SYMMETRIC BI-DERIVATIONS OF BCH-ALGEBRAS

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Abstract. The aim of this paper is to introduce the notion of left-right (resp. right-left) symmetric bi-derivation of BCH-algebras and some related properties are investigated.

1. Introduction

In 1966, Imai and Iseki introduced two classes of abstract algebras, BCK-algebra and BCI-algebras [6]. It is known that the class of BCI-algebras is a generalization of the class of BCK-algebras. In 1983, Hu and Li [3] introduced the notion of a BCH-algebra, which is a generalization of the notions of BCK-algebras and BCI-algebras. They have studied a few properties of these algebras. In this paper, we introduce the notion of left-right (resp. right-left) symmetric bi-derivations of BCH-algebras and investigate some properties of symmetric bi-derivations in a BCH-algebra. Moreover, we prove that the set of all symmetric bi-derivations on a medial BCH-algebra forms a semigroup under a suitably defined binary composition.

2. Preliminary

By a BCH-algebra, we mean an algebra \( (X, *, 0) \) with a single binary operation “\(*\)” that satisfies the following identities for any \( x, y, z \in X \):

- (BCH1) \( x * x = 0 \),
- (BCH2) \( x * y = 0 \) and \( y * x = 0 \) imply \( x = y \),
- (BCH3) \( (x * y) * z = (x * z) * y \), where \( x \leq y \) if and only if \( x * y = 0 \).

In a BCH-algebra, the following identities are true for all \( x, y \in X \):

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\[(x \ast (x \ast y)) \ast y = 0, \]
\[(x \ast 0) = 0 \implies x = 0, \]
\[(0 \ast (x \ast y)) = (0 \ast x) \ast (0 \ast y), \]
\[x \ast 0 = x, \]
\[(x \ast y) \ast x = 0 \ast y, \]
\[x \ast y = 0 \implies 0 \ast x = 0 \ast y. \]

**Definition 2.1.** Let \( I \) be a nonempty subset of a \( BCH \)-algebra \( X \). Then \( I \) is called an **ideal** of \( X \) if it satisfies:

(i) \( 0 \in I \),
(ii) \( x \ast y \in I \) and \( y \in I \) imply \( x \in I \).

**Definition 2.2.** A \( BCH \)-algebra is said to be **medial** if it satisfies

\[(x \ast y) \ast (z \ast w) = (x \ast z) \ast (y \ast w) \]

for all \( x, y, z, w \).

In a medial \( BCH \)-algebra, the following identity hold:

\[(x \ast (x \ast y)) = y \text{ for all } x, y \in X. \]

**Definition 2.3.** A \( BCH \)-algebra \( X \) is said to be **commutative** if \( y \ast (y \ast x) = x \ast (x \ast y) \) for all \( x, y \in X \). For a \( BCH \)-algebra \( X \), we denote \( x \land y = y \ast (y \ast x) \) for all \( x, y \in X \).

**Definition 2.4.** Let \( X \) be a \( BCH \)-algebra. A map \( d : X \to X \) is a **left-right derivation** (briefly, \( (l, r) \)-derivation) of \( X \) if it satisfies the identity

\[d(x \ast y) = (d(x) \ast y) \land (x \ast d(y)) \]

for all \( x, y \in X \). If \( d \) satisfies the identity

\[d(x \ast y) = (x \ast d(y)) \land (d(x) \ast y) \]

for all \( x, y \in X \), then \( d \) is a **right-left derivation** (briefly, \( (r, l) \)-derivation) of \( X \). Moreover, if \( d \) is both an \( (l, r) \) and \( (r, l) \)-derivation of \( X \), then \( d \) is a **derivation** of \( X \).

**Definition 2.5.** A \( BCH \)-algebra is said to be it associative if \( (x \ast y) \ast z = x \ast (y \ast z) \) for all \( x, y, z \in X \).

**Definition 2.6.** For any \( BCH \)-algebra, we define the set \( G(X) \) by as follows

\[G(X) = \{ x \in X | 0 \ast x = x \}. \]

**Definition 2.7.** Let \( X \) be a \( BCH \)-algebra. Then the set \( X_+ = \{ x \in X | 0 \ast x = 0 \} \) is called a **BCA-part** of \( X \).
3. Symmetric bi-derivations of $BCH$-algebras

In what follows, let $X$ denote a $BCH$-algebra unless otherwise specified.

**Definition 3.1.** Let $X, *, 0)$ be a $BCH$-algebra. Define a binary composition “$+$” on $X$ as follows:

$$x + y = x * (0 * y)$$

for any $x, y \in X$.

**Theorem 3.2.** In any medial $BCH$-algebra $(X, *, 0)$, if we define “$+$” as $x + y = x * (0 * y)$ for any $x, y \in X$, then the following properties hold:

1. $x + 0 = x = 0 + x$,
2. Addition is associative,
3. Addition is commutative,
4. Additive inverse of $x$ is $0 * x$.

**Proof.** (1) Let $X$ be a medial $BCH$-algebra and $x \in X$. Then

$$x + 0 = x * (0 * 0) = x * 0 = x = 0 * (0 * x) = 0 + x.$$

(2) Applying the definition of “$+$” repeatedly and simplifying, we have the result.

(3) For any $x, y \in X$,

$$x + y = 0 + (x + y) = (y * y) + (x * (0 * y))$$
$$= (y * y) * (0 * (x * (0 * y)))$$
$$= (y * y) * ((0 * x) * (0 * (0 * y)))$$
$$= (y * y) * ((0 * x) * y)$$
$$= (y * (0 * x)) * y = y * (0 * x)$$
$$= y * (0 * x) = y + x$$

(4) For any $x \in X$,

$$x + (0 * x) = x * (0 * (0 * x)) = x * x = 0.$$

Hence the additive inverse of $x$ is written as as $-x = 0 * x$.

**Definition 3.3.** Let $X$ be a medial $BCH$-algebra. If we define an addition “$+$” as $x + y = x * (0 * y)$ for all $x, y \in X$, then $(X, +)$ is an abelian group with identity $0$ and the additive inverse denoted by $-x = 0 * x$ for any $x \in X$. 

\[\Box\]
If we have a medial BCH-algebra \((X, *, 0)\), it follows from the above definition that \((X, +)\) is an abelian group with \(-y = 0 * y\) for any \(y \in X\). Then we have \(x - y = x * y\) for any \(x, y \in X\). On the other hand, if we choose an abelian group \((X, +)\) with an identity 0 and define \(x * y = x - y\), we obtain a medial BCH-algebra \((X, *, 0)\) where \(x + y = x * (0 * y)\) for any \(x, y \in X\).

Since \(x + (0 * y) = x * (0 * (0 * y)) = x * y\), for all \(x, y \in X\), we have \(x * y = x + (0 * y) = x - y\).

**Definition 3.4.** Let \(X, Y\) be BCH-algebras. An operation * on the Cartesian product \(X \times X\) of \(X, Y\) as follows: For \(x_1, x_2 \in X, y_1, y_2 \in Y\),

1. \((x_1, y_1) * (x_2, y_2) = (x_1 * x_2, y_1 * y_2)\),
2. \((0, 0) = 0\).

**Lemma 3.5.** A cartesian product of two BCH-algebras is again a BCH-algebra.

**Proof.** (1) For all \((x, y) \in X \times Y\), we have \((x, y) * (x, y) = (x * x, y * y) = (0, 0)\).

(2) For any \((x_1, y_1), (x_2, y_2) \in X \times Y\), let \((x_1, y_1) * (x_2, y_2) = (0, 0)\) and \((x_2, y_2) * (x_1, y_1) = (0, 0)\). Then we have \(x_1 * x_2 = 0\) and \(x_2 * x_1 = 0\), which means that \(x_1 = x_2\). Also, \(y_1 * y_2 = 0\) and \(y_2 * y_1 = 0\). Thus we get \(y_1 = y_2\). Hence \((x_1, y_1) = (x_2, y_2)\).

(3) For any \((x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y\), we get \(((x_1, y_1) * (x_2, y_2)) * (x_3, y_3) = ((x_1 * x_2) * x_3, (y_1 * y_2) * y_3) = ((x_1 * x_3) * x_2, (y_1 * y_3) * y_2) = ((x_1, y_1) * (x_3, y_3)) * (x_2, y_2)\).

**Definition 3.6.** Let \(X\) be a BCH-algebra. A map \(D : X \times X \rightarrow X\) is a symmetric map if \(D(x, y) = D(y, x)\) holds for all pairs of elements \(x, y \in X\).

**Example 3.7.** Let \(X = \{0, 1, 2, 3\}\) be a BCH-algebra with Cayley table as follows:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
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<tr>
<td>3</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

The map \(D : X \times X \rightarrow X\) defined by \(D(x, y) = x * (0 * y)\) is a symmetric map.

**Definition 3.8.** Let \(X\) be a BCH-algebra and let \(D : X \times X \rightarrow X\) be a symmetric mapping. A mapping \(d : X \rightarrow X\) defined by \(d(x) = D(x, x)\) is called a trace of \(D\).
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**Example 3.9.** In Example 3.4, \( d(0) = D(0,0) = 0 + 0 = 0, d(1) = D(1,1) = 1+1 = 0, d(2) = D(2,2) = 2+2 = 0, \)
\( d(3) = D(3,3) = 3+3 = 0. \)

**Definition 3.10.** Let \( X \) be a BCH-algebra and let \( D : X \times X \to X \) be a symmetric mapping. If \( D \) satisfies the identity, \( D(x \ast y, z) = (D(x, z) \ast y) \land (x \ast D(y, z)) \) for all \( x, y, z \in X \), then \( D \) is called a left-right symmetric bi-derivation (briefly, \((l, r)\)-symmetric bi-derivation) of \( X \).

If \( D \) satisfies the identity, \( D(x \ast y, z) = (x \ast D(y, z)) \land (D(x, z) \ast y) \) for all \( x, y, z \in X \), then \( D \) is called a right-left symmetric bi-derivation (briefly, \((r, l)\)-symmetric bi-derivation) of \( X \).

If \( D \) is both an \((l, r)\)-symmetric bi-derivation and an \((r, l)\)-symmetric bi-derivation, then \( D \) is called a symmetric bi-derivation of \( X \).

**Example 3.11.** In Example 3.4, define a mapping \( D : X \times X \to X \) by
\[
D(x, y) = \begin{cases} 
0 & \text{if } (x, y) = (0, 0) \\
0 & \text{if } (x, y) = (0, 1) \\
0 & \text{if } (x, y) = (1, 0) \\
2 & \text{if } (x, y) = (0, 2) \\
2 & \text{if } (x, y) = (2, 0) \\
1 & \text{if } (x, y) = (1, 1) \\
0 & \text{if } (x, y) = (2, 2) \\
2 & \text{if } (x, y) = (2, 1) \\
2 & \text{if } (x, y) = (1, 2) 
\end{cases}
\]

Then it is easily checked that \( D \) is a symmetric bi-derivation of \( X \).

**Example 3.12.** Let \( X = \{0, 1, 2\} \) be a BCH-algebra with Cayley table as follows:

\[
\begin{array}{c|ccc}
\ast & 0 & 1 & 2 \\
\hline
0 & 0 & 0 & 2 \\
1 & 1 & 0 & 2 \\
2 & 2 & 2 & 0 \\
\end{array}
\]

Define a map \( D : X \times X \to X \) by

\[
D(x, y) = \begin{cases} 
0 & \text{if } (x, y) = (0, 0) \\
0 & \text{if } (x, y) = (0, 1) \\
0 & \text{if } (x, y) = (1, 0) \\
2 & \text{if } (x, y) = (0, 2) \\
2 & \text{if } (x, y) = (2, 0) \\
1 & \text{if } (x, y) = (1, 1) \\
0 & \text{if } (x, y) = (2, 2) \\
2 & \text{if } (x, y) = (2, 1) \\
2 & \text{if } (x, y) = (1, 2) 
\end{cases}
\]

Then it is easily checked that \( D \) is a symmetric bi-derivation of \( X \).

**Proposition 3.13.** Let \( X \) be a medial BCH-algebra. Define a symmetric map \( D : X \times X \to X \) by \( D(x, y) = x + y \) for all \( x, y \in X \). Then \( D \) is a \((l, r)\)-symmetric bi-derivation of \( X \).
Proof. For all $x, y, z \in X$, we have

\[
D(x \ast y, z) = (x \ast y) + z = (x \ast y) \ast (0 \ast z) \\
= (x \ast (0 \ast z)) \ast y = (x + z) \ast y \quad (\because (x \ast y) \ast z = (x \ast z) \ast y) \\
= (x \ast (y + z)) \ast ((x \ast (y + z)) \ast (x + z) \ast y) \\
= (x + z) \ast (x \ast (y + z)) \\
= (x \ast z) \ast y = (x \ast (y + z)) \\
= (D(x, z) \ast y) \wedge (x \ast (D(y, z))).
\]

This proves that $D$ is a $(l, r)$-symmetric bi-derivation of $X$. \hfill \Box

Theorem 3.14. Let $X$ be an associative medial BCH-algebra. Then the symmetric map $D : X \times X \to X$ defined by $D(x, y) = x + y$ for all $x, y \in X$ is a symmetric bi-derivation of $X$.

Proof. By the above proposition, $D$ is a $(l, r)$-symmetric bi-derivation of $X$. For all $x, y, z \in X$, we have

\[
D(x \ast y, z) = (x \ast y) + z = (x \ast y) \ast (0 \ast z) \\
= (x \ast (0 \ast z)) \ast y = ((x \ast z) \ast y) \\
= (x \ast z) \ast y = (x \ast (y + z)).
\]

Also, we have for any $x, y, z \in X$,

\[
(x \ast D(y, z)) \wedge (D(x, z) \ast y) = x \ast D(y, z) \\
= (x \ast (y + z)) \ast ((x \ast z) \ast y) \\
= x \ast (y + z) \ast (x \ast z) \\
= (x \ast (y \ast z)) \ast (x \ast z) \\
= (x \ast y) \ast z.
\]

From (1) and (2), $D(x \ast y, z) = (x \ast D(y, z)) \wedge (D(x, z) \ast y)$ for all $x, y, z \in X$. This proves that $D$ is a $(r, l)$-symmetric bi-derivation, and so a symmetric bi-derivation of $X$. \hfill \Box

Proposition 3.15. Let $X$ be a medial BCH-algebra and let $D$ be a symmetric map. Then we have for any $x \in X$,

1. If $D$ is a $(l, r)$-symmetric bi-derivation of $X$ and $(x \ast z) \ast (y \ast z) = x \ast y$, then $D(x, y) = D(x, y) \wedge x$.
2. If $D$ is a $(r, l)$-symmetric bi-derivation of $X$, then $D(x, y) = x \wedge D(x, y)$ for all $x, y \in X$ if and only if $D(0, y) = 0$ for all $x \in X$. 
Symmetric bi-derivations of BCH-algebras

Proof. (1) Let $D$ be a $(l, r)$-symmetric bi-derivation of $X$. Then we have
\[
D(x, y) = D(x \ast 0, y) = D(x, y) \ast 0 \land (x \ast D(0, y)) = D(x, y) \land (x \ast D(0, y)) = (x \ast D(0, y)) \ast ((x \ast D(0, y)) \ast D(x, y)) = (x \ast D(0, y)) \ast ((x \ast D(x, y)) \ast D(0, y)) = x \ast (x \ast D(x, y)) \quad (\because (x \ast y) \ast z = (x \ast z) \ast y)
\]
\[
= D(x, y) = D(x, y) \land x.
\]

(2) Let $D$ be a $(r, l)$-symmetric bi-derivation of $X$ and $D(0, y) = 0$ for all $y \in X$. Then we have
\[
D(x, y) = D(x; y) = (x \ast 0, y) \land (D(x, y) \ast 0) = (x \ast 0) \land D(x, y) = x \land D(x, y).
\]
Conversely, if $D(x, y) = x \land D(x, y)$ for all $x, y \in X$, then
\[
D(0, y) = 0 \land D(0, y) = D(0, y) \ast D(0, y) = 0.
\]

\[\square\]

Proposition 3.16. Let $X$ be a medial BCH-algebra and let $D : X \times X \to X$ be a $(l, r)$-symmetric bi-derivation of $X$. Then
(1) $D(a, y) = D(0, y) \ast (0, a) = D(0, y) + a$ for all $a, x, y \in X$,
(2) $D(a + b, y) = D(a, y) + D(b, y) - D(0, y)$ for all $a, b, x, y \in X$,
(3) $D(a, y) = a$ if and only if $D(0, y) = 0$ for all $a, y \in X$.

Proof. (1) Let $(l, r)$-symmetric bi-derivation of $X$ and let $a = 0 \ast (0 \ast a)$. Then we have
\[
D(a, y) = D(0 \ast (0 \ast a), y) = (D(0, y) \ast (0 \ast a)) \land (0 \ast D(0 \ast a, y)) = D(0, y) \ast (0 \ast a) \quad (\because x \land y = x)
\]
\[
= D(0, y) + a
\]
for all $a, x, y \in X$, for any $a, x, y \in X$. 

\[\square\]
(2) By (1), we get for any \( a, b, y \in X \),
\[
D(a + b, y) = D(0, y) + a + b
\]
\[
= D(0, y) + a + D(0, y) + b - D(0, y)
\]
\[
= D(a, y) * D(b, y) - D(0, y).
\]

(3) Let \( D(a, y) = a \) for any \( a, y \in X \). Putting \( a = 0 \), then we get \( D(0, y) = 0 \) for any \( y \in X \). Conversely, if \( D(0, y) = 0 \), then \( D(a, y) = 0 + a = a \).

**Proposition 3.17.** Let \( X \) be a medial BCH-algebra and let \( D : X \times X \to X \) be a \((r, l)\)-symmetric bi-derivation of \( X \). Then

1. \( D(a, y) \in G(X) \) for any \( a \in G(X) \),
2. \( D(a, y) = a * D(0, y) = a + D(0, y) \) for any \( a, y \in X \),
3. \( D(a + b, y) = D(a, y) + D(b, y) - D(0, y) \) for all \( a, b, y \in X \),
4. \( D(a, y) = a \) for any \( a, y \in X \) if and only if \( D(0, y) = 0 \).

**Proof.**

(1) Let \( a \in G(X) \). Then \( 0 * a = a \), and so
\[
D(a, y) = D(0 * a, y)
\]
\[
= (0 * (D(a, y)) \wedge (D(0, y) * a)
\]
\[
= (D(0, y) * a) * ((D(0, y) * a) * (0 * D(a, y)))
\]
\[
= 0 * D(a, y).
\]
This implies that \( D(a, y) \in G(X) \).

(2) For any \( a, y \in X \), we get
\[
D(a, y) = D(a * 0, y)
\]
\[
= (a * (D(0, y)) \wedge (D(a, y) * 0)
\]
\[
= (a * (D(0, y)) \wedge D(a, y)
\]
\[
= D(a, y) * (D(a, y) * (a * D(0, y)))
\]
\[
= a * D(0, y).
\]
Again, for any \( a, y \in X \), we get
\[
D(a, y) = a * D(0, y)
\]
\[
= (a * (D(0, y)) \wedge (D(a, y) * 0)
\]
\[
= a * D(0 * (D(0, y)) \wedge (D(0, y) * 0)
\]
\[
= a * (0 * D(0, y))
\]
\[
= a + D(0, y).
\]
(3) For any $a, b, y$, we have
\[
D(a + b, y) = a + b + D(0, y)
= a + D(0, y) + b + D(0, y) - D(0, y)
= D(a, y) + D(b, y) - D(0, y).
\]

(4) If $D(0, y) = 0$, then $D(a, y) = D(a \ast 0, y) = a \ast D(0, y) = a \ast 0 = a$
by (2). Conversely, if $D(a, y) = a$ for any $a \in X$, we get $D(0, y) = 0$. □

**Definition 3.18.** Let $X$ be a BCH-algebra and let $D : X \times X \rightarrow X$
be a symmetric mapping. If $D(0, z) = 0$, for all $z \in X$, $D$ is called
componentwise regular. In particular, if $D(0, 0) = d(0) = 0$, $D$ is called
d-regular.

**Proposition 3.19.** Let $D$ be a $(r, l)$-symmetric bi-derivation of $X$
and $0 \ast x = 0$ for all $x \in X$. Then $D$ is d-regular.

**Proof.** Let $D$ be a system bi-derivation of $X$ and $0 \ast x = 0$ for all
$x \in X$. Then we have
\[
D(0, 0) = D(0 \ast x, 0) = (0 \ast D(x, 0)) \wedge (D(0, 0) \ast x)
= 0 \wedge (D(0, 0) \ast x)
= 0.
\]
Hence $D$ is d-regular. □

**Theorem 3.20.** Let $D$ be an $(l, r)$-symmetric bi-derivation of $X$. If
there exists $a \in X$ such that $D(x, z) \ast a = 0$ for all $x, z \in X$, then $D$
is componentwise regular.

**Proof.** Let $D(x, y) \ast a = 0$ for all $x, z \in X$. Then
\[
0 = D(x \ast a, z) \ast a = ((D(x \ast z) \ast a) \wedge (D(0, 0) \ast x)) \ast a
= (0 \wedge (D(0, 0) \ast x)) \ast a
= 0 \ast a,
\]
that is, $0 \leq a$, and so
\[
D(0, z) = D(0 \ast a, z)
= (D(0, z) \ast a) \wedge (0 \ast D(a, z))
= 0 \wedge (0 \ast D(a, z)) = 0.
\]
Hence $d$ is componentwise regular. □

**Corollary 3.21.** Let $D$ be an $(l, r)$-symmetric bi-derivation of $X$.
If there exists $a \in X$ such that $D(x, z) \ast a = 0$ for all $x, z \in X$, then $D$
is d-regular.
THEOREM 3.22. Let $D$ be an $(r, l)$-symmetric bi-derivation of $X$. If there exists $a \in X$ such that $a \ast D(x, z) = 0$ for all $x, z \in X$, then $D$ is componentwise regular.

Proof. Let $D(x, y) \ast a = 0$ for all $x, z \in X$. Then
\[
0 = a \ast D(x \ast a, z) = a \ast ((a \ast D(x \ast z)) \wedge (D(a, z) \ast x))
\]
\[
= a \ast (0 \wedge (D(a, z) \ast x))
\]
\[
= a \ast 0,
\]
This shows that
\[
D(0, z) = D(a \ast 0, z)
\]
\[
= (a \ast D(0, z)) \wedge (D(a, z) \ast 0)
\]
\[
= 0 \wedge D(a, z) = 0.
\]
Hence $D$ is componentwise regular.

COROLLARY 3.23. Let $D$ be an $(r, l)$-symmetric bi-derivation of $X$. If there exists $a \in X$ such that $a \ast D(x, z) = 0$ for all $x, z \in X$, then $D$ is $d$-regular.

Let $D$ be a symmetric bi-derivation of $X$ and $a \in X$. Define a set $Fix_a(X)$ by
\[
Fix_a(X) := \{ x \in X \mid D(x, a) = x \}
\]
for all $x \in X$.

PROPOSITION 3.24. Let $D$ be a symmetric bi-derivation of $X$. Then $Fix_a(X)$ is a subalgebra of $X$.

Proof. Let $x, y \in Fix_a(X)$. Then we have $D(x, a) = x$ and $D(y, a) = y$, and so
\[
D(x \ast y, a) = (D(x, a) \ast y) \wedge (x \ast D(y, a))
\]
\[
= (x \ast y) \wedge (x \ast y)
\]
\[
= (x \ast y) \ast ((x \ast y) \ast (x \ast y))
\]
\[
= (x \ast y) \ast 0 = x \ast y.
\]
Hence we get $x \ast y \in Fix_a(X)$. This completes the proof.

PROPOSITION 3.25. Let $D$ be a symmetric bi-derivation of $X$. If $x, y \in Fix_a(X)$, we obtain $x \wedge y \in Fix_a(X)$.
Hence we get $x \wedge y \in \text{Fix}_a(X)$. This completes the proof. □

**Proposition 3.26.** Let $X$ be a commutative $BCH$-algebra and $d$ a trace of $D$. Then, if $x \leq y$ for all $x, y \in X$, then $d(x \wedge y) = d(x)$.

**Proof.** Let $x \leq y$. Then we get $x \ast y = 0$ and

\[
d(x \wedge y) = D(x \wedge y, x \wedge y) = D(y \ast (y \ast x), y \ast (y \ast x)) = D(x \ast (x \ast y), x \ast (x \ast y)) = D(x, x) = d(x).
\]

This completes the proof. □

**Definition 3.27.** Let $X$ be a $BCH$-algebra. A self-map $d$ on $X$ is said to be isotone if $x \leq y$ implies $d(x) \leq d(y)$ for $x, y \in X$.

Let $\text{Der}(X)$ denote the set of all $(l, r)$-symmetric bi-derivation on $X$. Define the binary operation “$\wedge$” on $\text{Der}(X)$ as follows:

\[
(D_1 \wedge D_2)(x, y) = D_1(x \ast y) \wedge D_2(x, y)
\]

for any $D_1, D_2 \in \text{Der}(X)$ and $x, y \in X$.

**Proposition 3.28.** Let $D_1$ and $D_2$ are $(l, r)$-symmetric bi-derivations on $X$. Then $D_1 \wedge D_2$ is also a $(l, r)$-symmetric bi-derivation of $X$.

**Proof.** Let $D_1$ and $D_2$ are $(l, r)$-symmetric bi-derivations on $X$. Then

\[
(D_1 \wedge D_2)(x \ast y, z) = (D_1 \wedge D_2)(x, z) \ast (x \ast ((D_1 \wedge D_2)(y, z))).
\]

\[
(D_1 \wedge D_2)(x \ast y, z) = D_1(x \ast y, z) \wedge D_2(x \ast y, z) = D_2(x \ast y, z) \ast (D_2(x \ast y, z) \ast D_1(x \ast y, z)) = D_1(x \ast y, z)
\]

\[
= (D_1(x, z) \ast y) \wedge (x \ast D_1(y, z)) = (x \ast D_1(y, z)) \ast ((x \ast D_1(y, z)) \ast (D_1(x, z) \ast y)) = D_1(x, z) \ast y
\]

(1)
\[ ((D_1 \land D_2)(x, z) \ast y) \]
\[ = (x \ast (D_1 \land D_2)(y, z) \ast ((x \ast (D_1 \land D_2)(y, z)) \ast (D_1 \land D_2)(x, z) \ast y)) \]
\[ = (D_1(x, z) \land D_2(x \ast y, z) \ast (D_2(x \ast y, z) \ast D_1(x \ast y, z)) \]
\[ = D_1(x \ast y, z) \]
\[ = (D_1 \land D_2)(x, z) \ast y \]
\[ = (D_1(x, z) \land D_2(x, z)) \ast y \]
\[ = (D_2(x, z) \ast (D_2(x, z) \ast D_1(x, z))) \ast y \]
\[ = D_1(x, z) \ast y \] (2)

Combining (1) and (2), we prove that \( D_1 \land D_2 \) is a \((l, r)\)-symmetric bi-derivation of \( X \). \qed

**Proposition 3.29.** The binary composition “\( \land \)” defined on \( \text{Der}(X) \) is associative.

**Proof.** Let \( D_1, D_2 \) and \( D_2 \) are \((l, r)\)-symmetric bi-derivations on \( X \). Then

\[ ((D_1 \land D_2) \land D_3)(x \ast y, z) \]
\[ = ((D_1 \land D_2)(x \ast y, z)) \land D_3(x \ast y, z)) \]
\[ = (D_1(x, z) \ast y) \land D_3(x \ast y, z)) \]
\[ = (D_3(x \ast y, z) \ast (D_3(x, z) \ast D_1(x, z) \ast y) \]
\[ = D_1(x, z) \ast y \] (1)

\[ (D_1 \land (D_2 \land D_3))(x \ast y, z) \]
\[ = (D_1(x, z) \land (D_2 \land D_3)(x \ast y, z)) \]
\[ = (D_1(x, z) \land (D_2(x, z) \ast y) \]
\[ = (D_2(x, z) \ast y) \ast ((D_2(x, z) \ast y) \ast (D_1(x \ast y, z)) \]
\[ = D_1(x \ast y, z) \]
\[ = (D_1(x, z) \ast y) \land (x \ast D_1(y, z)) \]
\[ = (x \ast D_1(y, z)) \ast ((x \ast D_1(y, z)) \ast (D_1(x, z) \ast y) \]
\[ = D_1(x, z) \ast y. \] (1)

Combining (1) and (2), we have \((D_1 \land D_2) \land D_3 = D_1 \land (D_2 \land D_3)\), which implies that “\( \land \)” is associative. \qed
Combining the above two propositions, we obtain the following theorem.

**Theorem 3.30.** $\text{Der}(X)$ is a semigroup under the binary composition $\wedge$.

**References**


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