THE RIEMANN DELTA INTEGRAL ON TIME SCALES

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Abstract. In this paper, we define the extension \( f^* : [a, b] \to \mathbb{R} \) of a function \( f : [a, b]_T \to \mathbb{R} \) for a time scale \( T \) and show that \( f \) is Riemann delta integrable on \([a, b]_T\) if and only if \( f^* \) is Riemann integrable on \([a, b]\).

1. Introduction and preliminaries

Let \( T \) be a time scale, \( a < b \) be points in \( T \), and \([a, b]_T\) be the closed interval in \( T \). A partition \( P = \{(\xi_i, [t_{i-1}, t_i])\}_{i=1}^n \) of \([a, b]_T\) is a collection of tagged intervals such that

\[ a = t_0 < t_1 < \cdots < t_n = b, \quad t_i \in T \text{ for each } i = 1, 2, \ldots, n, \]

and \( \xi_i \) is an arbitrary point on \([t_{i-1}, t_i)_T\).

Let \( f \) be a real-valued bounded function on \([a, b]_T\). Let \( M_i = \sup\{f(t) : t \in [t_{i-1}, t_i)_T\} \) and \( m_i = \inf\{f(t) : t \in [t_{i-1}, t_i)_T\} \). The upper \( \Delta \)-sum \( \mathcal{S}_P(f) \) and the lower \( \Delta \)-sum \( \mathcal{S}_P(f) \) of \( f \) with respect to \( P \) are defined by

\[ \mathcal{S}_P(f) = \sum_{i=1}^n M_i(t_i - t_{i-1}), \quad \mathcal{S}_P(f) = \sum_{i=1}^n m_i(t_i - t_{i-1}). \]

Let \( \{(a_k, b_k)\}_{k=1}^\infty \) be the sequence of intervals contiguous to \([a, b]_T\) in \([a, b]_T\).

For a function \( f : [a, b]_T \to \mathbb{R} \), define the extension \( f^* : [a, b] \to \mathbb{R} \) of \( f \) by

\[ f^*(t) = \begin{cases} f(a_k) & \text{if } t \in (a_k, b_k) \text{ for some } k \\ f(t) & \text{if } t \in [a, b]_T. \end{cases} \]

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It is well-known [7] that \( f : [a, b] \to \mathbb{R} \) is McShane delta integrable on \([a, b]\) if and only if \( f^* : [a, b] \to \mathbb{R} \) is McShane integrable on \([a, b]\).

In this paper, we consider the Riemann delta integral and show that a function \( f : [a, b] \to \mathbb{R} \) is Riemann delta integrable on \([a, b] \to \mathbb{R}\) if and only if \( f^* : [a, b] \to \mathbb{R} \) is Riemann integrable on \([a, b] \to \mathbb{R}\).

2. The Riemann delta integral

**Definition 2.1.** For given \( \delta > 0 \), a partition \( \mathcal{P} = \{(\xi_i, [t_{i-1}, t_i])\}_{i=1}^n \) is a \( \delta \)-partition of \([a, b] \) if for each \( i \in \{1, 2, \ldots, n\} \) either \( t_i - t_{i-1} \leq \delta \) or \( t_i - t_{i-1} > \delta \) and \( \sigma(t_{i-1}) = t_i \), where \( \sigma(t) = \inf\{s \in \mathcal{T} : s > t\} \).

**Definition 2.2.** A bounded function \( f : [a, b] \to \mathbb{R} \) is Riemann delta integrable (or \( R_{\Delta} \)-integrable) on \([a, b] \to \mathbb{R}\) if there exists a number \( A \) such that for each \( \epsilon > 0 \) there exists \( \delta > 0 \) such that
\[
\left| \sum_{i=1}^n f(\xi_i)(t_i - t_{i-1}) - A \right| < \epsilon
\]
for every \( \delta \)-partition \( \mathcal{P} = \{(\xi_i, [t_{i-1}, t_i])\}_{i=1}^n \) of \([a, b] \). The number \( A \) is called the Riemann delta integral of \( f \) on \([a, b] \) and we write
\[
A = (R_{\Delta}) \int_a^b f.
\]

The following theorem gives a Cauchy criterion for \( R_{\Delta} \)-integrability.

**Theorem 2.3.** [3] A bounded function \( f : [a, b] \to \mathbb{R} \) is \( R_{\Delta} \)-integrable on \([a, b] \) if and only if for each \( \epsilon > 0 \) there exists a partition \( \mathcal{P} \) of \([a, b] \) such that
\[
\overline{\mathcal{S}}_{\mathcal{P}}(f) - \underline{\mathcal{S}}_{\mathcal{P}}(f) < \epsilon.
\]

Let \( \mathcal{P} = \{(\xi_i, [t_{i-1}, t_i])\}_{i=1}^n \) and \( \mathcal{Q} = \{(\eta_j, [x_{j-1}, x_j])\}_{j=1}^m \) be two partitions of \([a, b] \) (or \([a, b] \)). If \( \{t_0, t_1, \ldots, t_n\} \subset \{x_0, x_1, \ldots, x_m\} \), then we say that \( \mathcal{Q} \) is a refinement of \( \mathcal{P} \) and we denote \( \mathcal{Q} \geq \mathcal{P} \).

Recall that \( f : [a, b] \to \mathbb{R} \) is Riemann integrable on \([a, b] \) with value \( A \) for each \( \epsilon > 0 \) there exists a partition \( \mathcal{P} = \{(\xi_i, [t_{i-1}, t_i])\} \) of \([a, b] \) such that
\[
\left| \sum_{j} f(\eta_j)(x_j - x_{j-1}) - A \right| < \epsilon
\]
for every refinement \( \mathcal{Q} = \{(\xi_i, [x_{j-1}, x_j])\} \) of \( \mathcal{P} \).

**Theorem 2.4.** A bounded function \( f : [a, b] \to \mathbb{R} \) is \( R_{\Delta} \)-integrable on \([a, b] \) if and only if \( f^* : [a, b] \to \mathbb{R} \) is Riemann integrable on \([a, b] \). In that case, \( (R) \int_a^b f^* = (R_{\Delta}) \int_a^b f \).
Proof. Let $f : [a, b]_T \to \mathbb{R}$ be $R_\Delta$-integrable on $[a, b]_T$ and let $\epsilon > 0$. Then there exists a partition $\mathcal{P}_0 = \{(\xi^0_j, [t^0_{j-1}, t^0_j])\}_{j=1}^m$ of $[a, b]_T$ such that

\begin{equation}
(2.1) \quad \left| \sum_{i=1}^n f(\xi_i)(t_i - t_{i-1}) - (R_\Delta) \int_a^b f \right| < \epsilon
\end{equation}

for every partition $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i])\}_{i=1}^n \geq \mathcal{P}_0$ of $[a, b]_T$. Assume that $\mathcal{P}' = \{(\xi_i', [t_{i-1}', t_i'])\}_{i=1}^n$ is a partition of $[a, b]$ with $\mathcal{P}' \geq \mathcal{P}_0$, where we regard $\mathcal{P}_0$ as a partition of $[a, b]$. If $i \leq n$, then there is a unique $j \leq m$ such that $[t_{i-1}', t_i'] \subseteq [t^0_{j-1}, t^0_j]$ and there is a $\xi''_i \in [t^0_{j-1}, t^0_j]_T$ with $f^*(\xi_i') = f(\xi''_i)$. For each $j \leq m$, there are $i_{1j}, i_{2j} \leq n$ such that $[t_{i_{1j}-1}', t_{i_{1j}}'], [t_{i_{2j}-1}', t_{i_{2j}}'] \subseteq [t^0_{j-1}, t^0_j]$ and

\[ f(\xi''_{i_{1j}}) = \min \left\{ f(\xi''_i) \right\}_{[t_{i-1}', t_i'] \subseteq [t^0_{j-1}, t^0_j]}, \quad f(\xi''_{i_{2j}}) = \max \left\{ f(\xi''_i) \right\}_{[t_{i-1}', t_i'] \subseteq [t^0_{j-1}, t^0_j]} \]

By (2.1), we have

\begin{equation}
(2.2) \quad \sum_{i=1}^n f^*(\xi_i')(t_i' - t_{i-1}')
\begin{align*}
&= \sum_{j=1}^m \sum_{[t_{i-1}', t_i'] \subseteq [t^0_{j-1}, t^0_j]} f(\xi''_i)(t_i' - t_{i-1}') \\
&= \sum_{j=1}^m \left( \sum_{[t_{i-1}', t_i'] \subseteq [t^0_{j-1}, t^0_j]} f(\xi''_i)(t^0_j - t^0_{j-1}) \right) (t^0_j - t^0_{j-1}) \\
&\leq \sum_{j=1}^m f(\xi''_{i_{1j}})(t^0_j - t^0_{j-1}) \\
&< \sum_{j=1}^m f(\xi''_{i_{2j}})(t^0_j - t^0_{j-1}) + 2\epsilon.
\end{align*}
\end{equation}

Similarly, we have

\begin{equation}
(2.3) \quad \sum_{i=1}^n f^*(\xi_i')(t_i' - t_{i-1}') > \sum_{j=1}^m f(\xi''_j)(t^0_j - t^0_{j-1}) - 2\epsilon.
\end{equation}

From (2.1), (2.2), (2.3) we have
such that 
\[ \int_a^b f = \lim_{n \to \infty} \sum_{i=1}^n f^*(\xi_i) \Delta x_i \]

Then \( Q \) is a refinement of \( P \).

Put \( \epsilon > 0 \). Let \( f^* : [a, b] \to \mathbb{R} \) be Riemann integrable on \([a, b] \). Let \( \epsilon > 0 \). Then there exists a partition \( P = [x_i, y_i] \) of \([a, b] \) such that

Thus \( f^* \) is Riemann integrable on \([a, b] \) and \( \int_a^b f^* = \int_a^b f \).

Conversely, suppose that \( f^* : [a, b] \to \mathbb{R} \) is Riemann integrable on \([a, b] \). Let \( \epsilon > 0 \). Then there exists a partition \( P = [x_i, y_i] \) of \([a, b] \) such that

\[ \mathcal{S}_P(f^*) - \mathcal{S}_P(f^*) < \epsilon. \]

Let \( \{(a_k, b_k)\} \) be the sequence of intervals contiguous to \([a, b]_T \) in \([a, b] \). Put

\[ A = \{i \mid [x_i, y_i] \subset [a_k, b_k] \text{ for some } k \in \mathbb{N}, i = 1, 2, \ldots, n\}, \]

\[ B = \{1, 2, \ldots, n\} - A. \]

We see that \([x_i, y_i] \neq \emptyset\) for each \( i \in B \). Put

\[ s_i = \inf[x_i, y_i], \quad t_i = \sup[x_i, y_i] \text{ for each } i \in B. \]

Put \( B_1 = \{i \in B \mid x_i < s_i\}, \quad B_2 = \{i \in B \mid t_i < y_i\} \)

\[ B_3 = \{i \in B \mid s_i < t_i\}. \]

Let \( K = \{k \in \mathbb{N} \mid [x_i, y_i] \subset [a_k, b_k] \text{ for some } i \in A\} \)

\[ \cup \{k \in \mathbb{N} \mid [x_i, s_i] \subset [a_k, b_k] \text{ for some } i \in B_1\} \]

\[ \cup \{k \in \mathbb{N} \mid [t_i, y_i] \subset [a_k, b_k] \text{ for some } i \in B_2\}. \]

Then the partition

\[ P' = \{[x_i, y_i] \mid i \in A\} \cup \{[x_i, s_i] \mid i \in B_1\} \cup \{[t_i, y_i] \mid i \in B_2\} \]

\[ \cup \{[s_i, t_i] \mid i \in B_3\} \]

is a refinement of \( P \). Hence, \( \mathcal{S}_{P'}(f^*) - \mathcal{S}_{P'}(f^*) < \epsilon. \)

Put \( P'' = \{[s_i, t_i] \mid i \in B_3\}, \quad Q = \{[a_k, b_k] \mid k \in K\} \cup P''. \)

Then \( Q \) is a partition of \([a, b]_T \) and

\[ \mathcal{S}_Q(f) - \mathcal{S}_Q(f) = \mathcal{S}_{P''}(f) - \mathcal{S}_{P''}(f) \]

\[ \mathcal{S}_{P'}(f^*) - \mathcal{S}_{P'}(f^*) < \epsilon. \]
By Theorem 2.3, $f$ is $R_{\Delta}$-integrable on $[a, b]_T$. □

**Theorem 2.5.** Let $f$ be a bounded $R_{\Delta}$-integrable function on $[a, b]_T$. Then $f$ is $R_{\Delta}$-integrable on every subinterval $[c, d]_T$ of $[a, b]_T$.

**Proof.** Let $f$ be a bounded $R_{\Delta}$-integrable function on $[a, b]_T$. By Theorem 2.4, $f^*: [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$. By the property of the Riemann integral, $f^*$ is Riemann integrable on every subinterval $[c, d] \subset [a, b]$. By Theorem 2.4, $f$ is $R_{\Delta}$-integrable on every subinterval $[c, d]_T \subset [a, b]_T$. □

**Theorem 2.6.** Let $f$ and $g$ be $R_{\Delta}$-integrable on $[a, b]_T$ and $\alpha, \beta$ be real numbers. Then $\alpha f + \beta g$ is $R_{\Delta}$-integrable on $[a, b]_T$ and

$$(R_{\Delta}) \int_a^b (\alpha f + \beta g) = \alpha (R_{\Delta}) \int_a^b f + \beta (R_{\Delta}) \int_a^b g.$$ 

**Proof.** Let $f$ and $g$ be $R_{\Delta}$-integrable on $[a, b]_T$. By Theorem 2.4, $\alpha f^* + \beta g^*$ is Riemann integrable on $[a, b]$ and

$$(R) \int_a^b (\alpha f^* + \beta g^*) = \alpha (R) \int_a^b f^* + \beta (R) \int_a^b g^*.$$ 

Hence, $\alpha f + \beta g$ is $R_{\Delta}$-integrable on $[a, b]_T$ and

$$(R_{\Delta}) \int_a^b (\alpha f + \beta g) = \alpha (R_{\Delta}) \int_a^b f + \beta (R_{\Delta}) \int_a^b g.$$ 

□

**Theorem 2.7.** Let $f$ be a bounded function on $[a, b]_T$ and let $c \in T$ with $a < c < b$. If $f$ is $R_{\Delta}$-integrable on each of intervals $[a, c]_T$ and $[c, b]_T$, then $f$ is $R_{\Delta}$-integrable on $[a, b]_T$ and

$$(R_{\Delta}) \int_a^b f = (R_{\Delta}) \int_a^c f + (R_{\Delta}) \int_c^b f.$$ 

**Proof.** If $f$ is $R_{\Delta}$-integrable on $[a, c]_T$ and $[c, b]_T$, then $f^*$ is Riemann integrable on $[a, c]$ and $[c, b]$. By the property of the Riemann integral, $f^*$ is Riemann integrable on $[a, b]$ and

$$(R) \int_a^b f^* = (R) \int_a^c f^* + (R) \int_c^b f^*.$$ 

By Theorem 2.4, $f$ is $R_{\Delta}$-integrable on $[a, b]_T$ and

$$(R_{\Delta}) \int_a^b f = (R_{\Delta}) \int_a^c f + (R_{\Delta}) \int_c^b f.$$ 

□
Theorem 2.8. Let \( \{f_n\} \) be a sequence of \( R_\Delta \)-integrable functions on \([a, b]_T\) such that \( f_n \to f \) uniformly on \([a, b]_T\). Then \( f \) is \( R_\Delta \)-integrable on \([a, b]_T\) and
\[
(R_\Delta) \int_a^b f = \lim_{n \to \infty} (R_\Delta) \int_a^b f_n.
\]

Proof. Let \( \{f_n\} \) be a sequence of \( R_\Delta \)-integrable functions on \([a, b]_T\) such that \( f_n \to f \) uniformly on \([a, b]_T\). By Theorem 2.4, \( \{f_n^*\} \) is a sequence of Riemann integrable functions on \([a, b]\) such that \( f_n^* \to f^* \) uniformly on \([a, b]\).

By the property of Riemann integral, \( f^* \) is Riemann integrable on \([a, b]\) and
\[
(R) \int_a^b f^* = \lim_{n \to \infty} (R) \int_a^b f_n^*.
\]

By Theorem 2.4, \( f \) is \( R_\Delta \)-integrable on \([a, b]_T\) and
\[
(R_\Delta) \int_a^b f = \lim_{n \to \infty} (R_\Delta) \int_a^b f_n.
\]

\[\square\]

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