STABILITY OF A GENERALIZED POLYNOMIAL FUNCTIONAL EQUATION OF DEGREE 2 IN NON-ARCHIMEDEAN NORMED SPACES

CHANG-JU LEE* AND YANG-HI LEE**

Abstract. In this paper, we investigate the stability for the functional equation
\[ f(3x + y) - 3f(2x + y) + 3f(x + y) - f(y) = 0 \]
in the sense of M. S. Moslehian and Th. M. Rassias.

1. Introduction

The stability problem of the functional equation was formulated by S. M. Ulam [16] in 1940. D. H. Hyers [4], T. Aoki [1] and Th. M. Rassias [15] made important role to study the stability of the functional equation. During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians, see [2], [3], [6]-[13].

By a non-Archimedean field, we mean a field \( K \) equipped with a function (valuation) \( | \cdot | \) from \( K \) into \([0, \infty)\) such that \( |r| = 0 \) if and only if \( r = 0 \), \( |rs| = |r||s| \), and \( |r + s| \leq \max\{|r|, |s|\} \) for all \( r, s \in K \). Clearly \( |1| = |-1| \) and \( |n| \leq 1 \) for all \( n \in \mathbb{N} \). Let \( X \) be a vector space over a scalar field \( K \) with a non-Archimedean non-trivial valuation \( | \cdot | \). A function \( \| \cdot \| : X \to \mathbb{R} \) is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(i) \( \|x\| = 0 \) if and only if \( x = 0 \);
(ii) \( \|rx\| = |r|\|x\| \) (\( r \in K, x \in X \));.

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(iii) the strong triangle inequality, namely,
\[ \|x + y\| \leq \max\{\|x\|, \|y\|\} \]
for all \(x, y \in X\) and \(r \in \mathbb{K}\). Then \((X, \| \cdot \|)\) is called a non-Archimedean normed space. Due to the fact that
\[ \|x_n - x_m\| \leq \max\{|x_{j+1} - x_j| : m \leq j \leq n - 1\}(n > m), \]
a sequence \(\{x_n\}\) is Cauchy if and only if \(\{x_{n+1} - x_n\}\) converges to zero in a non-Archimedean normed space. By a complete non-Archimedean space, we mean one in which every Cauchy sequence is convergent.

\[ f(x + y) = f(x) + f(y) \]
and the quadratic functional equation
\[ f(x + y) + f(x - y) - 2f(x) - 2f(y) = 0 \]
in non-Archimedean normed spaces.

Now we consider the generalized polynomial functional equation of degree 2
\[ f(3x + y) - 3f(2x + y) + 3f(x + y) - f(y) = 0 \]
whose solution is called a general quadratic mapping. In 2009, the second author [9] obtained a stability of the generalized polynomial functional equation of degree 2 by taking and composing an additive mapping \(A\) and a quadratic mapping \(Q\) to prove the existence of a general quadratic function \(F\) which is close to the given function \(f\). In his processing, \(A\) is approximate to the odd part \(\frac{f(x) - f(-x)}{2}\) of \(f\) and \(Q\) is close to the even part \(\frac{f(x) + f(-x)}{2} - f(0)\) of it, respectively.

In this paper, we get a general stability result of the generalized polynomial functional equation of degree 2 in non-Archimedean normed spaces.

2. Stability of the generalized polynomial functional equation of degree 2

In this section, we prove the generalized Hyers-Ulam stability of the generalized polynomial functional equation of degree 2. Throughout this section, we assume that \(X\) is a non-Archimedean normed space and \(Y\) is a complete non-Archimedean space.

For a given mapping \(f : X \rightarrow Y\), we use the abbreviation
\[ Df(x, y) := f(3x + y) - 3f(2x + y) + 3f(x + y) - f(y) \]
for all $x, y \in X$.

**Lemma 2.1.** (Lemma 3.1 in [5]) If $f : X \to Y$ is a mapping such that $Df(x, y) = 0$ for all $x, y \in X \setminus \{0\}$, then $f$ is a general quadratic mapping.

**Theorem 2.2.** Let $\varphi : (X \setminus \{0\})^2 \to [0, \infty)$ be a function such that

\[
\lim_{n \to \infty} \frac{\varphi(2^n x, 2^n y)}{|4^n|} = 0
\]

for all $x, y \in X \setminus \{0\}$ and let for each $x \in X \setminus \{0\}$ the limit

\[
\lim_{n \to \infty} \max_{0 \leq j < n} \left\{ \frac{\varphi(2^j x, -2^j x)}{|2| \cdot |4|^{j+1}}, \frac{\varphi(-2^j x, 2^j x)}{|2| \cdot |4|^{j+1}} \right\},
\]

denoted by $\tilde{\varphi}(x)$, exists. Suppose that $f : X \to Y$ is a mapping satisfying the inequality

\[
\|Df(x, y)\| \leq \varphi(x, y)
\]

for all $x, y \in X \setminus \{0\}$. Then there exists a unique general quadratic mapping $T : X \to Y$ such that

\[
\|f(x) - T(x)\| \leq \tilde{\varphi}(x)
\]

for all $x \in X \setminus \{0\}$ with $T(0) = f(0)$. In particular, $T$ is given by

\[
T(x) = \lim_{n \to \infty} \frac{f(2^n x) + f(-2^n x) - 2f(0)}{2 \cdot 4^n} + \lim_{n \to \infty} \frac{f(2^n x) - f(-2^n x)}{2^{n+1}} + f(0)
\]

for all $x \in X$.

**Proof.** Let $J_n f : X \to Y$ be a mapping defined by

\[
J_n f(x) = \frac{f(2^n x) + f(-2^n x) - 2f(0)}{2 \cdot 4^n} + \frac{f(2^n x) - f(-2^n x)}{2^{n+1}} + f(0)
\]

for all $x \in X$ and all $n \in \mathbb{N}$. Notice that $J_0 f(x) = f(x)$ and
\[ \|J_j f(x) - J_{j+1} f(x)\| = \left\| - \frac{Df(2^j x, -2^j x)}{2 \cdot 4^{j+1}} - \frac{Df(-2^j x, 2^j x)}{2 \cdot 4^{j+1}} \right\| \\
\quad - \frac{Df(2^j x, -2^j x)}{2^{j+2}} + \frac{Df(-2^j x, 2^j x)}{2^{j+2}} \right\| \leq \max \left\{ \frac{\|Df(2^j x, -2^j x)\|}{|2| \cdot |4|^{j+1}}, \frac{\|Df(-2^j x, 2^j x)\|}{|2| \cdot |4|^{j+1}} \right\} \]

(2.5)

for all \( x \in X \setminus \{0\} \) and all \( j \geq 0 \). It follows from (2.5) and (2.1) that the sequence \( \{J_n f(x)\} \) is Cauchy for all \( x \in X \setminus \{0\} \). Since \( Y \) is complete and \( J_n f(0) = f(0) \) for all \( n \in \mathbb{N} \), we conclude that \( \{J_n f(x)\} \) is convergent for all \( x \in X \). Set

\[ T(x) := \lim_{n \to \infty} J_n f(x). \]

One can show that

\[ \|J_n f(x) - f(x)\| = \left\| \sum_{j=0}^{n-1} J_j f(x) - J_{j+1} f(x) \right\| \leq \max_{0 \leq j < n} \left\{ \frac{\varphi(2^j x, -2^j x)}{|2| \cdot |4|^{j+1}}, \frac{\varphi(-2^j x, 2^j x)}{|2| \cdot |4|^{j+1}} \right\} \]

(2.6)

for all \( n \in \mathbb{N} \) and all \( x \in X \setminus \{0\} \). By taking \( n \) to approach infinity in (2.6) and using (2.2) one obtains (2.4). Replacing \( x \) and \( y \) by \( 2^n x \) and \( 2^n y \), respectively, in (2.3) we get

\[ \|DJ_n f(x, y)\| = \left\| \frac{Df(2^n x, 2^n y) - Df(-2^n x, -2^n y)}{2^n+1} \right\| \\
\quad + \frac{Df(2^n x, 2^n y) + Df(-2^n x, -2^n y)}{2^{2n+1}} \leq \max \left\{ \frac{\varphi(2^n x, 2^n y)}{|2|^{n+1}}, \frac{\varphi(-2^n x, -2^n y)}{|2|^{n+1}}, \frac{\varphi(2^n x, 2^n y)}{|2| \cdot |4|^{n}}, \frac{\varphi(-2^n x, -2^n y)}{|2| \cdot |4|^{n}} \right\} \]

for all \( x, y \in X \setminus \{0\} \) and all \( n \in \mathbb{N} \). Taking the limit as \( n \to \infty \) and using (2.1) and Lemma 2.1 we get \( DT(x, y) = 0 \) for all \( x, y \neq 0 \) and so \( T \) is a
Let $y$ satisfy the inequality and it is easy to see that $\tilde{T}$. For all $x, y$

\begin{equation}
\frac{DT'(2^j x, -2^j x)}{2 \cdot 4^{j+1}} - \frac{DT'(-2^j x, 2^j x)}{2 \cdot 4^{j+1}}
\end{equation}

\begin{equation}
- \frac{DT'(2^j x, -2^j x)}{2^{j+2}} + \frac{DT'(-2^j x, 2^j x)}{2^{j+2}} + J_k T'(x)
\end{equation}

for any $k \in N$ and so

\begin{equation}
\|T(x) - T'(x)\| = \lim_{k \to \infty} \|J_k T(x) - J_k T'(x)\|
\end{equation}

\begin{equation}
\leq \lim_{k \to \infty} \max\{\|J_k T(x) - J_k f(x)\|, \|J_k f(x) - J_k T'(x)\|\}
\end{equation}

\begin{equation}
\leq \lim_{k \to \infty} |2|^{-2k-1} \max\{\|T(2^k x) - f(2^k x)\|, \|T(-2^k x) - f(-2^k x)\|,
\end{equation}

\begin{equation}
\|f(2^k x) - T'(2^k x)\|, \|f(-2^k x) - T'(-2^k x)\|\}
\end{equation}

\begin{equation}
\leq \lim_{k \to \infty} \lim_{n \to \infty} \max_{k \leq j < n + k} \left\{ \frac{\varphi(2^j x, -2^j x)}{|4^j + 2^n|}, \frac{\varphi(-2^j x, 2^j x)}{|4^j + 2^n|} \right\}
\end{equation}

\begin{equation}
= 0
\end{equation}

for all $x \in X \setminus \{0\}$. Since $T(0) = f(0) = T'(0)$, we get $T(x) = T'(x)$ for all $x \in X$. This completes the proof of the uniqueness of $T$. \hfill \square

Corollary 2.3. Let $2 < r$ be a real number and $|2| < 1$. If $f : X \to Y$ satisfies the inequality

\begin{equation}
\|Df(x, y)\| \leq \theta(\|x\|^r + \|y\|^r)
\end{equation}

for all $x, y \in X$, then there exists a unique general quadratic mapping $T : X \to Y$ such that

\begin{equation}
\|f(x) - T(x)\| \leq 2\theta|2|^{-3}\|x\|^r
\end{equation}

for all $x \in X$ with $T(0) = f(0)$.

Proof. Let $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$. Since $|2| < 1$ and $r - 2 > 0$,

\begin{equation}
\lim_{n \to \infty} |4|^{-n} \varphi(2^nx, 2^ny) = \lim_{n \to \infty} |2|^{n(r-2)} \varphi(x, y) = 0
\end{equation}

for all $x, y \in X$. Therefore the conditions of Theorem 2.2 are fulfilled and it is easy to see that $\varphi(x) = 2\theta|2|^{-3}\|x\|^r$. By Theorem 2.2 there is
a unique general quadratic mapping $T : X \to Y$ satisfying (2.7) with $T(0) = f(0)$.

**Theorem 2.4.** Let $\varphi : (X \setminus \{0\})^2 \to [0, \infty)$ be a function such that

$$(2.8) \lim_{n \to \infty} 2^n \varphi(2^{-n}x, 2^{-n}y) = 0$$

for all $x, y \in X \setminus \{0\}$ and let for each $x \in X \setminus \{0\}$ the limit

$$(2.9) \lim_{n \to \infty} \max_{0 \leq j < n} \left\{ |2|^{j-1} \varphi \left( \frac{x}{2^{j+1}}, \frac{-x}{2^{j+1}} \right), |2|^{j-1} \varphi \left( \frac{-x}{2^{j+1}}, \frac{x}{2^{j+1}} \right) \right\},$$

denoted by $\tilde{\varphi}(x)$, exists. Suppose that $f : X \to Y$ is a mapping satisfying the inequality

$$(2.10) \|Df(x, y)\| \leq \varphi(x, y)$$

for all $x, y \in X \setminus \{0\}$. Then there exists a unique general quadratic mapping $T : X \to Y$ such that

$$(2.11) \|f(x) - T(x)\| \leq \tilde{\varphi}(x)$$

for all $x \in X \setminus \{0\}$ with $T(0) = f(0)$. In particular, $T$ is given by

$$T(x) = \lim_{n \to \infty} \frac{4^n}{2} \left( f\left( \frac{x}{2^n} \right) + f\left( \frac{-x}{2^n} \right) - 2f(0) \right) + \lim_{n \to \infty} 2^{n-1} \left( f\left( \frac{x}{2^n} \right) - f\left( \frac{-x}{2^n} \right) \right) + f(0)$$

for all $x \in X$.

**Proof.** Let $J_n f : X \to Y$ be a mapping defined by

$$J_n f(x) = \frac{4^n}{2} \left( f(2^{-n}x) + f(-2^{-n}x) - 2f(0) \right) + 2^{n-1} \left( f\left( \frac{x}{2^n} \right) - f\left( \frac{-x}{2^n} \right) \right) + f(0)$$

for all $x \in X$ and $n \in \mathbb{N}$. Notice that $J_0 f(x) = f(x)$ and

$$\|J_n f(x) - J_{n+1} f(x)\| \leq \max \left\{ |2|^{j-1} \varphi \left( \frac{x}{2^{j+1}}, \frac{-x}{2^{j+1}} \right), |2|^{j-1} \varphi \left( \frac{-x}{2^{j+1}}, \frac{x}{2^{j+1}} \right) \right\}$$

for all $x \in X$. Therefore, for all $x \in X \setminus \{0\}$, we have

$$\lim_{n \to \infty} \frac{4^n}{2} \left( f\left( \frac{x}{2^n} \right) + f\left( \frac{-x}{2^n} \right) - 2f(0) \right) = \lim_{n \to \infty} \frac{4^n}{2} \left( f\left( \frac{x}{2^n} \right) - f\left( \frac{-x}{2^n} \right) \right) = f(0),$$

and

$$\lim_{n \to \infty} 2^{n-1} \left( f\left( \frac{x}{2^n} \right) - f\left( \frac{-x}{2^n} \right) \right) = 0.$$
for all \( x \in X \setminus \{0\} \) and all \( j \geq 0 \). It follows from (2.12) and (2.8) that the sequence \( \{ J_n f(x) \} \) is Cauchy for all \( x \in X \setminus \{0\} \). Since \( Y \) is complete and \( J_n f(0) = f(0) \), we conclude that \( \{ J_n f(x) \} \) is convergent for all \( x \in X \).

Set

\[
T(x) := \lim_{n \to \infty} J_n f(x)
\]

for all \( x \in X \). Using induction one can show that

\[
\| J_n f(x) - f(x) \| \leq \max_{0 \leq j < n} \left\{ |2|^{j-1} \phi \left( \frac{x}{2^{j+1}}, \frac{-x}{2^{j+1}} \right), |2|^{j-1} \phi \left( \frac{-x}{2^{j+1}}, \frac{x}{2^{j+1}} \right) \right\}
\]

for all \( n \in \mathbb{N} \) and all \( x \in X \setminus \{0\} \). By taking \( n \) to approach infinity in (2.13) and using (2.9) one obtains (2.11). Replacing \( x \) and \( y \) by \( 2^{-n}x \) and \( 2^{-n}y \), respectively, in (2.10) we get

\[
\| D J_n f(x, y) \| = \left\| 2^{n-1} D f \left( \frac{x}{2^n}, \frac{y}{2^n} \right) - 2^{n-1} D f \left( \frac{-x}{2^n}, \frac{-y}{2^n} \right) + 2^{2n-1} D f \left( \frac{x}{2^n}, \frac{y}{2^n} \right) + 2^{2n-1} D f \left( \frac{-x}{2^n}, \frac{-y}{2^n} \right) \right\|
\]

\[
\leq \max \left\{ |2|^{n-1} \phi \left( \frac{x}{2^n}, \frac{y}{2^n} \right), |2|^{n-1} \phi \left( \frac{-x}{2^n}, \frac{-y}{2^n} \right) \right\}
\]

for all \( x, y \in X \setminus \{0\} \). Taking the limit as \( n \to \infty \) and using (2.8) and Lemma 2.1 we get \( DT(x, y) = 0 \) for all \( x, y \neq 0 \) and so \( T \) is a general quadratic mapping. Now we are going to prove the uniqueness of \( T \). If \( T' \) is another general quadratic mapping satisfying (2.11) with \( T'(0) = f(0) \), then

\[
T'(x) - J_k T'(x) = \sum_{j=0}^{k-1} \left( (2^{2j-1} + 2^{j-1}) DT' \left( \frac{x}{2^{j+1}}, \frac{-x}{2^{j+1}} \right) \right. \\
+ (2^{2j-1} - 2^{j-1}) DT' \left( \frac{-x}{2^{j+1}}, \frac{x}{2^{j+1}} \right) \\
= 0
\]

for any \( k \in \mathbb{N} \) and so
\[ \|T(x) - T'(x)\| = \lim_{k \to \infty} \|J_k T(x) - J_k T'(x)\| \]
\[ \leq \lim_{k \to \infty} \max \{ \|J_k T(x) - J_k f(x)\|, \|J_k f(x) - J_k T'(x)\| \} \]
\[ \leq \lim_{k \to \infty} |2|^{k-1} \max \left\{ \left\| T \left( \frac{x}{2^k} \right) - f \left( \frac{x}{2^k} \right) \right\|, \left\| T \left( -\frac{x}{2^k} \right) - f \left( -\frac{x}{2^k} \right) \right\|, \left\| f \left( \frac{x}{2^k} \right) - T' \left( \frac{x}{2^k} \right) \right\|, \left\| f \left( -\frac{x}{2^k} \right) - T' \left( -\frac{x}{2^k} \right) \right\| \right\} \]
\[ \leq \lim_{k \to \infty} |2|^{k-1} \tilde{\phi} \left( \frac{x}{2^k} \right) \]
\[ = 0 \]
for all \( x \in X \setminus \{0\} \). Since \( T(0) = f(0) = T'(0) \), we get \( T(x) = T'(x) \) for all \( x \in X \). This completes the proof of the uniqueness of \( T \).

**Corollary 2.5.** Let \( r < 1 \) be a real number and \( |2| < 1 \). If \( f : X \to Y \) satisfies the inequality
\[ \|Df(x, y)\| \leq \theta(\|x\|^r + \|y\|^r) \]
for all \( x, y \in X \setminus \{0\} \), then there exists a unique general quadratic mapping \( T : X \to Y \) such that
\[ \|f(x) - T(x)\| \leq 2\theta |2|^{-1-r} \|x\|^r \]
for all \( x \in X \setminus \{0\} \) with \( T(0) = f(0) \).

**Proof.** Let \( \varphi(x, y) = \theta(\|x\|^r + \|y\|^r) \). Since \( |2| < 1 \) and \( 1 - r > 0 \), we get
\[ \lim_{n \to \infty} |2|^n \varphi(2^{-n}x, 2^{-n}y) = \lim_{n \to \infty} |2|^n |2|^{-1-r} \varphi(x, y) = 0 \]
for all \( x, y \in X \setminus \{0\} \). Therefore the conditions of Theorem 2.4 are fulfilled and it is easy to see that \( \tilde{\varphi}(x) = 2\theta |2|^{-1-r} \|x\|^r \). By Theorem 2.4 there is a unique general quadratic mapping \( T : X \to Y \) satisfying (2.14) with \( T(0) = f(0) \).

**Theorem 2.6.** Let \( \varphi : (X \setminus \{0\})^2 \to [0, \infty) \) be a function such that
\[ \lim_{n \to \infty} |4|^n \varphi(2^{-n}x, 2^{-n}y) = 0 \]
and
\[ \lim_{n \to \infty} \frac{\varphi(2^nx, 2^ny)}{|2|^n} = 0 \]
for all $x, y \in X \setminus \{0\}$ and let for each $x \in X \setminus \{0\}$ the limit
\[
\lim_{n \to \infty} \max_{0 \leq j < n} \left\{ |2|^{2j-1} \varphi \left( \frac{x}{2^{j+1}}, \frac{x}{2^{j+1}} \right), |2|^{2j-1} \varphi \left( \frac{-x}{2^{j+1}}, \frac{x}{2^{j+1}} \right) \right\},
\]
(2.17)
denoted by $\tilde{\varphi}(x)$, exists. Suppose that $f : X \to Y$ is a mapping satisfying the inequality
\[
\|Df(x, y)\| \leq \varphi(x, y)
\]
(2.18) for all $x, y \in X \setminus \{0\}$. Then there exists a unique general quadratic mapping $T : X \to Y$ such that
\[
\|f(x) - T(x)\| \leq \tilde{\varphi}(x)
\]
(2.19) for all $x \in X \setminus \{0\}$ with $T(0) = f(0)$. In particular, $T$ is given by
\[
T(x) = \lim_{n \to \infty} \frac{4n}{2} \left( f \left( \frac{x}{2^n} \right) + f \left( \frac{-x}{2^n} \right) - 2f(0) \right) + \lim_{n \to \infty} \frac{f(2^n x) - f(-2^n x)}{2^{n+1}} + f(0)
\]
for all $x \in X$.

Proof. Let $J_n f : X \to Y$ be a mapping defined by
\[
J_n f(x) = \lim_{n \to \infty} \frac{4n}{2} \left( f(2^{-n} x) + f(-2^{-n} x) - 2f(0) \right) + \frac{f(2^n x) - f(-2^n x)}{2^{n+1}} + f(0)
\]
for all $x \in X$ and $n \in \mathbb{N}$. Notice that $J_0 f(x) = f(x)$ and
\[
\|J_j f(x) - J_{j+1} f(x)\| = \left\| 2^{j-1} Df \left( \frac{x}{2^{j+1}}, \frac{-x}{2^{j+1}} \right) + 2^{j-1} Df \left( \frac{-x}{2^{j+1}}, \frac{x}{2^{j+1}} \right) \right\|
\]
\[
\leq \max \left\{ |2|^{2j-1} \varphi \left( \frac{x}{2^{j+1}}, \frac{-x}{2^{j+1}} \right), |2|^{2j-1} \varphi \left( \frac{-x}{2^{j+1}}, \frac{x}{2^{j+1}} \right) \right\},
\]
(2.20)
for all $x \in X \setminus \{0\}$ and all $j \geq 0$. It follows from (2.15), (2.16) and (2.20) that the sequence $\{J_n f(x)\}$ is Cauchy for all $x \in X \setminus \{0\}$. Since $Y$ is
complete and \( J_n f(0) = f(0) \) for all \( n \in \mathbb{N} \), we conclude that \( \{ J_n f(x) \} \) is convergent for all \( x \in X \). Set
\[
T(x) := \lim_{n \to \infty} J_n f(x).
\]
From (2.20) we have
\[
\| J_n f(x) - f(x) \| \leq \max_{0 \leq j < n} \left\{ |2|^{2j-1} \varphi \left( \frac{x}{2^j + 1}, \frac{-x}{2^j + 1} \right), |2|^{2j-1} \varphi \left( \frac{-x}{2^j + 1}, \frac{x}{2^j + 1} \right), \right. \]
\[
\frac{\varphi(2^j x, 2^j x)}{|2|^{j+2}}, \frac{\varphi(-2^j x, 2^j x)}{|2|^{j+2}} \right\}
\]
for all \( n \in \mathbb{N} \) and all \( x \in X \setminus \{0\} \). By taking \( n \) to approach infinity in (2.21) and using (2.17) one obtains (2.19). By using (2.18) we get
\[
\| DJ_n f(x, y) \| = \left\| \frac{Df(2^n x, 2^n y) - Df(-2^n x, -2^n y)}{2^{n+1}} \right. \]
\[
+ 2^{2n-1} Df \left( \frac{x}{2^n}, \frac{y}{2^n} \right) + 2^{2n-1} Df \left( \frac{-x}{2^n}, \frac{-y}{2^n} \right) \right\| \]
\[
\leq \max \left\{ \varphi \left( 2^n x, 2^n y \right), \varphi \left( -2^n x, -2^n y \right), \frac{\varphi(2^n x, 2^n y)}{|2|^{n+1}}, \frac{\varphi(-2^n x, -2^n y)}{|2|^{n+1}}, \right. \]
\[
|2|^{2n-1} \varphi \left( \frac{x}{2^n}, \frac{y}{2^n} \right), |2|^{2n-1} \varphi \left( \frac{-x}{2^n}, \frac{-y}{2^n} \right) \right\}
\]
for all \( x, y \in X \setminus \{0\} \) and all \( n \in \mathbb{N} \). Taking the limit as \( n \to \infty \) and using (2.15), (2.16) and Lemma 2.1 we get \( DT(x, y) = 0 \) for all \( x, y \neq 0 \) and so \( T \) is a general quadratic mapping. Now we are going to prove the uniqueness of \( T \). Assume that \( T' \) is another general quadratic mapping satisfying (2.19) with \( T'(0) = f(0) \). Then
\[
T'(x) = \sum_{j=0}^{k-1} \left( 2^{2j-1} DT' \left( \frac{x}{2^j+1}, \frac{-x}{2^j+1} \right) + 2^{2j-1} DT' \left( \frac{-x}{2^j+1}, \frac{x}{2^j+1} \right) \right. \]
\[
- \frac{DT'(2^j x, 2^j x)}{2^{j+2}} + \frac{DT'(-2^j x, 2^j x)}{2^{j+2}} \right) + J_k T'(x)
\]
for any \( k \in \mathbb{N} \) and so.
\[ ||T(x) - T'(x)|| = \lim_{k \to \infty} ||J_{2k}T(x) - J_{2k}T'(x)|| \leq \lim_{k \to \infty} \max\{||J_{2k}T(x) - J_{2k}f(x)||, ||J_{2k}f(x) - J_{2k}T'(x)||\} \]
\[ \leq \lim_{k \to \infty} \max\left\{ \left\| \frac{||T(2^{2k}x) - f(2^{2k}x)||}{|2|^{2k+1}} \right\|, \left\| \frac{||T(-2^{2k}x) - f(-2^{2k}x)||}{|2|^{2k+1}} \right\|, \left\| f(2^{2k}x) - T'(2^{2k}x) \right\|, \left\| f(-2^{2k}x) - T'(-2^{2k}x) \right\| \right\} \]
\[ \leq \lim_{k \to \infty} \max_{0 \leq j < n} \left\{ \left| 2 \right|^{j-2k-2} \varphi \left( \frac{2^{2k}x}{2^{j+1}}, \frac{-2^{2k}x}{2^{j+1}} \right) \right\} \]
\[ \leq \lim_{k \to \infty} \max_{0 \leq j < n} \left\{ \max_{0 \leq j < k} \left\{ \left| 2 \right|^{j-2k-2} \varphi \left( \frac{2^{2k}x}{2^{j+1}}, \frac{-2^{2k}x}{2^{j+1}} \right) \right\}, \max_{k \leq j < 2k} \left\{ \left| 2 \right|^{j-2k-2} \varphi \left( \frac{2^{2k}x}{2^{j+1}}, \frac{-2^{2k}x}{2^{j+1}} \right) \right\}, \max_{2k \leq j < n} \left\{ \left| 2 \right|^{j-2k-2} \varphi \left( \frac{2^{2k}x}{2^{j+1}}, \frac{-2^{2k}x}{2^{j+1}} \right) \right\} \right\} \]

(2.22)

for all \( x \in X \setminus \{0\} \) and all \( k \in \mathbb{N} \). On the other hand, we have the inequalities

\[ \lim_{k \to \infty} \lim_{n \to \infty} \max_{0 \leq j < n} \left\{ \left| 2 \right|^{j-2k-2} \varphi \left( \frac{2^{2k}x}{2^{j+1}}, \frac{-2^{2k}x}{2^{j+1}} \right) \right\} \]
\[ \lim_{k \to \infty} \max_{k \leq j < 2k} \left\{ |2|^{-4} \max_{0 \leq j < k} \left\{ \frac{\varphi(2^j x, 2^j x)}{|2|^j} \right\}, |2|^{k-4} \max_{0 \leq j < k} \left\{ \frac{\varphi(2^j x, 2^j x)}{|2|^j} \right\} \right\} \]
\[ = 0, \]
\[ \lim_{k \to \infty} \lim_{n \to \infty} \max_{0 \leq j < n} \left\{ |2|^{-2k-2} \max_{n \leq j < n-2k} \left\{ |4|^j \varphi \left( \frac{x}{2j+1}, -\frac{x}{2j+1} \right) \right\} \right\} \]
\[ = 0, \]
\[ \lim_{k \to \infty} \lim_{n \to \infty} \max_{2k \leq j < n-2k} \left\{ \frac{|\varphi(2^j x, -2^j x)|}{|2|^j+3} \right\} \]
\[ = 0, \]
\[ \lim_{k \to \infty} \lim_{n \to \infty} \max_{2k \leq j < n-2k} \left\{ |2|^{4k-2} \varphi \left( \frac{-x}{2^{2k+1}}, \frac{x}{2^{2k+1}} \right) \right\} \]
\[ \leq \lim_{k \to \infty} \lim_{n \to \infty} \max_{0 \leq j < k} \left\{ \frac{\varphi(-2^j x, 2^j x)}{|2|^j+4k+3} \right\}, \]
\[ \leq \lim_{k \to \infty} \lim_{n \to \infty} \max_{k \leq j < 2k} \left\{ \frac{\varphi(-2^{j-2k} x, 2^{j-2k} x)}{|2|^{j-4k+3}} \right\}, \]
\[ \leq \lim_{k \to \infty} \max_{k+1 \leq j < 2k+1} \left\{ |4|^j \varphi \left( \frac{x}{2j}, -\frac{x}{2j} \right) \right\}, \]
\[ \leq \lim_{n \to \infty} \max_{0 \leq j < n-2k} \left\{ \frac{\varphi(2^j x, 2^j x)}{|2|^j} \right\} \]
\[ = 0 \]
for all \( x \in X \setminus \{0\} \) and all \( k \in \mathbb{N} \). So the right hand side of (2.22) tends to 0 as \( k \to \infty \). Since \( T(0) = f(0) = T'(0) \), we conclude that \( T(x) = T'(x) \) for all \( x \in X \). This completes the proof of the uniqueness of \( T \).

**Corollary 2.7.** Let \( 1 < r < 2 \) be a real number and \( |2| < 1 \). If \( f : X \to Y \) satisfies the inequality
\[
\|Df(x, y)\| \leq \theta(\|x\|^r + \|y\|^r)
\]
for all \( x, y \in X \setminus \{0\} \), then there exists a unique general quadratic mapping \( T : X \to Y \) such that
\[
\|f(x) - T(x)\| \leq 2\theta|2|^{1-r}\|x\|^r
\]
for all \( x \in X \setminus \{0\} \) with \( T(0) = f(0) \).

**Proof.** Let \( \varphi(x, y) = \theta(\|x\|^r + \|y\|^r) \). Since \( |2| < 1 \) and \( 1 < r < 2 \), we have
\[
\lim_{n \to \infty} |2|^n\varphi(2^{-n}x, 2^{-n}y) = \lim_{n \to \infty} |2|^{n(2-r)}\varphi(x, y) = 0
\]
and
\[
\lim_{n \to \infty} \frac{\varphi(2^n x, 2^n y)}{|2|^n} = \lim_{n \to \infty} |2|^{n(r-1)}\varphi(x, y) = 0
\]
for all \( x, y \in X \). Therefore the conditions of Theorem 2.6 are fulfilled and it is easy to see that \( \varphi(x) = 2\theta|2|^{1-r}\|x\|^r \). By Theorem 2.6 there is a unique quadratic mapping \( T : X \to Y \) satisfying (2.23) with \( T(0) = f(0) \).

\[
\begin{align*}
\text{References} \\
\end{align*}
\]


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