REPRESENTATIONS OF THE AUTOMORPHISM GROUP OF A SUPERSINGULAR K3 SURFACE OF ARTIN-INVARIANT 1 OVER ODD CHARACTERISTIC

JUNMYEONG JANG*

Abstract. In this paper, we prove that the image of the representation of the automorphism group of a supersingular K3 surface of Artin-invariant 1 over odd characteristic \( p \) on the global two forms is a finite cyclic group of order \( p + 1 \). Using this result, we deduce, for such a K3 surface, there exists an automorphism which cannot be lifted over a field of characteristic 0.

1. Introduction

Let \( k \) be an algebraically closed field and \( X \) be an algebraic K3 surface defined over \( k \). \( H^0(X, \Omega^2_{X/k}) \) is a one dimensional \( k \)-space and the canonical representation of the automorphism group of \( X \) on \( H^0(X, \Omega^2_{X/k}) \)

\[ \rho : \text{Aut} \ X \rightarrow GL(H^0(X, \Omega^2_{X/k})) \]

is a character. If the characteristic of \( k \) is not 2, the image of \( \rho \) is a finite cyclic group. ([14], [7]) We let \( N \) be the order of \( \text{Im} \rho \). If the characteristic of \( k \) is 0 or \( X \) is of finite height over odd characteristic, \( \phi(N) \) is at most 20. ([13], [7]) Here \( \phi \) is the Euler \( \phi \)-function. If \( X \) is a supersingular K3 surface of Artin-invariant \( \sigma \) over odd characteristic, \( N \) divides \( p^\sigma + 1 \). ([14], Prop. 2.4) For the definition of height and Artin-invariant, see section 2. In this paper, we prove that for a supersingular K3 surface of Artin-invariant 1 over odd characteristic, the order of \( \text{Im} \rho \) is \( p+1 \).

Theorem 3.3. Let \( X \) be a supersingular K3 surface of Artin invariant 1 over an algebraically closed field \( k \) of odd characteristic \( p \). Then

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Im $\rho$ is a cyclic group of order $p + 1$.

Assume $k$ is an algebraically closed field of odd characteristic and $(R, m)$ is a discrete valuation ring of characteristic 0 whose residue field $R/m$ is isomorphic to $k$. In this case $p \in m$. A proper connected smooth scheme $X$ over $R$ is a lifting of $X$ over $R$ if the base change $X \otimes k$ is isomorphic to $X$. Every K3 surface defined over $k$ has a lifting over the ring of Witt vectors. ([3], [15])

Let $\alpha : X \to X$ be an automorphism of $X$. A lifting of $(X, \alpha)$ over $R$ is a pair $(X, a)$ such that $X$ is a lifting of $X$ over $R$ and $a$ is an $R$-isomorphism of $X$ satisfying $a \otimes k$ is equal to $\alpha$ under the identification $X \otimes k = X$. If $\alpha$ is of finite order and the order of $\alpha$ is prime to $p$, $\alpha$ has a lifting over $W$. ([8]) When $X$ is a K3 surface of finite height, we let $T_l$ be the $l$-adic transcendental lattice of $X$. In other words, $T_l(X)$ is the orthogonal complement of the embedding

$$ NS(X) \otimes \mathbb{Z}_l \hookrightarrow H^2_{\text{et}}(X, \mathbb{Z}_l). $$

The representation $\chi_l : \text{Aut } X \to O(T_l(X))$ has a finite image. ([7], Prop. 2.5) If the order of $\chi_l(\alpha)$ is not divisible by $p$, $(X, \alpha)$ has a lifting over $W$. ([8]) Since the characteristic polynomial of $\chi_l(\alpha)$ has integer coefficients ([4], 3.7.3), if $p \geq 23$ the order of $\chi_l(\alpha)$ is not divisible by $p$ for any $\alpha \in \text{Aut } X$.

From Theorem 3.3, we obtain the following.

**Corollary 3.5.** If $\phi(p+1) > 20$, a supersingular K3 surface of Artin-invariant 1 over $k$ has an automorphism which does not have a lifting over a field of characteristic 0.

2. Preliminary: supersingular K3 surfaces

In this section we review some properties of supersingular K3 surfaces. Let $k$ be an algebraically closed field of odd characteristic $p$. $W = W(k)$ is the ring of Witt vectors of $k$ and $K$ is the fraction field of $W$. $W$ and $K$ are equipped with the natural Frobenius operators

$$ \sigma : W \to W, \quad \sigma : K \to K. $$

Let $X$ be a K3 surface defined over $k$. The second crystalline cohomology of $X/k$, $H^2_{\text{cris}}(X/W)$ is a free $W$-module of rank 22 equipped with a Frobenius linear endomorphism

$$ F : H^2_{\text{cris}}(X/W) \to H^2_{\text{cris}}(X/W). $$
$H^2_{cris}(X/K) = H^2_{cris}(X/W) \otimes K$ has the induced Frobenius-linear automorphism

$$\mathbf{F} : H^2_{cris}(X/K) \to H^2_{cris}(X/K).$$

By the Dieudonné-Manin theorem, an $F$-isocrystal $(H^2_{cris}(X/K), \mathbf{F})$ has a decomposition

$$(H^2_{cris}(X/K), \mathbf{F}) = \bigoplus K[T]/(T^{r_i} - p^{s_i}).$$

Here $K[T]$ is a Frobenius semi-commutative polynomial ring satisfying $Ta = \sigma(a)T$ for any $a \in K$. Under the identification, the operator $\mathbf{F}$ corresponds to the multiplication by $T$ on the right hand side. We say the rational number $s_i/r_i$ is a Newton slope of $H^2_{cris}(X/K)$. The length of Newton slope $s_i/r_i$ is $r_i$. The sum of the lengths of all the slopes is equal to the dimension of $H^2_{cris}(X/K)$. If the only slopes of $H^2_{cris}(X/K)$ is 1, we say the height of $X$ is $\infty$ or $X$ is supersingular. If $X$ is not supersingular, there exists an integer $h$ ($1 \leq h \leq 10$) such that $1 - 1/h$, $1$, $1 + 1/h$ are slopes of $H^2_{cris}(X/K)$ of length $h$, $22 - 2h$, $h$ respectively. In this case, the height of $X$ is $h$.

The Neron-Severi group of $X$, $NS(X)$ is a finite free abelian group equipped with a lattice structure given by the intersection theory. The Picard number of $X$, $\rho(X)$ is the rank of $NS(X)$. The Neron-Severi lattice of a K3 surface is even by the Riemann-Roch theorem and the signature of $NS(X)$ is $(1, \rho(X) - 1)$ by the Hodge index theorem. The Picard number of $X$ is at most the length of the Newton slope 1. ([5]) It follows that $\rho(X) \leq 22 - 2h$ if the height of $X$ is $h < \infty$. Also it is known that $X$ is supersingular if and only if $\rho(X) = 22$. ([2], [10]) Note that over a field of characteristic 0, the Picard number of a K3 surface is at most 20 since $h^{1,1} = 20$ for a K3 surface.

For a lattice $L$, we denote the discriminant of $L$ and the discriminant group $L^*/L$ by $d(L)$ and $A(L)$ respectively. Let $d(X) = d(NS(X)) \in \mathbb{Z}$ be the discriminant of the lattice $NS(X)$ and $A(X) = (NS(X))^*/NS(X)$ be the discriminant group of $NS(X)$. When $X$ is a supersingular K3 surface over $k$, $d(X) = -p^{2\sigma}$ for an integer $1 \leq \sigma \leq 10$. We say $\sigma$ is the Artin-invariant of $X$. It is known that the moduli space of K3 surfaces of height $h < \infty$ is $20 - h$ dimensional and the moduli space of supersingular K3 surfaces of Artin-invariant $\sigma$ is $\sigma - 1$ dimensional. ([1]) Moreover a supersingular K3 surface of Artin-invariant 1 is unique up to isomorphism and it is isomorphic to the Kummer surface of the self-product of a supersingular elliptic curve. For a supersingular K3 surface $X$, the lattice structure of $NS(X)$ is determined by the Artin-invariant and the base characteristic $p$. ([17]) Let us denote the Neron-Severi lattice of a
supersingular K3 surface of Artin-invariant $\sigma$ over a field of characteristic $p$ by $\Lambda_{p,\sigma}$. After tensor product with $\mathbb{Z}_p$, we obtain a decomposition of $\mathbb{Z}_p$-lattice

$$\Lambda_{p,\sigma} \otimes \mathbb{Z}_p = E_0(p) \oplus E_1.$$  

Here $E_0$ and $E_1$ are unimodular $\mathbb{Z}_p$-lattices of rank $2\sigma$ and $22 - 2\sigma$ respectively. And $d(E_0) = (−1)^\sigma \delta$ and $d(E_1) = (−1)^{\sigma+1} \delta$, where $\delta$ is a non-square unit of $\mathbb{Z}_p$. Note that a unimodular $\mathbb{Z}_p$-lattice is uniquely determined up to isomorphism by the rank and the discriminant, square or non-square. It follows that

$$A(\Lambda_{p,\sigma}) = A(\Lambda_{p,\sigma} \otimes \mathbb{Z}_p) = A(E_0(p)) = E_0(p)/pE_0(p)$$

is a $2\sigma$-dimensional quadratic $\mathbb{Z}/p$ space. Note that the discriminant of $A(\Lambda_{p,\sigma})$ is a $(−1)^\sigma$ times non-square and $A(\Lambda_{p,\sigma})$ does not contain a $\sigma$-dimensional isotropic $\mathbb{Z}/p$-subspace.

By the flat Kummer sequence

$$0 \to \mu_{p^n} \to \mathbb{G}_{m,X} \overset{p^n}{\to} \mathbb{G}_{m,X} \to 0,$$

we have a canonical inclusion

$$\text{NS}(X) \otimes \mathbb{Z}_p \hookrightarrow H^2_{\text{fl}}(X, \mathbb{Z}_p(1)).$$

Also there exists an exact sequence ([5])

$$0 \to H^2_{\text{fl}}(X, \mathbb{Z}_p(1)) \to H^2_{\text{cris}}(X/W) \overset{id-p}{\to} H^2_{\text{cris}}(X/W).$$

Composing two embeddings, we obtain the cycle map

$$\text{NS}(X) \otimes W \hookrightarrow H^2_{\text{cris}}(X/W).$$

If $X$ is a supersingular K3 surface, the cycle map is an embedding of $W$-lattices of the same rank. Since $H^2_{\text{cris}}(X/W)$ is unimodular by the Poincaré duality,

$$\text{NS}(X) \otimes W \subset H^2_{\text{cris}}(X/W) \subset (\text{NS}(X) \otimes W)^*$$

and $K_X = H^2_{\text{cris}}(X/W)/(\text{NS}(X) \otimes W)$ is a $\sigma$-dimensional isotropic $k$-subspace of the discriminant group

$$A(\text{NS}(X) \otimes W) = (\text{NS}(X) \otimes W)^*/(\text{NS}(X) \otimes W) = A(\text{NS}(X)) \otimes k.$$

$K_X$ is equipped with a Frobenius-inverse linear operator $V : K_X \to K_X$ such that $V^\sigma = 0$ and we can choose $x \in K_X$ such that $\{x, Vx, \ldots, V^{\sigma-1}x\}$ is a basis of $K_X$. ([14]) Also we have a canonical isomorphism ([6], Prop.2.2)

$$K_X/VK_X \simeq H^2(X, \mathcal{O}_X).$$
Let $\mathcal{P} = \{ x \in NS(X) \otimes \mathbb{R} | (x, x) > 0 \}$, the positive cone of $X$ and $\Delta = \{ v \in NS(X) | (v, v) = -2 \}$, the set of roots of $X$. For any $v \in \Delta$, let $s_v$ be the reflection with respect to $v$,

$$s_v : u \mapsto u + (u, v)v.$$ 

Let $W_X$ be the subgroup of the orthogonal group of $NS(X)$ generated by all the reflections $s_v, (v \in \Delta)$ and $-id$. Let $\mathcal{P}^0 = \{ v \in \mathcal{P} | (v, w) \neq 0, \forall w \in \Delta \}$. It is known that the $W_X$ acts simply transitively on the set of connected components of $\mathcal{P}^0$. Moreover, the connected component which contains an ample divisor is the ample cone of $X$. ([16], [11]) It follows that $v \in NS(X)$ represents an ample divisor if and only if $(v, v) > 0$ and $(v, w) > 0$ for all effective $w \in \Delta$. If $(v, v) > 0$ and $(v, w) \neq 0$ for any $w \in \Delta$, there exists a unique element $\gamma$ in $W_X$ such that $\gamma(v)$ represents an ample divisor.

**Theorem 2.1** (Crystalline Torelli theorem, [16], p.371). Let $X$ and $Y$ be supersingular K3 surfaces defined over $k$. Assume $\Psi : NS(X) \rightarrow NS(Y)$ is an isometry. If $\Psi$ takes the ample cone of $NS(X) \otimes \mathbb{R}$ into the ample cone of $NS(Y) \otimes \mathbb{R}$ and $K_X$ into $K_Y$, then there exists a unique isomorphism $\psi : Y \rightarrow X$ such that $\Psi = \psi^*$. 

**Remark 2.2.** Our definition of $K_X$ is slightly different from the definition of the period space in [16]. However, in the statement of the crystalline Torelli theorem, we can use $K_X$ instead of the period space.

For a supersingular K3 surface $X$, let

$$\rho^{-1} : Aut X \rightarrow GL(H^2(X, \mathcal{O}_X))$$

and

$$\chi : Aut X \rightarrow GL(A(NS(X)))$$

be the representation of the automorphism group of $X$ on $H^2(X, \mathcal{O}_X)$ and $A(NS(X))$ respectively. Since $\rho^{-1}$ is a character, $\rho^{-1}$ is isomorphic to the representation

$$\rho : Aut X \rightarrow GL(H^0(X, \Omega^2_X/\mathcal{O}_X)).$$

Any automorphism of $X$ preserves $K_X$ in $A(NS(X)) \otimes k$, so there is a canonical projection

$$pr : \text{Im} \chi \rightarrow \text{Im} \rho^{-1} \simeq \text{Im} \rho.$$ 

**Proposition 2.3** ([7], Prop.2.1). $pr$ is an isomorphism. In particular, both of $\text{Im} \chi$ and $\text{Im} \rho$ are finite cyclic groups.

For the order of $\text{Im} \rho$, the following is known.
Proposition 2.4 ([14], Prop.2.4). If the Artin-invariant of $X$ is $\sigma$, the order of $\text{Im} \rho$ divides $p^\sigma + 1$.

3. Proof

Assume $k$ is an algebraically closed field of odd characteristic $p$ and $X$ is a supersingular K3 surface of Artin-invariant 1. Then $A(X)$ is a 2 dimensional space over $\mathbb{F}_p$ equipped with a non degenerate quadratic form $q = q_{A(X)}$. Here $\mathbb{F}_p$ is a prime field of characteristic $p$. Also we can see $A(X)$ does not contain a non-zero isotropic vector over $\mathbb{F}_p$. Let us choose an orthogonal basis of $A(X)$, $\{x, y\}$ such that $x \cdot x = 1, y \cdot y = -\delta$ and $x \cdot y = 0$. The following lemma is well-known. We present a proof using the zeta function.

Lemma 3.1. For any $\alpha \in \mathbb{F}_p^*$, the cardinality of the set $\{v = ax + by \in A(X) | a, b \in \mathbb{F}_p, q(v) = \alpha \}$ is $p + 1$.

Proof. Let $C$ be the smooth conic $X^2 - \delta Y^2 - \alpha Z^2$ in $\mathbb{P}_k^2$. Let $Z_C(t)$ be the zeta function of $C$. Since $Z_C(t) = \frac{1}{(1-t)(1-pt)}$, $|C(\mathbb{F}_p)| = p + 1$. For any $(X, Y) \neq (0, 0)$, $X^2 - \delta Y^2 \neq 0$, so each point of $C(\mathbb{F}_p)$ gives a distinct solution $(X/Z)^2 - \delta (Y/Z)^2 = \alpha$. This completes the proof.

Lemma 3.2. The special orthogonal group $SO(q)$ is a finite cyclic group of order $p + 1$.

Proof. Assume $\gamma \in O(q)$. We have $p + 1$ choices of $\gamma(x)$ by Lemma 3.1. Because $\gamma(x) \cdot \gamma(y) = 0$ and $\gamma(y) \cdot \gamma(y) = -\delta$, for each choice of $\gamma(x)$, there are two possibilities of $\gamma(y)$. Therefore the order of $O(q)$ is $2(p + 1)$ and the order of $SO(q)$ is $p + 1$. There are two isotropic lines in $A(X) \otimes \mathbb{F}_p^2$. Any $\gamma \in O(q)$ fixes or interchanges two isotropic lines and $\gamma \in O(q)$ is contained in $SO(q)$ if and only if $\gamma$ fixes the isotropic lines. Let $v \in A(X) \otimes \mathbb{F}_p^2$ be an isotropic vector. The character $\lambda : SO(q) \rightarrow k^*$ defined by $\gamma(v) = \lambda(\gamma)v$ is an injection. It follows that $SO(q) = \mathbb{Z}/(p + 1)$.

Theorem 3.3. Let $X$ be a supersingular K3 surface of Artin-invariant 1 over an algebraically closed field $k$ of odd characteristic $p$. Then $\text{Im} \rho$ is a cyclic group of order $p + 1$.

Proof. Since $NS(X)$ is even indefinite of rank 22 and $A(NS(X)) = (\mathbb{Z}/p)^2$, the canonical map

$$\pi : O(NS(X)) \rightarrow O(q)$$


is surjective. ([12], 1.14.2) Assume $\gamma \in \pi^{-1}(SO(q))$. Since $K_X$ is an isotropic line of $A(X) \otimes k$, $\gamma$ preserves $K_X$. By [18], p.456 there is a decomposition

$$NS(X) = U \oplus H^{(p)} \oplus E_8^2.$$  

Here $U$ is an even unimodular hyperbolic lattice of rank 2 and $E_8$ is a negative definite unimodular root lattice. $H^{(p)}$ is the maximal order of the quaternion algebra over $\mathbb{Q}$ which is ramified only at $p$ and $\infty$. The lattice structure of $H^{(p)}$ is induced by the trace map of the quaternion algebra. $H^{(p)}$ is a negative definite even lattice of rank 4 and $A(H^{(p)}) = A(NS(X)) = (\mathbb{Z}/p)^2$.

The Weyl group $W_X \subset O(NS(X))$ is generated by $-id$ and reflections $s_v (v \in \Delta)$. For $v \in \Delta$, $v \cdot v = -2$, so $\mathbb{Z}_p v$ is a unimodular sublattice of $NS(X) \otimes \mathbb{Z}_p$ and we have a decomposition

$$NS(X) \otimes \mathbb{Z}_p = M \oplus \mathbb{Z}_p v,$$

where $M$ is the orthogonal complement of $\mathbb{Z}_p v$ in $NS(X) \otimes \mathbb{Z}_p$. Then $s_v|NS(X) \otimes \mathbb{Z}_p = id \oplus -id$ with respect to the decomposition. Since $A(M) = A(X)$, $s_v|A(X) = id$ and $s_v|K_X = id$. The positive cone $\mathcal{P}$ has two connected components. Since $s_v$ fixes a positive vector, $s_v$ fixes connected components of $\mathcal{P}$. On the other hand, $-id$ interchanges the connected components of $\mathcal{P}$. Let $\iota = id \oplus -id \oplus id \in O(NS(X))$ for the decomposition

$$NS(X) \otimes \mathbb{Z}_p = U \oplus H^{(p)} \oplus E_8^2.$$  

$\iota$ preserves the connected components of $\mathcal{P}$ and $\iota|A(NS(X)) = -id$. Assume $\psi \in O(NS(X))$ and $\pi(\psi) \in SO(q)$. There exists a unique $\gamma \in W_X \cup W_X \cdot \iota$ such that $\gamma \circ \psi$ preserves the ample cone and $\pi(\gamma \circ \psi) = \pi(\psi)$. Since $\gamma \circ \psi$ preserves the ample cone and $K_X$, $\gamma \circ \psi \in Aut X \subset O(NS(X))$ by the crystalline Torelli theorem. Therefore

$$\text{Im } \chi = \pi(\text{Aut } X) = \pi(\pi^{-1}(SO(q))) = SO(q)$$

and $\text{Im } \rho \simeq \text{Im } \chi$ is a cyclic group of order $p + 1$. 

\textbf{Remark 3.4.} For $\sigma > 1$, there exists a supersingular K3 surface of Artin-invariant $\sigma$ over $k$ such that the order of $\text{Im } \chi$ is equal to or less than 2. ([9], Theorem 1.7)

\textbf{Corollary 3.5.} If $\phi(p + 1) > 20$, a supersingular K3 surface of Artin-invariant 1 over $k$ has an automorphism which can not be lifted over a field of characteristic 0.
Proof. Let $\alpha$ be an automorphism of $X$ such that the order of $\rho(\alpha)$ is $p + 1$. Assume $R$ is a discrete valuation ring of characteristic 0 whose residue field is isomorphic to $k$ and $(X, \alpha)$ is a lifting of $(X, \alpha)$ over $R$. Let $F$ be the fraction field of $R$ and $X_F = X \otimes F$. $X_F$ is a K3 surface defined over $F$ and $H^0(X, \Omega^2_{X/R})$ is a free $R$-module of rank 1. The order of $\alpha^*|H^0(X, \Omega^2_{X/R})$ is equal to the order of $\alpha^*|H^0(X_F, \Omega^2_{X_F/F})$. Since $\alpha^*|H^0(X, \Omega^2_{X/R})$ is a multiplication by a root of unity, the order of $\alpha^*|H^0(X, \Omega^2_{X/R})$ is a $p$-power times of the order of $\alpha^*|H^0(X, \Omega^2_{X/k})$. But the $\phi$ value of the order of $\alpha^*|H^0(X_F, \Omega^2_{X_F/F})$ is at most 20 and it is a contradiction. Therefore $(X, \alpha)$ is not liftable. 

References

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Department of Mathematics
University of Ulsan
Ulsan 680-749, Republic of Korea
E-mail: jmjang@ulsan.ac.kr