A FAMILY OF CHARACTERISTIC CONNECTIONS

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Abstract. The characteristic connection is a good substitute for Levi-Civita connection in studying non-integrable geometries. In this paper we consider the homogeneous space $U(3)/(U(1) \times U(1) \times U(1))$ with a one-parameter family of Hermitian structures. We prove that the one-parameter family of Hermitian structures admit a characteristic connection. We also compute the torsion of the characteristic connections.

1. Introduction

The non-integrable geometries are studied by many mathematicians ([5], [6], [9]) and a very important tool in studying non-integrable geometries is the characteristic connection ([7]). The characteristic connection is a metric connection with a skew symmetric torsion which preserves a given $G$-structure. So, the characteristic connection is a good substitute for the Levi-Civita connection on a manifold with a non-integrable geometry where the holonomy group with respect to the Levi-Civita connection is the whole group $SO(n)$ and the geometric structure is not preserved by the Levi-Civita connection.

Recently, many geometric properties with respect to the characteristic connection are discussed. Eigenvalues of the generalised Dirac operators, Dirac operators with respect to the characteristic connection, and other geometric properties concerning the eigenvalue estimates are investigated ([3], [4], [10], [11]). In discussing the above geometric properties, examples of manifolds with geometric structures admitting a characteristic connection are needed. But not every geometric structure admits a characteristic connection.

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In [4], [12], an example of 6-dimensional homogeneous manifold with
a characteristic connection is given, which is in fact a nearly kähler man-
ifold. Furthermore, this manifold admits a split holonomy, a geometric
structure concerning the condition of the torsion and the holonomy. We
then use the generalized Dirac eigenvalue estimate for the split holonomy
([4]).

In this paper we prove that the homogeneous space $U(3)/(U(1) \times
U(1) \times U(1))$ admits a one-parameter family of Hermitian structures
with characteristic connection and compute the characteristic connec-
tions concretely.

It is well known that the difference of the characteristic connection,
denoted by $\nabla^ch$, from the Levi-Civita connection, denoted by $\nabla^g$, is the
torsion of the characteristic connection ([8]):

$$\nabla^ch_XY = \nabla^g_XY + \frac{1}{2}T(X,Y).$$

So, it suffices to compute the Torsion $T$ for the characteristic connection
$\nabla^ch$. For the torsion $T$ we use the following formula which is available
for 6-dimensional almost hermitian manifolds (Theorem 4.2 [2]).

$$(1.1) \quad T(X, Y, -) = N(X, Y) + d\Omega(JX, JY, J-).$$

In Section 2, we prove that the homogeneous space $U(3)/(U(1) \times
U(1) \times U(1))$ admits a one-parameter family of Hermitian structures
with characteristic connection.

In section 3, we compute the characteristic connections $\nabla^ch$ of $(M, g_t, J)$.

2. Characteristic connections

We begin with a well-known metric family for a homogeneous reductive
space. We refer to [1] and [12] for more informations.

Let $G := U(3)$ and $H := U(1) \times U(1) \times U(1) \subset G$ diagonally embed-
ded. Then $M := G/H$ is a 6-dimensional manifold with

$\mathfrak{g} = \mathfrak{u}(3) = \{ A \in M_3(\mathbb{C}) : A + \bar{A}^t = 0 \}, \quad \mathfrak{h} = \{ A \in \mathfrak{u}(3) : A \text{ is diagonal} \}.$

We define an Ad $(G)$-invariant inner product $\beta := -\frac{1}{2}\Re(\text{tr} AB), \quad A, B \in \mathfrak{u}(3)$ and decompose $\mathfrak{m} = \mathfrak{h}^\perp$ into

$$m_1 := \left\{ \begin{bmatrix} 0 & a & b \\ -\bar{a} & 0 & 0 \\ -\bar{b} & 0 & 0 \end{bmatrix} : a, b \in \mathbb{C} \right\}, \quad m_2 := \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & -\bar{c} & 0 \end{bmatrix} : c \in \mathbb{C} \right\}.$$
A family of characteristic connections

Then an Ad ($H$)-invariant inner product on $m$ defined by

$$\beta_t := \beta_m^{1 \times m_1} + 2t \beta_m^{2 \times m_2}$$

induces a left invariant metric $g_t$ on $G/H$ for each $t > 0$.

Let $D_{kl} = (d_{ij})$ be the $n \times n$ matrix with zero entries except that its $(k,l)$-entry is 1. Furthermore, let $E_{kl} := D_{kl} - D_{lk}$ for $k \neq l$ and $S_{kl} := i(D_{kl} + D_{lk})$. Then

$$\{e_1 := E_{12}, e_2 := S_{12}, e_3 := E_{13}, e_4 := S_{13}, e_5 := \frac{1}{\sqrt{2t}}E_{23}, e_6 := \frac{1}{\sqrt{2t}}S_{23}\}$$

is an orthonormal basis of $m$ with respect to $\beta_t$. As basis for $h$ we take $H_k = S_{kk}/2, k = 1, 2, 3$.

We now recall the Nijenhuis tensor

$$N(X,Y) = [JX,JY] - J[X,JY] - J[JX,Y] - [X,Y]$$

where $J^2 = -Id$ in an almost complex structure. Then it holds

$$N(X,Y) = N(Y,X),$$
$$N(X,JY) = -JN(X,Y) = N(JX,Y).$$

(2.1)

The Nijenhuis tensor $N$ as (2,1)-tensor field is already skew-symmetric from the definition, so the $N$-tensor as (3,0)-tensor is totally skew-symmetric if

$$N(X,Y,Z) = -N(X,Z,Y), \text{ for } X,Y,Z \in TM^{2n}.$$

We now consider a 2-form $\Omega$ and $J$ on $G/H$ as follows:

$$\Omega(X,Y) := e_{12} + e_{34} - e_{56} =: g_t(JX,Y) \text{ with } J^2 = -Id,$$

where $e_{ij} = e_i \wedge e_j$.

Then by computation,

$$J(e_1) = e_2, J(e_2) = -e_1, J(e_3) = e_4,$$
$$J(e_4) = -e_3, J(e_5) = -e_6, J(e_6) = e_5.$$  

(2.3)

**Theorem 2.1.** On $M = U(3)/(U(1) \times U(1) \times U(1))$ we consider a metric family $g_t$ and an almost complex structure $J$ as above. Then the characteristic connection exists for all $t > 0$.

**Proof.** It is well known that a 6-dimensional almost Hermitian manifold $(M^6, g, J)$ admits a characteristic connection if and only if its Nijenhuis tensor $N$ is totally skew-symmetric ([2] Theorem 4.2). We will actually show that in our case the tensor $N \equiv 0$.

For $X \in TM^{2n}$, (2.1) implies

$$N(X,JX) = -JN(X,X) = 0,$$
so by (2.3) we have
\[ N(e_1, e_2) = N(e_3, e_4) = N(e_5, e_6) = 0. \]

Now we compute
\[
N(e_1, e_3) = [Je_1, Je_3] - J[e_1, Je_3] - J[Je_1, e_3] - [e_1, e_3] \\
= -\sqrt{2t}e_5 - J(-\sqrt{2t}e_6) - J(\sqrt{2t}e_6) + \sqrt{2t}e_5 \\
= 0, \\
N(e_1, e_5) = [Je_1, Je_5] - J[e_1, Je_5] - J[Je_1, e_5] - [e_1, e_5] \\
= 1/\sqrt{2t}e_3 + J(1/\sqrt{2t}e_4) - J(1/\sqrt{2t}e_4) - 1/\sqrt{2t}e_3 \\
= 0, \\
= 1/\sqrt{2t}e_1 + J(1/\sqrt{2t}e_2) + J(1/\sqrt{2t}e_2) + 1/\sqrt{2t}e_1 \\
= 0.
\]

Here we use the following relations:
\[
[e_2, e_4] = [e_1, e_3] = -\sqrt{2t}e_5 \quad \text{and} \quad [e_1, e_4] = -[e_2, e_3] = -\sqrt{2t}e_6, \\
[e_1, e_5] = -[e_2, e_6] = 1/\sqrt{2t}e_3 \quad \text{and} \quad [e_1, e_6] = [e_2, e_5] = 1/\sqrt{2t}e_4, \\
[e_4, e_6] = [e_3, e_5] = -1/\sqrt{2t}e_1 \quad \text{and} \quad [e_3, e_6] = [e_4, e_5] = -1/\sqrt{2t}e_2.
\]

By (2.1), (2.3) we obtain
\[
N(e_1, e_4) = -JN(e_1, e_4) = N(e_2, e_4) = JN(e_2, e_3) = 0, \\
N(e_1, e_5) = JN(e_1, e_6) = -N(e_2, e_6) = JN(e_2, e_5) = 0, \\
N(e_3, e_5) = JN(e_3, e_6) = N(e_4, e_6) = -JN(e_4, e_5) = 0.
\]

Therefore, we have
\[
N(e_1, e_4) = N(e_2, e_3) = 0, \\
N(e_1, e_6) = N(e_2, e_5) = 0, \\
N(e_3, e_6) = N(e_4, e_5) = 0.
\]

So, we actually have \( N \equiv 0 \) and \( N \) is totally skew-symmetric. \( \square \)
3. The torsion of the characteristic connection

For the further computations we recall the following (X.2 [13]):

- The map $\Lambda_t$, which implies the Levi-Civita connection, is uniquely characterized by (X.2 [13]),

$$\Lambda_t(X)Y - \Lambda_t(Y)X = [X, Y]_m, \quad (3.1)$$

$$\beta_t(\Lambda_t(X)Y, Z) + \beta_t(Y, \Lambda_t(X)Z) = 0. \quad (3.2)$$

- For the metric $g_t, t > 0$, the map $\Lambda_t: m \rightarrow so(m)$ is defined by

$$\Lambda_t(X)Y = \frac{1}{2}[X, Y]_m, \quad \Lambda_t(X)B = t[X, B], \quad \Lambda_t(A)Y = (1 - t)[A, Y], \quad \Lambda_t(A)B = 0, \quad (3.3)$$

for $X, Y \in m_1, A, B \in m_2$. By direct computations we can check that the map $\Lambda_t(X)$ defined as (3.3) satisfies the conditions (3.1) and (3.2).

Let $(M, g)$ be a manifold with a characteristic connection. We denote the Levi-Civita connection and the characteristic connection by $\nabla^g$ and $\nabla^{ch}$, respectively. Then, it is well known that for $X, Y \in TM$ (see [8])

$$\nabla^{ch}_X Y = \nabla^g_X Y + \frac{1}{2} T(X, Y)$$

for some $(2,1)$-tensor $T$ which is known to be the torsion of the characteristic connection. So, it suffices to compute the above torsion $T$ for the characteristic connection $\nabla^{ch}$.

Furthermore, in a $6$-dimensional almost Hermitian manifold $(M, g, J)$, the torsion for $\nabla^{ch}$ satisfies (Theorem 4.2 [2])

$$T(X, Y, -) = N(X, Y) + d\Omega(JX, JY, J-) \quad (3.4)$$

Here the $(2,1)$-tensors $T, \Omega$ are considered as $(3,0)$-tensors. That is, for $T$ we define $T(X, Y, Z) = g(T(X, Y), Z)$, similarly for $N$.

For $g_t$ as above we now compute the Levi-Civita connection $\nabla^{gt}$ using (3.3). The map $\Lambda_{g_t}$ (see (3.3)) is simply denoted by $\Lambda_t$ and we consider $E_{ij}$ with respect to the orthonormal basis $e_i$ of $m$. So, $E_{ij}$ actually maps $e_i$ to $-e_j$. We recall the following lemma ([1], [12]).
Lemma 3.1. We identify $m$ with $\mathbb{R}^6$ and take $E_{ij}$ defined above as basis of $\mathfrak{so}(m)$. Then
\[
\Lambda_t(e_1) = \sqrt{t/2}(E_{35} + E_{46}), \quad \Lambda_t(e_2) = \sqrt{t/2}(E_{45} - E_{36}),
\]
\[
\Lambda_t(e_3) = \sqrt{t/2}(E_{26} - E_{15}), \quad \Lambda_t(e_4) = -\sqrt{t/2}(E_{16} + E_{25}),
\]
\[
\Lambda_t(e_5) = \frac{1-t}{\sqrt{2}t}(E_{13} + E_{24}), \quad \Lambda_t(e_6) = \frac{1-t}{\sqrt{2}t}(E_{14} - E_{23}).
\]

Proof. We compute $\Lambda(e_1)$. By (3.3) $\Lambda_t(e_1)e_i = \frac{1}{2}[e_1, e_i]_m^2$ for $i = 1, \cdots, 4$ and $\Lambda_t(e_1)e_j = t[e_1, e_j]$ for $j = 5, 6$. Hence,
\[
\Lambda_t(e_1)e_1 = 0,
\]
\[
\Lambda_t(e_1)e_2 = \frac{1}{2}[e_1, e_2]_m^2 = 0,
\]
\[
\Lambda_t(e_1)e_3 = \frac{1}{2}[e_1, e_3]_m^2 = -\sqrt{t/2}e_5,
\]
\[
\Lambda_t(e_1)e_4 = \frac{1}{2}[e_1, e_4]_m^2 = -\sqrt{t/2}e_6,
\]
\[
\Lambda_t(e_1)e_5 = t[e_1, e_5] = \sqrt{t/2}e_3,
\]
\[
\Lambda_t(e_1)e_6 = t[e_1, e_6] = \sqrt{t/2}e_4.
\]
We consider $E_{ij}$ which maps $e_i$ of $m$ and $e_j$ to $e_i$. Then we have
\[
\Lambda(e_1) = \sqrt{t/2}(E_{35} + E_{46}).
\]
We obtain the other results by similar computations. \hfill \Box

Theorem 3.1. The manifold $(M, g_t, J)$, $t > 0$ as above admits a characteristic connection
\[
\nabla^\mathfrak{g}_XY = \nabla^g_XY + \frac{1}{2}T(X, Y, -),
\]
with $T = (\sqrt{2t} - \sqrt{t/2} + \frac{1}{\sqrt{2}t})(e_{145} - e_{235}) - \sqrt{2}(e_{136} + e_{246})$, where $e_{ijk}$ means $e_i \wedge e_j \wedge e_k$ and $\nabla^t := \nabla^g$.

Proof. By (3.4), we need to compute $d\Omega$ and $N$ in $(M, g, J)$.

i) First $d\Omega$ is given by
\[
d\Omega = \sum_i e_i \wedge \nabla^g_{e_i} \Omega,
\]
so we compute $\nabla^g_{e_i} \Omega$, $\Omega = e_{12} + e_{34} - e_{56}, i = 1, \cdots, 6$. It is well known that the three 2-forms $w = e_{12}, e_{34}, e_{56}$ are invariant under the isotropy
representation. It is well known that ([12])

\[
\nabla^g_{e_i} w = \Lambda_t(e_i) w = \sum_j (e_j \cup \Lambda_t(e_i)) \land (e_j \cup w).
\]

(3.5)

Note that \(E_{ij}\) maps \(e_i\) to \(-e_j\), so \(E_{ij}\) can be identified with the two form \(-e_{ij}\).

From Lemma 3.1 \(\Lambda_t(e_1) = \sqrt{t/2}(E_{35} + E_{46})\) identified with \(-\sqrt{t/2}(e_{35} + e_{46})\), so \(e_j \cup \Lambda_t(e_1) = 0\) for \(j = 1, 2\) and (3.5) implies

\[
\nabla^t_{e_1} e_1 e_{12} = \nabla^t_{e_2} e_1 e_{12} = 0.
\]

Similarly

\[
\nabla^t_{e_3} e_{34} = \nabla^t_{e_4} e_{34} = 0
\]

and

\[
\nabla^t_{e_5} e_{56} = \nabla^t_{e_6} e_{56} = 0.
\]

Now

\[
\nabla^t_{e_1} e_{34} = \sum_j (e_j \cup \Lambda(e_1)) \land (e_j \cup e_{34})
\]

\[
= -\sqrt{t/2} \sum_j (e_j \cup (e_{35} + e_{46})) \land (e_j \cup e_{34})
\]

\[
= -\sqrt{t/2} \left( (e_3 \cup (e_{35} + e_{46})) \land (e_3 \cup e_{34})
\right.
\]

\[
+ \left( (e_4 \cup (e_{35} + e_{46})) \land (e_4 \cup e_{34}) \right)
\]

\[
= -\sqrt{t/2}(e_5 \land e_4 - e_6 \land e_3)
\]

\[
= \sqrt{t/2}(e_{45} - e_{36})
\]

and

\[
\nabla^t_{e_1} e_{56} = \sum_j (e_j \cup \Lambda(e_1)) \land (e_j \cup e_{56})
\]

\[
= -\sqrt{t/2} \sum_j (e_j \cup (e_{35} + e_{46})) \land (e_j \cup e_{56})
\]

\[
= -\sqrt{t/2} \left( (e_5 \cup (e_{35} + e_{46})) \land (e_5 \cup e_{56})
\right.
\]

\[
+ \left( (e_6 \cup (e_{35} + e_{46})) \land (e_6 \cup e_{56}) \right)
\]

\[
= -\sqrt{t/2}(-e_3 \land e_6 + e_4 \land e_5)
\]

\[
= \sqrt{t/2}(e_{36} - e_{45}).
\]
Similarly,
\begin{align*}
\nabla_t e_2 e_{34} &= -\sqrt{t/2} \sum_j (e_j \downarrow (e_{45} - e_{36})) \land (e_j \downarrow e_{34}) = -\sqrt{t/2} (e_{46} + e_{35}). \\
\nabla_t e_2 e_{56} &= -\sqrt{t/2} \sum_j (e_j \downarrow (e_{45} - e_{36})) \land (e_j \downarrow e_{56}) = \sqrt{t/2} (e_{46} + e_{35}). \\
\nabla_t e_3 e_{12} &= -\sqrt{t/2} \sum_j (e_j \downarrow (e_{26} - e_{15})) \land (e_j \downarrow e_{12}) = -\sqrt{t/2} (e_{16} + e_{25}). \\
\nabla_t e_3 e_{56} &= -\sqrt{t/2} \sum_j (e_j \downarrow (e_{26} - e_{15})) \land (e_j \downarrow e_{56}) = -\sqrt{t/2} (e_{16} + e_{25}). \\
\nabla_t e_4 e_{12} &= \sqrt{t/2} \sum_j (e_j \downarrow (e_{16} + e_{25})) \land (e_j \downarrow e_{12}) = \sqrt{t/2} (e_{15} - e_{26}). \\
\nabla_t e_4 e_{56} &= \sqrt{t/2} \sum_j (e_j \downarrow (e_{16} + e_{25})) \land (e_j \downarrow e_{56}) = \sqrt{t/2} (e_{15} - e_{26}). \\
\nabla_t e_5 e_{12} &= -\sqrt{t/2} \sum_j (e_j \downarrow (e_{13} + e_{24})) \land (e_j \downarrow e_{12}) = \sqrt{t/2} (e_{23} - e_{14}). \\
\nabla_t e_5 e_{34} &= -\frac{1-t}{\sqrt{2t}} \sum_j (e_j \downarrow (e_{13} + e_{24})) \land (e_j \downarrow e_{34}) = \frac{1-t}{\sqrt{2t}} (e_{14} - e_{23}). \\
\nabla_t e_5 e_{12} &= -\frac{1-t}{\sqrt{2t}} \sum_j (e_j \downarrow (e_{14} - e_{23})) \land (e_j \downarrow e_{12}) = \frac{1-t}{\sqrt{2t}} (e_{13} + e_{24}). \\
\nabla_t e_5 e_{34} &= -\frac{1-t}{\sqrt{2t}} \sum_j (e_j \downarrow (e_{14} - e_{23})) \land (e_j \downarrow e_{34}) = -\frac{1-t}{\sqrt{2t}} (e_{13} + e_{24}). \\
\end{align*}

Note that
\begin{align*}
\nabla_t e_1 e_{34} + \nabla_t e_1 e_{56} &= \nabla_t e_2 e_{34} + \nabla_t e_2 e_{56} = 0, \\
\nabla_t e_3 e_{12} &= \nabla_t e_3 e_{56}, \quad \nabla_t e_4 e_{12} = \nabla_t e_4 e_{56}. 
\end{align*}
So, we have

\[ d\Omega = \sum_i e_i \wedge \nabla^t_{e_i} \Omega \]

\[ = \sum_i e_i \wedge \nabla^t_{e_i} (e_{12} + e_{34} - e_{56}) \]

\[ = e_1 \wedge \nabla^t_{e_1} e_{34} - e_1 \wedge \nabla^t_{e_1} e_{56} + e_2 \wedge \nabla^t_{e_2} e_{34} - e_2 \wedge \nabla^t_{e_2} e_{56} \]

\[ + e_3 \wedge \nabla^t_{e_3} e_{12} - e_3 \wedge \nabla^t_{e_3} e_{56} + e_4 \wedge \nabla^t_{e_4} e_{12} - e_4 \wedge \nabla^t_{e_4} e_{56} + e_5 \wedge \nabla^t_{e_5} e_{12} + e_5 \wedge \nabla^t_{e_5} e_{34} + e_6 \wedge \nabla^t_{e_6} e_{12} + e_6 \wedge \nabla^t_{e_6} e_{34} \]

\[ = 2(e_1 \wedge \nabla^t_{e_1} e_{34} + e_2 \wedge \nabla^t_{e_2} e_{34}) \]

\[ + e_5 \wedge \nabla^t_{e_5} e_{12} + e_5 \wedge \nabla^t_{e_5} e_{34} + e_6 \wedge \nabla^t_{e_6} e_{12} + e_6 \wedge \nabla^t_{e_6} e_{34} \]

\[ = 2(\sqrt{t/2}e_1 \wedge (e_{45} - e_{36}) - \sqrt{t/2}e_2 \wedge (e_{46} + e_{35})) \]

\[ + \sqrt{t/2}e_5 \wedge (e_{23} - e_{14}) + \frac{1-t}{\sqrt{2t}}e_5 \wedge (e_{14} - e_{23}) \]

\[ + \frac{1-t}{\sqrt{2t}}e_6 \wedge (e_{13} + e_{24}) - \frac{1-t}{\sqrt{2t}}e_6 \wedge (e_{13} + e_{24}) \]

\[ = \sqrt{2t}(e_{145} - e_{136} - e_{246} - e_{235}) + \sqrt{t/2}(e_{235} - e_{145}) \]

\[ + \frac{1-t}{\sqrt{2t}}(e_{145} - e_{235}) \]

\[ = (\sqrt{2t} - \sqrt{t/2} + \frac{1-t}{\sqrt{2t}})(e_{145} - e_{235}) - \sqrt{2t}(e_{136} + e_{246}) \]

And from (2.3)

\[ d\Omega(J) = (\sqrt{2t} - \sqrt{t/2} + \frac{1-t}{\sqrt{2t}})(e_{145} - e_{136}) + \sqrt{2t}(e_{245} + e_{135}) \]

In the proof of Theorem 2.1 we see that \( N \equiv 0 \) which means

\[ T = N + d\Omega(J) = (\sqrt{2t} - \sqrt{t/2} + \frac{1-t}{\sqrt{2t}})(e_{145} - e_{235}) - \sqrt{2t}(e_{136} + e_{246}) \]

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