**ENTROPY, POSITIVELY CONTINUUM-WISE EXPANSIVENESS AND SHADOWING**

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Abstract. Let $(X, d)$ be a compact metric space, and let $f : X \to X$ be a continuous map. We consider that if a positively continuum-wise expansiveness continuous map $f$ has the positively shadowing property in the nonwandering set, then the topological entropy is positive.

1. Introduction

Let $f : X \to X$ be a continuous map of a compact metric space $(X, d)$. For the dynamic properties (shadowing, positively measure expansive) and entropy, Morales [3] proved that if a homeomorphism $f : X \to X$ has the shadowing property in the nonwandering set and it is positively measure expansive then the topological entropy is positive. In the paper, we consider that if a continuous map $f : X \to X$ has the positively shadowing property in the nonwandering set and it is another type of expansiveness then the topological entropy is positive.

We say that a point $x \in X$ is *nonwandering* if for any neighborhood $U$ of $x$, there is $n > 0$ such that $f^n(U) \cap U \neq \emptyset$. Denote by $\Omega(f)$ the set of all nonwandering points of $f$. For given $x, y \in M$, we write $x \rightsquigarrow y$ if for any $\delta > 0$, there is a finite $\delta$-pseudo orbit $\{x_i\}_{i=a}^{b} (a < b)$ of $f$ such that $x_a = x$ and $x_b = y$. The set of points $\{x \in M : x \rightsquigarrow x\}$ is called the *chain recurrent set* of $f$ and is denoted by $\mathcal{CR}(f)$. Then it is known that $\Omega(f) \subset \mathcal{CR}(f)$. For any $\delta > 0$, a sequence $\{x_i\}_{i \in \mathbb{Z}_+}$ is a $\delta$-pseudo orbit of $f$ if

Received October 15, 2015; Accepted November 05, 2015.

2010 Mathematics Subject Classification: Primary 37B40; Secondary 37B05.

Key words and phrases: positively continuum-wise expansive, positively measure expansive, entropy, shadowing.

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*This work is supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Science, ICT & Future Planning (No. 2014R1A1A1A05002124).
Let $\Lambda \subset X$ be a closed $f$-invariant set. We say that $f$ has the \textit{positively shadowing property} on $\Lambda$ if for any $\epsilon > 0$, there is $\delta > 0$ such that for any $\delta$-pseudo orbit $\{x_i\}_{i \geq 0} \subset \Lambda$ there is $y \in X$ such that $d(f^i(y), x_i) < \epsilon$ for $i \geq 0$.

Then the shadowing points can be in $\Lambda$ or is in $X$. On the other hand, we say that $f$ has the \textit{positively shadowing property} in $\Lambda$ if for any $\epsilon > 0$ there is $\delta > 0$ such that for any $\delta$-pseudo orbit $\{x_i\}_{i \geq 0} \subset \Lambda$ there is $y \in \Lambda$ such that $d(f^i(y), x_i) < \epsilon$ for $i \geq 0$.

If $\Lambda = X$ then we say that $f$ has the \textit{positively shadowing property}.

Recall Bowen’s definition of topological entropy on a closed set $A$.

For $n \in \mathbb{N}$ and $\epsilon > 0$, a set $B \subset A$ is said to be $(B, n, \epsilon)$-separated for $f$ if for any distinct two points $a, b \in A$ there is $k \in \{0, 1, \ldots, n - 1\}$ such that $d(f^k(a), f^k(b)) > \epsilon$. Let $\Phi_f(A, n, \epsilon)$ denote the maximal cardinality of a $(A, n, \epsilon)$-separated set for $f$ contained in $A$. Since $X$ is compact, $\Phi_f(A, n, \epsilon)$ is finite. The topological entropy of $f$ on $A$ is defined by

$$h(f, A) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log \Phi_f(A, n, \epsilon)}{n}.$$ 

We can denote $h(f) = h(f, A)$ if there is no confusion.

Given $x \in X$ and $\delta > 0$, we define the \textit{dynamical $\delta$-ball} as follows, $\Gamma_\delta(x) = \{y \in X : d(f^i(x), f^i(y)) \leq \delta \text{ for all } i \geq 0\}$. Let $\mathcal{M}(X)$ be the set of all Borel probability measures on $X$ and let $\mathcal{M}^*(X)$ be the set of nonatomic measures $\mu \in \mathcal{M}(X)$. We say that $f$ is \textit{positively measure expansive} (or, \textit{positively $\mu$-expansive}) if there is $\delta > 0$ (called an\textit{ expansive constant}) such that $\mu(\Gamma_\delta(x)) = 0$ for any $\mu \in \mathcal{M}^*(X)$. Note that in [3, Lemma 8], Morales proved that a homeomorphism $f : X \to X$ is positively measure expansive if and only if it is positively invariant measure expansive.

Kato [2] introduced positively continuum-wise expansiveness which is a more general notion of positively measure expansiveness (see [1, Lemma 2.3]). A continuous map $f$ on $X$ is said to be \textit{positively continuum-wise expansive} if there is a constant $e > 0$ such that for any nondegenerate (is not singleton) continuum $A$ there is an integer $n \geq 0$ such that $\text{diam} f^n(A) \geq e$, where $\text{diam} A = \sup\{d(x, y) : x, y \in A\}$ for any subset $A$ of $X$. Here the constant $e$ is called a \textit{positively continuum-wise expansive constant} for $f$. In this paper, we prove that if $f$ has the positively shadowing property in $\Omega(f)$ and $f$ is positively continuum-wise expansive, then the topological entropy is positive.
We say that \( f \) is \textit{equicontinuous} if for any \( \epsilon > 0 \) there is \( \delta > 0 \) such that for any \( x, y \in X \) if \( d(x, y) < \delta \) then \( d(f^i(x), f^i(y)) < \epsilon \) for all \( i \in \mathbb{Z}_+ \). It is known that an equicontinuous map \( f : X \to X \) has \( h(f) = 0 \).

\textbf{Lemma 1.1.} \cite[Corollary 6]{[4]} \textit{If \( f \) has the positively shadowing property and \( h(f) = 0 \) then \( f|_{\Omega(f)} : \Omega(f) \to \Omega(f) \) is an equicontinuous homeomorphism.}

The following was proved in \cite[Lemma 10]{[3]} for a homeomorphism \( f : X \to X \). However, in the paper, we consider for a continuous map \( f : X \to X \).

\textbf{Lemma 1.2.} If a continuous map \( f : X \to X \) is equicontinuous then \( f \) is not positively measure expansive.

\textit{Proof.} Suppose, by contradiction that an equicontinuous map \( f \) is positively measure \( \mu \)-expansive for some \( \mu \in \mathcal{M}^*(X) \). Let \( \epsilon > 0 \) be the number of the positively measure expansive constant of \( \mu \). Since \( f \) is equicontinuous, there is \( \delta > 0 \) such that \( B[x, \delta] \subset \Gamma_\epsilon(x) \) for all \( x \in X \), where \( B[x, \delta] = \{ y \in X : d(x, y) \leq \delta \} \) is a closed \( \delta \)-ball centered at \( x \). Note that \( f \) is positively measure expansive if and only if \( f|_{\Omega(f)} \) is positively measure expansive. Thus we know \( \mu(B[x, \delta]) = 0 \) for \( x \in \Omega(f) \). Since \( X \) is compact, we can take finite points \( x_1, x_2, \ldots, x_n \) such that \( X = \bigcup_{i=1}^n B[x_i, \delta] \). Then we know

\[ \mu(X) \leq \sum_{i=1}^n \mu(B[x_i, \delta]). \]

Since \( f \) is positively measure expansive, \( \mu(\Gamma_\epsilon(x)) = 0 \) and so, \( \mu(B[x, \delta]) = 0 \). Thus \( \sum_{i=1}^n \mu(B[x_i, \delta]) = 0 \), which is a contradiction \( \mu(X) \neq 0 \). \( \blacksquare \)

\textbf{Lemma 1.3.} A continuous map \( f : X \to X \) is positively continuum-wise expansive if and only if there is \( \delta > 0 \) such that for \( x \in X \), if a continuum \( C \subset \Gamma_\delta(x) \) then \( C \) is a singleton.

\textit{Proof.} The proof is similar to \cite[Lemma 2.2]{[1]}. \( \blacksquare \)

For a closed invariant set \( \Lambda \), it is known that a continuous map \( f : X \to X \) is continuum-wise expansive if and only if \( f|_{\Lambda} \) is continuum-wise expansive.

\textbf{Lemma 1.4.} Let \( \Lambda \subset X \) be a closed \( f \)-invariant set. If \( f|_{\Lambda} \) is equicontinuous then \( f \) is not positively continuum-wise expansive.
Proof. Let $f|\Lambda$ be equicontinuous. To derive a contradiction, we may assume that $f$ is positively continuum-wise expansive. Let $e > 0$ be a positively continuum-wise expansive constant of $f$. Since $f$ is an equicontinuous, there is $\delta \in (0, e)$ such that if for any $x, y \in \Lambda$ with $d(x, y) < \delta$ then $d(f^i(x), f^i(y)) < e$ for all $i \in \mathbb{Z}_+$. We set $C_\delta(x) = \{y \in \Lambda \setminus \{x\} : d(x, y) \leq \delta\}$. Then it is clear $C_\delta(x) \subset \Gamma_e(x) = \{y \in \Lambda : d(f^i(x), f^i(y)) \leq e \text{ for all } i \geq 0\}$. Since $f$ is positively continuum-wise expansive, $f|\Lambda$ is positively continuum-wise expansive. By Lemma 1.3, $C_\delta(x)$ must be a singleton which is a contradiction since $C_\delta(x)$ is not singleton. Thus if $f|\Lambda$ is equicontinuous, then $f$ is not positively continuum-wise expansive.

Theorem 1.5. Let $f : X \to X$ be a continuous map and $f$ have the positively shadowing property in $\Omega(f)$. If $f$ is positively continuum-wise expansive then the topological entropy $h(f)$ is positive.

Proof. Let $f$ have the shadowing property in $\Omega(f)$ and $f$ be positively continuum-wise expansive. Suppose, by contradiction, that $h(f) = 0$. Since $f$ has the positively shadowing property in $\Omega(f)$ and $h(f) = 0$, by Lemma 1.1 $f|\Omega(f)$ is equicontinuous. Since $\Omega(f)$ is closed and invariant, by Lemma 1.4, $f$ is not positively continuum-wise expansive which is a contradiction.

References

Entropy, positively continuum-wise expansiveness and shadowing

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