof. \(\square\)

References


* Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea
E-mail: jhpark2003@gmail.com