ON HENSTROCK INTEGRALS OF INTERVAL-VALUED FUNCTIONS ON TIME SCALES

WON TAE OH* AND JU HAN YOON**

Abstract. In this paper we introduce the interval-valued Henstock integral on time scales and investigate some properties of these integrals.

1. Introduction and preliminaries

The Henstock integral for real functions was first defined by Henstock [2] in 1963. The Henstock integral is more powerful and simpler than the Lebesgue, Wiener and Feynman integrals. The Henstock delta integral on time scales was introduced by Allan Peterson and Bevan Thompson [5] in 2006. In 2000, Congxin Wu and Zengtai Gong introduced the concept of the Henstock integral of interval-valued functions [6].

In this paper we introduce the concept of the Henstock delta integral of interval-valued function on time scales and investigate some properties of the integral.

A time scale $T$ is a nonempty closed subset of real number $\mathbb{R}$ with the subspace topology inherited from the standard topology of $\mathbb{R}$. For $t \in T$ we define the forward jump operator $\sigma(t) = \inf \{s \in T : s > t\}$ where $\inf \phi = \sup \{T \}$, while the backward jump operator $\rho(t) = \sup \{s \in T : s < t\}$ where $\sup \phi = \inf \{T \}$. If $\sigma(t) > t$, we say that $t$ is right-scattered, while if $\rho(t) < t$, we say that $t$ is left-scattered. If $\sigma(t) = t$, we say that $t$ is right-dense, while if $\rho(t) = t$, we say that $t$ is left-dense. The forward graininess function $\mu(t)$ of $t \in T$ is defined by $\mu(t) = \sigma(t) - t$, while the backward graininess function $\nu(t)$ of $t \in T$ is defined by $\nu(t) = t - \rho(t)$. For $a, b \in T$ we denote the closed interval $[a, b]_T = \{t \in T : a \leq t \leq b\}$.
δ = (δL, δR) is a Δ-gauge on [a, b]T if δL(t) > 0 on (a, b)T, δR(t) > 0 on (a, b)T, δL(a) ≥ 0, δR(b) ≥ 0 and δR(t) ≥ μ(t) for each t ∈ [a, b]T.

A collection P = \{([t_{i-1}, t_i], ξ_i) : 1 ≤ i ≤ n\} of tagged intervals is δ-fine Henstock partition of [a, b]T if U^n_{i=1}[t_{i-1}, t_i] = [a, b]T, [t_{i-1}, t_i]T ⊂ [ξ_i − δL(ξ_i), ξ_i + δR(ξ_i)] and ξ_i ∈ [t_{i-1}, t_i]T for each i = 1, 2, ..., n.

**Definition 1.1 ([5]).** A function f : [a, b] → \mathbb{R} is Henstock delta integrable (or HΔ-integrable) on [a, b] if there exists a number A such that for each ε > 0 there exists a Δ-gauge δ on [a, b] such that 

\[
\left| \sum_{i=1}^{n} f(ξ_i)(t_i - t_{i-1}) - A \right| < ε
\]

for every δ-fine Henstock partition P = \{([t_{i-1}, t_i], ξ_i) : 1 ≤ i ≤ n\} of [a, b]. The number A is called the Henstock delta integral of f on [a, b] and we write A = (HΔ) \int_{a}^{b} f.

**Definition 1.2.** Let I_R = \{I = [I^−, I^+] : I^− = \min\{x : x ∈ I\}, I^+ = \max\{x : x ∈ I\}\} is the closed bounded interval on the real \mathbb{R}, where I^− = \min\{x : x ∈ I\}, I^+ = \max\{x : x ∈ I\}.

For A, B, C ∈ I_R, we define A ≤ B iff A^− ≤ B^− and A^+ ≤ B^+, A + B = C iff A^− + B^− = C^− and A^+ + B^+ = C^+, and AB = \{ab : a ∈ A, b ∈ B\}, where (AB)^− = \min(A^−B^−, A^−B^+, A^+B^−, A^+B^+) and (AB)^+ = \max(A^−B^−, A^−B^+, A^+B^−, A^+B^+). Define d(A, B) = max(‖A^− − B^−‖, ‖A^+ − B^+‖) as the distance between A and B.

**Definition 1.3 ([6]).** An interval-valued function F : [a, b] → I_R is Henstock integrable to I_0 ∈ I_R on [a, b] if for every ε > 0 there exists a gauge δ on [a, b] such that 

\[
d\left( \sum_{i=1}^{n} F(ξ_i)(t_i - t_{i-1}), I_0 \right) < ε
\]

whenever P = \{([t_{i-1}, t_i], ξ_i) : 1 ≤ i ≤ n\} of [a, b] is a δ-fine Henstock partition of [a, b]. We write (IH) \int_{a}^{b} F(x)dx = I_0 and F ∈ IH[a, b].

2. The interval-valued Henstock delta integral on time scales

In this section, we will define the Henstock integral of interval-valued function on time scales and investigate some properties of the integral.

**Definition 2.1.** An interval-valued function F : [a, b]T → I_R is Henstock delta integrable to I_0 ∈ I_R on [a, b]T if for every ε > 0 there
exists a $\Delta$-gauge $\delta$ on $[a, b]_T$ such that
\[
d \left( \sum_{i=1}^{n} F(\xi_i)(t_i - t_{i-1}), I_0 \right) < \epsilon
\]
whenever $P = \{(t_{i-1}, t_i]_T, \xi_i) : 1 \leq i \leq n \}$ is a $\delta$-fine Henstock partition of $[a, b]_T$. We write $(IH_{\Delta}) \int_{a}^{b} F(x)dx = I_0$ and $F \in IH_{\Delta}[a, b]_T$.

**Remark 2.2.** It is clear, if $F(x) = F^-(x) = F^+(x)$ for all $x \in [a, b]$, then Definition 2.1 implies the real-valued Henstock integral on $[a, b]$.

**Remark 2.3.** If $F \in IH_{\Delta}[a, b]_T$, then the integral is unique.

**Theorem 2.4.** An interval-valued function $F : [a, b]_T \longrightarrow I_\mathbb{R}$ is Henstock delta integrable on $[a, b]_T$ if and only if $F^-, F^+ \in H_{\Delta}[a, b]_T$ and
\[
(IH_{\Delta}) \int_{a}^{b} F(x)dx = \left[ (H_{\Delta}) \int_{a}^{b} F^-(x)dx, (H_{\Delta}) \int_{a}^{b} F^+(x)dx \right],
\]
where $F(x) = [F^-(x), F^+(x)]$.

**Proof.** Let $F \in IH_{\Delta}[a, b]_T$. Then there exists an interval $I_0 = [I_0^-, I_0^+]$ with the property that for each $\epsilon > 0$ there exists a $\Delta$-gauge $\delta$ on $[a, b]_T$ such that
\[
d \left( \sum_{i=1}^{n} F(\xi_i)(t_i - t_{i-1}), I_0 \right) < \epsilon
\]
whenever $P = \{(t_{i-1}, t_i]_T, \xi_i) : 1 \leq i \leq n \}$ is a $\delta$-fine Henstock partition of $[a, b]_T$.

Let $P = \{(t_{i-1}, t_i]_T, \xi_i) : 1 \leq i \leq n \}$ be a $\delta$-fine Henstock partition of $[a, b]_T$. Since
\[
d \left( \sum_{i=1}^{n} F(\xi_i)(t_i - t_{i-1}), I_0 \right) = \max \left( \left| \sum_{i=1}^{n} F^- (\xi_i)(t_i - t_{i-1}) - I_0^- \right|, \left| \sum_{i=1}^{n} F^+ (\xi_i)(t_i - t_{i-1}) - I_0^+ \right| \right),
\]
\[
\left| \sum_{i=1}^{n} F^- (\xi_i)(t_i - t_{i-1}) - I_0^- \right| < \epsilon, \left| \sum_{i=1}^{n} F^+ (\xi_i)(t_i - t_{i-1}) - I_0^+ \right| < \epsilon.
\]
Conversely, let \( F^-, F^+ \in H_{\Delta}[a,b]_T \) then there exist \( H_1, H_2 \in \mathbb{R} \) with the property that given \( \Delta \)-gauge \( \delta \) on \([a,b]_T\) such that
\[
\left| \sum_{i=1}^{n} F^-(\xi_i)(t_i - t_{i-1}) - H_1 \right| < \epsilon, \left| \sum_{i=1}^{n} F^+(\xi_i)(t_i - t_{i-1}) - H_2 \right| < \epsilon.
\]
whenever \( P = \{(t_{i-1}, t_i]_T, \xi_i) : 1 \leq i \leq n \} \) is a \( \delta \)-fine Henstock partition of \([a,b]_T\). We define \( I_0 = [H_1, H_2] \), then if \( P = \{(t_{i-1}, t_i]_T, \xi_i) : 1 \leq i \leq n \} \) is a \( \delta \)-fine Henstock partition of \([a,b]_T\). We have
\[
d \left( \sum_{i=1}^{n} F(\xi_i)(t_i - t_{i-1}), I_0 \right) < \epsilon.
\]
Hence \( F : [a,b]_T \to \mathbb{I}_{\mathbb{R}} \) is Henstock delta integrable on \([a,b]_T\).

From Theorem 2.4 and the properties of Henstock delta integral ([6]), we can easily obtain the following theorems.

**Theorem 2.5.** Let \( F, G \in IH_{\Delta}[a,b]_T \) and \( \beta, \gamma \in \mathbb{R} \). Then

1. \( \beta F + \gamma G \in IH_{\Delta}[a,b]_T \) and \( (IH_{\Delta}) \int_a^b (\beta F + \gamma G)\,dx = \beta (IH_{\Delta}) \int_a^b F\,dx + \gamma (IH_{\Delta}) \int_a^b G\,dx \)
2. If \( F(x) \leq G(x) \) a.e. in \([a,b]_T\), then \( (IH_{\Delta}) \int_a^b F\,dx \leq (IH_{\Delta}) \int_a^b G\,dx \)

**Theorem 2.6.** Let \( F \in IH_{\Delta}[a,c]_T \) and \( F \in IH_{\Delta}[c,b]_T \). Then \( F \in IH_{\Delta}[a,b]_T \) and
\[
(IH_{\Delta}) \int_a^b F\,dx = (IH_{\Delta}) \int_c^b F\,dx + (IH_{\Delta}) \int_c^c F\,dx
\]

**Theorem 2.7.** Let \( F, G \in IH_{\Delta}[a,b]_T \) and \( d(F,G) \) is Lebesgue delta integrable on \([a,b]_T\). Then
\[
d \left( (IH_{\Delta}) \int_a^b F\,dx, (IH_{\Delta}) \int_a^b G\,dx \right) \leq (L_{\Delta}) \int_a^b d(F,G)\,dx
\]

**Proof.** By definition of distance, we have
\[
\begin{align*}
d\left((IH_\Delta) \int_a^b Fdx, (IH_\Delta) \int_a^b Gdx\right) \\
= \max \left(\left|\left((IH_\Delta) \int_a^b Fdx\right)^- - \left((IH_\Delta) \int_a^b Gdx\right)^-\right|, \left|\left((IH_\Delta) \int_a^b Fdx\right)^+ - \left((IH_\Delta) \int_a^b Gdx\right)^+\right|\right) \\
= \max \left(\left|(IH_\Delta) \int_a^b (F^- - G^-)dx\right|, \left|(IH_\Delta) \int_a^b (F^+ - G^+)dx\right|\right) \\
= \max \left(\left|(L_\Delta) \int_a^b |F^- - G^-| dx\right|, \left|(L_\Delta) \int_a^b |F^+ - G^+| dx\right|\right) \\
\leq (L_\Delta) \int_a^b d(F, G)dx.
\end{align*}
\]

### 3. The Henstock delta integral of fuzzy number valued functions

**Definition 3.1 ([1]).** Let \( \tilde{A} \in F(\mathbb{R}) \) be a fuzzy subset on \( \mathbb{R} \). If for any \( \lambda \in [0, 1] \), \( A_\lambda = [A^-_\lambda, A^+_\lambda] \) and \( A_1 \neq \emptyset \), where \( A_\lambda = \{x : \tilde{A}(x) \geq \lambda\} \), then \( \tilde{A} \) is called a fuzzy number.

Let \( \tilde{\mathbb{R}} \) denote the set of all fuzzy numbers.

**Definition 3.2 ([3]).** Let \( \tilde{A}, \tilde{B} \in \tilde{\mathbb{R}} \), we define \( \tilde{A} \leq \tilde{B} \) iff \( A_\lambda \leq B_\lambda \) for all \( \lambda \in (0, 1] \), \( \tilde{A} + \tilde{B} = \tilde{C} \) iff \( A_\lambda + B_\lambda = C_\lambda \) for any \( \lambda \in (0, 1] \), \( \tilde{A} \cdot \tilde{B} = \tilde{D} \) iff \( A_\lambda \cdot B_\lambda = D_\lambda \) for any \( \lambda \in (0, 1] \). For \( D(\tilde{A}, \tilde{B}) = \sup_{\lambda \in [0, 1]} d(A_\lambda, B_\lambda) \) is called the distance between \( \tilde{A}, \tilde{B} \).

**Lemma 3.3 ([1]).** If a mapping \( H : [0, 1] \rightarrow I_\mathbb{R}, \lambda \mapsto H(\lambda) = [m_\lambda, n_\lambda] \), satisfies \( [m_{\lambda_1}, n_{\lambda_1}] \supset [m_{\lambda_2}, n_{\lambda_2}] \) when \( \lambda_1 < \lambda_2 \), then
\[
\tilde{A} := \bigcup_{\lambda \in (0, 1]} \lambda H(\lambda) \in \tilde{\mathbb{R}}
\]
and
\[
A_\lambda = \bigcap_{n=1}^{\infty} H(\lambda_n),
\]
where \( \lambda_n = [1 - 1/(n + 1)] \lambda. \)

**Definition 3.4.** Let \( \tilde{F} : [a, b] \to \tilde{\mathbb{R}} \). If the interval-valued function \( F_\lambda(x) = [F^-_\lambda(x), F^+_\lambda(x)] \) is Henstock delta integrable on \([a, b] \) for any \( \lambda \in (0, 1] \), then we say that \( \tilde{F}(x) \) is Henstock delta integrable on \([a, b] \) and the integral is defined by Henstock delta integral is defined by

\[
\left( F \Delta \right) \int_a^b \tilde{F}(x)dx := \bigcup_{\lambda \in (0, 1]} \lambda\left( I \Delta \right) \int_a^b F_\lambda(x)dx
\]

We will write \( \tilde{F} \in F \Delta [a, b]. \)

**Theorem 3.5.** \( \tilde{F} \in F \Delta [a, b], \) then \( \left( F \Delta \right) \int_a^b \tilde{F}(x)dx \in \tilde{\mathbb{R}} \) and

\[
\left( (F \Delta) \int_a^b \tilde{F}(x)dx \right)_\lambda = \bigcap_{n=1}^\infty \left( I \Delta \right) \int_a^b F_{\lambda_n}(x)dx,
\]

where \( \lambda_n = [1 - 1/(n + 1)] \lambda. \)

**Proof.** Let \( H : (0, 1] \to \mathbb{R} \) be defined by

\[
H(\lambda) = \left( (H \Delta) \int_a^b F^-_\lambda(x)dx, (H \Delta) \int_a^b F^+_\lambda(x)dx \right).
\]

Since \( F^-_\lambda(x) \) and \( F^+_\lambda(x) \) are increasing and decreasing on \( \lambda \), respectively, therefore, when \( 0 < \lambda_1 \leq \lambda_2 \leq 1 \) we have \( F^-_{\lambda_1}(x) \leq F^-_{\lambda_2}(x), F^+_{\lambda_1}(x) \geq F^+_{\lambda_2}(x) \) on \([a, b] \). Thus from Theorem 2.5, we have

\[
\left( (H \Delta) \int_a^b F^-_{\lambda}(x)dx, (H \Delta) \int_a^b F^+_{\lambda}(x)dx \right) \supset \left( (H \Delta) \int_a^b F^-_{\lambda_1}(x)dx, (H \Delta) \int_a^b F^+_{\lambda_1}(x)dx \right).
\]

Using Theorem 2.5 and Lemma 3.3 we obtain

\[
\left( I \Delta \right) \int_a^b \tilde{F}(x)dx := \bigcup_{\lambda \in (0, 1]} \lambda \left( (H \Delta) \int_a^b F^-_\lambda(x)dx, (H \Delta) \int_a^b F^+_\lambda(x)dx \right) \in \tilde{\mathbb{R}}
\]

and for all \( \lambda \in (0, 1), \)

\[
\left( (F \Delta) \int_a^b \tilde{F}(x)dx \right)_\lambda = \bigcap_{n=1}^\infty \left( I \Delta \right) \int_a^b F_{\lambda_n}(x)dx,
\]
where $\lambda_n = [1 - 1/(n + 1)]\lambda$.

Using Theorem 3.5 and the properties of $(IH)$ integral, we can obtain the properties of $(FH_\Delta)$ integral. For examples, we get the linearity, monotonicity and interval additivity properties of $(FH_\Delta)$ integral.

References


*Department of Mathematics
Chungbuk National University
Cheongju 361-763, Republic of Korea
E-mail: wntoh@chungbuk.ac.kr

**Department of Mathematics Education
Chungbuk National University
Cheongju 361-763, Republic of Korea
E-mail: yoonjh@cbnu.ac.kr